DETERMINISTIC AND STOCHASTIC FRONT PROPAGATION

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03/21/2010-03/26/2010

1 Overview of the Field

Front propagation, driven by reaction, diffusion and transport, appears as one of the central features in various phenomena in combustion, chemistry, biology and physics. Because of its ubiquitous character, the study of front dynamics is carried out in almost all branches of science. The mathematical questions that arise in front propagation, and in particular, in reaction-diffusion equations often turn out to be at the leading edge of nonlinear analysis. Methods from nonlinear partial differential equations, dynamical systems and ordinary differential equations are often all required in the modern treatment of reaction-diffusion equations. Moreover, stochastic aspects become more and more important, both in order to take into account the complexity of the physical and biological environments, and to obtain reduced stochastic models due to the intractability of the full models.

The study of propagation phenomena in reaction-diffusion equations has a long history, starting with the seminal papers of KPP (Kolmogorov, Petrovskii and Piskunov, 1937) and Fisher (1937) for the Fisher-KPP equation, where large-time convergence to travelling waves was proved for the first time. Apart from some isolated important results (Kanel, 1956-1962) the subject really started to develop in the 70's. Since then there has been a profusion of works: it is appropriate to mention the contributions of Aronson-Weinberger (1975-1977), and Fife and McLeod (1977). The subject has since then experienced a fast growth, in relation with modelling issues: reaction-diffusion equations are commonly used to describe phase transitions in various contexts in physics and chemistry, and flame propagation in combustion, and also play a central role in modelling biological invasions in various situations (population dynamics, physiology, wound healing, tumor growth).

2 Recent Developments and Open Problems

There have been significant advances in the subject in the last 10 years: new, and unconventional problems coming from various sciences; new ideas, new methods and new connections between different fields.

- *New notions of propagation.* In a completely heterogeneous, unstructured or random medium, the classical notion of travelling front, that is, a profile moving at constant speed in a reference frame is no longer applicable. However, being able to address front propagation in heterogeneous environments, and to take into account other phenomena, such as transport and interaction with the environment, is an outstanding problem arising in ecology. The need for a new notion had already been made clear by Hamel and Nadirashvili (2001) in the construction of KPP fronts having no propagation velocity; several definitions (Shen 2001, Matano 2002) to handle quasi-periodic or stationary ergodic media were given. However the most general definition is due to Berestycki and Hamel (2005); it encompasses all the previous ones and reduces in constant, or periodic media to the already known notions: travelling or pulsating fronts. Since then, the field has grown rapidly; for instance: existence of bistable fronts in the whole space, passing an obstacle; existence and global stabilty of generalised fronts for combustion type (including random environments) nonlinearities is now established. Surprising nonexistence results in Fisher-KPP type equations have recently been proved (Nolen-Roquejoffre-Ryzhik-Zlatos 2010).

- *Singular limits and free boundary problems; homogenisation.* A first type of singular limit generating a free boundary problem is the approximation of geometric motions (eikonal equations, mean curvature motions) by reaction-diffusion equations: let us mention for instance Evans-Souganidis 1989, DeMottoni-Schatzman 1990, Barles-Soner-Souganidis 1993. Such singular limits have recently been found to be relevant in biological models, especially in selection/mutation dynamics models (Perthame-Souganidis 2005, Barles-Perthame 2007). Another type of problem is to describe the boundary between two regions where the unknown function satisfies a PDE, together with continuity conditions and jump conditions on the gradient; such problems can also be interpreted in terms of singular limits (Berestycki-Caffarelli-Nirenberg 1990). The major contributions of Caffarelli at the beginning of the 90's have made available methods that are now applied to inhomogeneous problems (Caffarelli and deLaLave 2001; Caffarelli, Lee and Mellet 2004). These ideas, combied with viscosity solutions methods, have allowed treatment of nonlinear homogenisation problems in random environments, a milestone being the work of Caffarelli, Souganidis, Wang (2005).

- *Nonlocal effects.* Motivated by a number of applications: turbulence, plasma physics, chemistry, ecology and even finance; and having strong links with probability theory, the study of reaction-diffusion equations involving nonlocal pseudo-differential operators rather than the classical diffusiion has recently emerged. An emblematic operator is the fractional Laplacian, which arises in many models, and for which many heuristic ideas are available from the physical literature. The field is rapidly expanding: let us quote, for instance: the classification of layer solutions by Cabré and Sola-Morales (2005), the fractional obstacle problem (Caffarelli-Salsa-Silvestre 2007) and its random homogenisation (Caffarelli-Mellet 2008), the boundary reaction model (Caffarelli-Roquejoffre-Sire, 2008). Another direction in the field is the study of nonlocal geometric motions (Caffarelli-Souganidis 2008) and its approximation by fractional reaction-diffusion equations (Imbert-Souganidis 2008); the corresponding minimal surface theory (regularity and Hausdorff dimension of the singular set) has recently been carried out.

