Banff International Research Station
for Mathematical Innovation and Discovery

10w5125 Some mathematical problems of material science: effects of multiple scales and extreme aspect ratios
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ABSTRACTS

Ellen ShiTing Bao

Title: Gradient Estimates for Elliptic Equation and System from Composite Media

Abstract:
We establish both upper and lower bounds of the gradient estimates for solutions to the perfect conductivity problem in the case where perfect (stiff) conductors are closely spaced inside an open bounded domain. These results give the optimal blow-up rates of the stress for conductors with arbitrary shape and in all dimensions. We also obtain an upper bound of the gradient estimates to the insulated case. As for system we recently derive local gradient estimates when the solution takes constant values on the inclusions.

Leonid Berlyand

Title: Flux norm approach to finite-demensiona homogenization approximation with non-separated scales and high contrast.

Abstract:
Homogenization Theory deals with mathematical models of strongly inhomogeneous media composed of Classicalmaterials with vastly different properties described by PDEs with rapidly oscillating coefficients of the form $A(x/\epsilon)$, $\epsilon \to 0$.

The goal is to approximate this problem by a homogenized (simpler) PDE with slowly varying coefficients that do not depend on the small parameter ϵ. The original problem has two scales: fine $O(\epsilon)$ and coarse $O(1)$, whereas the homogenized problem has only coarse scale.

The homogenization of PDEs with periodic or ergodic coefficients and well separated scales is now well understood. In a joint work with H. Owhadi (Caltech) we consider the most general case of arbitrary meausrable coefficients. The homogenization of PDEs with periodic or ergodic coefficients and well separated scales is now well understood. In a joint work with H. Owhadi (Caltech) we consider the most general case of arbitrary meausrable coefficients. Specifically, we study divergence-form scalar elliptic equations and vectorial equations with such coefficients. For these problems we establish two

finite-dimensional approximations of solutions, which we refer to as finite-dimensional homogenization approximations. We introduce a flux norm and establish the error estimate in this norm with an explicit and {\it optimal} error constant {\it independent of the contrast} and regularity of the coefficients. A proper generalization of the notion of cell problem is the key technical issue in our consideration.

Virginie Bonnaillie

Title: Interactions between moderately close inclusions for the Laplace equation and applications in mechanics

Abstract:
The presence of small inclusions or surface defects alters the solution of the Laplace equation posed in a reference domain. If the characteristic size of the perturbation is small, one can expect the solution of the problem posed on the perturbed geometry to be close to the solution of the reference shape. An asymptotic expansion with respect to that small parameter (the characteristic size of the perturbation) can then be performed. The case of a single inclusion, centered either in the reference domain either on its boundary has been deeply studied. The techniques rely on the notion of profile, a normalized solution of the Laplace equation in the exterior domain obtained by blow-up of the perturbation. It is used in a fast variable to describe the local behavior of the solution in the perturbed domain. Convergence of the asymptotic expansion is obtained thanks to the decay of the profile at infinity. In this talk, I will first briefly talk about the case of multiples inclusions, in particular about the intermediate cases, where the inclusions are moderately close, i.e. the distance between them is a third intermediary scale between the size of the inclusion and the size of the whole object. One can expect to have a weak interaction between the two inclusions. In a joint work with M. Dambrine, S. Tordeux and G. Vial, we have quantified this effect and provided a completed asymptotic description of the solution of Laplace equation. In a second time, I will talk about the numerical computation of the profile. In order to have an accurate computation, we have looked for a transparent boundary condition for an exterior boundary value problem in planar linear elasticity. The goal is to bound the infinite domain by a large ``box" to make numerical approximations possible (typically a large ball of radius R). The solution of the problem set in this bounded domain has to be close to the original solution ; the convergence is expected as R goes to infinity. Precisely, with D. Brancherie, M. Dambrine and G. Vial, we have considered the case of a linear elastic material in the exterior of a bounded domain on the boundary of which the displacement is prescribed. In that case, cancelling the leading singular parts at infinity of the solution leads to the approximate boundary condition of Ventcell's type set on the circle of radius R. The physical parameters E and are such that the quantity in front of the Laplace-Beltrami operator is nonnegative: Young's modulus E is nonnegative and Poisson's coefficient takes values in the interval (-1, 0.5). In that case, the usual variational approach for treating Ventcell's boundary condition are not at hands. With M. Dambrine, F. rau and G. Vial, we proved that this problem is well posed provided the truncation radius R is chosen big enough.

Eric Bonnetier

Title: Super-resolution in structured media, and small volume asymptotics

Abstract :
We consider the Helmholtz equation in a medium that contains a periodic array of scatterers, surrounded by a reference medium. We show that a dielectric defect, placed in the region where the scatterers are distributed, is felt in the far field as a dipole, that radiates in an effective medium. Its refractive index combines the material properties of the ambient medium and of the scatterers. Their presence thus affects the focusing properties of light around the defect. We discuss how this mechanism may be related to super-resolution phenomena that have been observed in recent time-reversal experiments.

Kurt Bryan

Title: Blow up of solutions to the heat equation with nonlinear boundary conditions

Abstract:
I'll discuss the behavior of solutions to the heat equation on boundeddomains when the boundary condition on the solution "u" is of the form du/dn = f(u) for a nonlinear function f. In particular, if the function f grows rapidly enough, most solutions blow up in finite time. I'll summarize a few known results and then present some work Michael Vogelius and I have done on the problem.

Yves Capdeboscq

Title: About Extremal Structures in Conductivity and Elasticity in the Small Volume Fraction Limit

Abstract:
The effect of perturbations of small volume on the Lam^[%G\303\257\302\277\302\275^[%@ Coefficients of an elastic body can be used to detect imperfections of small volume. In the analysis of such perturbations, an elastic moment tensor appears.
This quantity also appears naturally in the theory of composites, when one investigates inclusion shapes of minimal energy. In the context of inverse problems, it is relevant to know if for a given volume, one can distinguish thick inclusions from thin ones. We will discuss recent progresses on the characterization of inclusion shapes corresponding to extremal elastic moment tensor.

Juan Davila

Title: Bistable boundary reactions in two dimensions

Abstract:
Let $\Omega \subset \mathbb R^2$ be a bounded domain with smooth boundary. We consider the problem $$ \Delta u = 0 \quad \hbox{in } \Omega, \quad \frac{\partial u}{\partial \nu} = \frac1\varepsilon f(u) \quad \hbox{on } \partial\Omega,$$
where ν is the unit normal exterior vector, $\varepsilon > 0$ is a small parameter and f is a bistable nonlinearity like $f(u) = \sin(\pi u) $ or $f(u) = (1-u^2) u$. We construct solutions that develop multiple transitions from -1 to 1 and vice versa along a connected component of the boundary $\partial \Omega$. We also construct an explicit solution when Ω is the disk and $f(u) = \sin(\pi u) $.

Maria-Carme Calderer

Title: Gels: modeling, analysis and computing

Abstract:
This work is motivated by applications of polymers in body-implantable biomedical devices. Biomedical devices (e.g., pacemakers, bone replacement devices, artificial skin, artificial cornea) become gels upon implanting into the body. Stresses at the boundaries or at the interfaces between different materials of the device may cause device failure within its intended lifetime. We model these systems as immiscible and incompressible mixtures of fluid and nonlinear elastic solid. The relevant physical mechanisms include transport, diffusion, dissipation, elasticity and surface effects. The permeability properties of the gel play a main role in formulating the boundary conditions of the problem. The presence of multiple lenght and time scales and their relative sizes is a main feature of the model. We explore well-posedness and analyze mixed finite element methods to simulate the governing systems of equations.

Mikyoung Lim

Title: Blow-up of Electric Fields between Closely Spaced Spherical Perfect Conductors

Abstract:
The electric field increases toward infinity in the narrow region between closely adjacent perfect conductors as they approach each other. We establish optimal estimates for the electric field associated with the distance between two spherical conductors in n-dimensional spaces for n which is equal to or greater than 2. The novelty of these estimates is that they explicitly describe the dependency of the blow-up rate on the geometric parameters: the radii of the conductors.

Marta Lewicka

Title: Scaling laws and reduced theories of prestrained elastic films.

Abstract:
We will present the recent results concerning analysis and rigorous derivation of plate and shell models for thin films exhibiting residual stress at free equilibria. Examples of such structures include growing tissues (e.g. leaves). There, it is conjectured that the cell division results in the formation of non-Euclidean 'target metrics', leading to the tissue's complicated morphogenesis.

The variational analysis of these phenomena departs from the model of 3d non-Euclidean elastic energy, which measures the pointwise deviation of the given deformation from orientation preserving realizations of the target metric. We analyze the scaling of the energy minimizers in terms of the reference plate's thickness and rigorously derive the corresponding limiting theories, as the vanishing thickness Γ-limits. The theories are differentiated by the embeddability properties of the target metrics - in the same spirit as different scalings of external forces lead to a hierarchy of nonlinear elastic plate theories as recently displayed by Friesecke, James and Muller.

The resulting Euler-Lagrange equations describe the leading order displacements in a thin tissue which tries to adapt itself to an internally imposed metric. For the growth tensor which is a specific perturbation of Id, these equations turn out to be exactly the new system proposed by Mahadevan and Liang. They are an extension of the von-Karman equations, with the correction incorporating the pre-strain encoded in the perturbation of the growth tensor.

Some new relationships with non-smooth isometric embeddings of 2d metrics (on the mid-plate) into R^3 are also exhibited.

Shari Moskow

Title: Scattering and resonances of thin structures of high contrast

Abstract:
We consider the problem of calculating scattering and resonances of a thin membrane with a variable dielectric properties at a high contrast with respect to the background. For the scattering, we derive approximate solutions based on perturbation analysis. The approximate solutions consist of a leading order term, plus a corrector, each of which solves a problem which is of one dimension less than the original. We also derive a limiting resonance problem as the thickness goes to zero, and for the case of a simple resonance, find a first order correction. Since the limiting problem and the correction are in one less space dimension, the approach could be very efficient for calculating resonance frequencies and optical losses in photonic band gap type structures.

Convergence estimates are proved for the asymptotics. We also consider a second approach to resonance calculation, based on the finite element method with a truncated perfectly matched layer, which is not restricted to thin structures. We demonstrate the use of these methods in numerical calculations which further illustrate their differences. The asymptotic method finds resonance by solving a dense, but small, nonlinear eigenvalue problem, whereas the finite element method yields a large, but linear and sparse generalized eigenvalue problem. Both methods reproduce a localized defect mode found previously by finite difference time domain methods.

This talk is based on joint work with J. Gopalakrishnan, F. Santosa and J. Zhang

Monica Musso

Title: Finite-energy sign-changing entire solutions for semilinear elliptic problems

Abstract:
During this talk we will present two results of construction of new finite-energy sign changing solutions for some classical semilinear elliptic problem. First, we consider the problem of the existence of finite energy solitary waves for nonlinear Klein-Gordon or Schrodinger type equations \[\Delta u - u + f(u) =0 \quad \mbox{in} \quad \R^N, \, u \in H^1 (\R^N) \, ,\] where $N\geq 2$. Under natural conditions on the nonlinearity f, we prove the existence of {\em infinitely many nonradial solutions} in any dimension {\em $N \geq 2$}. Our result complements earlier works of Bartsch and Willem ($N=4$ or $N \geq 6$) and Lorca-Ubilla ($N=5$) where solutions invariant under the action of $O(2) \times O(N-2)$ are constructed. In contrast, the solutions we construct are invariant under the action of $D_k \times O(N-2)$ where $D_k \subset O(2)$ denotes the subgroup generated by the rotation of angle ${2\pi} / k$, for some integer $k\geq 7$, but they are not invariant under the action of $O(2) \times O(N-2)$. This work is in collaboration with F. Pacard and J. Wei.

A second result is the following: We construct sequences of sign changing solutions for some conformally invariant semilinear elliptic equation which is defined on S^n, with $n \geq 4$. The solutions we obtain have large energy and concentrate along some special submanifolds of S^n. For example, in dimension $n \geq 4$ we obtain sequences of solutions whose energy concentrates along one great circle or finitely many great circles which are linked ({\em Hopf links}). The solutions we obtain are not invariant under the action of of $O(k) \times O(n+1-k)$, for any $k=1, \ldots, n$. In dimension $n \geq 5$, we obtain sequences of solutions whose energy concentrates along a two dimensional torus (a {\em Clifford torus}). This work is in collaboration with M. del Pino, F. Pacard and A. Pistoia.

Hoai-Minh Nguyen

Title: Cloaking for the Helmholtz equation.

Abstract:
In the talk I will present estimates for the Helmholtz equation involving a small inclusion in 2D or 3D for a fixed frequency and their application for cloaking. It times permits estimates for an arbitrary frequency will be discussed (this part is a joint work with M. Vogelius).

Stephen P. Shipman

Title: Convergent power series for fields in sub-wavelength plasmonic crystals

Abstract:
For subwavelength plasmonic crystals consisting of plasmonic rods with frequency-dependent dielectric permittivity embedded in a host medium with unit permittivity, we obtain a convergent power series expansion of fields on the first branch of the dispersion relation. The plasma frequency scales inversely to the cell size, making the dielectric permittivity in the rods large and negative. The expressions for the series coefficients are explicitly related to the solutions of higher-order cell problems and the geometry of the rods. Within the radius of convergence, we compute the dispersion relation and the fields and define dynamic effective properties in a rigorous manner. The convergence proof requires the use of properties of the Catalan numbers to show that the series coefficients are exponentially bounded in the H1 Sobolev norm.

Darko Volkov

Title: An eigenvalue problem for elastic cracks in free space

Abstract:
We study an eigenvalue problem for the linear elasticity equations in three-dimensional space. The problem is defined in the whole space cut by a planar crack. The eigenvalue appears in a linear condition relating the traction to the jump in displacements across the crack. We prove for such problems that an eigenspace containing eigenfunctions, which do not average to zero over the crack is in general not simple. Then we prove for a more constrained eigenvalues problem, where the direction of slip over the crack is imposed, that the first eigenspace is in that case simple.

Juncheng Wei

Title: On Serrin's overdetermined problem for an epigraph in large dimensions

Abstract:
Using the method of moving planes, Serrin (1981) proved that if $\Omega $ is bounded, then the following overdetermined problem \[\Delta u+ f(u)=0 \ \mbox{in} \ \Omega, u=0 \ \mbox{on} \ \partial \Omega, \frac{\partial u}{\partial \nu} =Constant \ \mbox{on} \ \partial \Omega \] is either a ball or a cylinder. When $\Omega $ is unbounded and $\partial \Omega $ is an Lipschitz graph, Berestycki, Cafferalli and Nirenberg proved that $\partial \Omega$ must be a hyperplane. In this talk, we find a smooth domain $\R^N$, $N\geg 9$, of the form $\Omega = \{ (x',x_N)\ /\ x_N > F(x')\}$, different from a half-space, such that the equation $\Delta u + f(u) = 0 \ \hbox{in} \ \Omega $ has a positive, bounded solution u with $u=0$ and $\frac{\partial u}{\partial \nu} \, = \, constant$ on $\partial \Omega$. Relations to De Giorgi Conjecture, minimal surfaces and minimal graphs will be discussed. (Joint work with M. del Pino, F. Pacard and M. Kowalczyk)

Maria G. Westdickenberg

Title: Slow Motion of Gradient Flows

Abstract:
Sometimes physical systems exhibit ``metastability,'' in the sense that states get drawn toward so--called metastable states and are trapped near them for a very long time. A familiar example is the one--dimensional Allen Cahn equation: initial data is drawn quickly to a ``multi--kink'' state and the subsequent evolution is exponentially slow. The slow coarsening has been analyzed by Carr \& Pego, Fusco \& Hale, Bronsard \& Kohn, and X. Chen.

In general, what causes metastability? Our main idea is to convert information about the energy landscape (statics) into information about the coarsening rate (dynamics). We give sufficient conditions for a gradient flow system to exhibit metastability. We then apply this abstract framework to give a new analysis of the 1--d Allen Cahn equation. The central ingredient is to establish a certain nonlinear energy--energy--dissipation relationship. One benefit of the method is that it gives a natural proof of the fact that exponential closeness to the multi--kink state is not only propagated, but also generated.

This work is joint with Felix Otto, University of Bonn.

Xiaojing Xu

Title: Estimates for Steady Stokes Equations with Discontinuous Coefficients

Abstract:
In the talk, we show $C^{1, \alpha}$ estimates of solutions of the steady Stokes equations with discontinuous ceofficients. This paper is a continuation of Li and Nirenberg's paper [Estimates for elliptic systems from composite material, Communications on Pure and Applied Mathematics, LVI (2003), 0892-0925]. We mainly present a new perturbation lemma and a regularity estimate.