- *Solutions to long-standing open problems for the Allen-Cahn equation.* One of the most notable recent results is certainly the solution of the de Giorgi conjecture. The conjecture is based on the Simons theorem of 1967: a global minimal surface in R^N is a hyperplane if $N < 8$; in higher dimensions there are (Bombieri, de Giorgi, Giusti 1969) minimal cones. According to the de Giorgi conjecture, the solutions of $\Delta u + u - u^3 = 0$ in R^N which are increasing in x_N and converge to 1 as $x_N \to \pm\infty$, are planar, at least up to dimension 8. A vigorous effort from many mathematicians (Ghoussoub-Gui 1998, Ambrosio-Cabré 2000, Ghoussoub-Gui 200, and finally Savin 2006) has settled the 'nonexistence' part. A very recent work of Del Pino, Kowalczyk and Wei proved the existence of nonplanar solutions of the equation for $N > 8$, satisfying the de Giorgi requirements. This almost closes de Giorgi conjecture - some difficult open problems remain, for instance: what happens if the convergence to 1 as $x_N \to \pm\infty$ is dropped?.

- *New notions of eigenvalues.* It is well-known that an elliptic operator L in a bounded domain has a unique eigenvalue – the one with least real part – whose eigenfunction is positive. This result breaks down when the domain is unbounded, and two quantities (the first generalised eigenvalues of L) become relevant: $\lambda_1(L)$ – the highest λ such that the equation $Lu - \lambda u = 0$ has a positive supersolution, and $\lambda_1'(L)$ – the lowest λ such that the equation $Lu - \lambda u = 0$ has a positive subsolution. These quantities, which do not coincide in general - but which do coincide in the case of a bounded domain – were introduced by Pinsky, Berestycki-Hamel-Roques (2005), and their qualitative properties studied by Berestycki-Rossi (2006). These notions seem at first sight to be completely unrelated to front propagation; however they appear to be the key to the mathematical analysis, by Berestycki and Rossi (2008), of a model of adapting species to climate change (Berestycki-Diekman-Naglekerke-Zegeling 2008).

- *New large-time behaviour results in reaction-diffusion systems.* It is commonplace to say that, due to the

lack of maximum principle, reaction-diffusion systems are difficult to treat. Gradient-like reaction-diffusion systems with diagonal diffusion, where the nonlinear function is a gradient are an important class of such systems, which in particular include the Ginzburg-Landau equations. They happen to possess a family of Lyapounov functionals essentially indexed by the velocity that no one really knew how to use, until some important works of Risler (2007), Gallay-Risler (2007). They used this family in an involved fashion to obtain both existence and global stability of waves, and a result of this type would probably be difficult to obtain by another method. These ideas were successfully applied to damped hyperbolic equations as well (Gallay, Joly 2008).
In a completely different spirit, a generalised entropy principle, discovered by Perthame-Michel-Mischler (2005) has enabled advances in the treatment of the Keller-Segel model, a reduced model for the aggregation of colonies of bacteria which can also serve as a rough description of invasive sclerosis in the human brain (Perthame, Calvez 2007).
- *Reaction-diffusion equations coupled with convection.* The coupling between the front propagation and fluid motion is a common physical feature and the qualitative effects of such coupling have been actively studied recently. This includes the front speed-up by an incompressible fluid flow, improved mixing by a flow (Constantin-Kiselev-Ryzhik-Zlatos), eigenvalue enhancement (Berestycki-Hamel-Nadirashvili) and effect on the explosion threshold – all by prescribed flows. Even more interesting physically are problems where a fluid flow equation is coupled to a reaction-diffusion equation for temperature, such as the existence of traveling fronts in the Boussinesq approximation. Yet another important class of coupled fluid-front problems involves a Fokker-Planck equation for a distribution of particles coupled to a fluid equation (Otto-Tzavaras, Constantin-Masmoudi) – these arise in nematic phase transitions.

3 Presentation Highlights

3.1 Objectives

This workshop followed the 2006 workshop 'Reaction-diffusion equations and free boundary problems'. We had initially planned the same format: five general broadly-oriented lecture series of 2 hours, given by E. Brunet, L. Caffarelli, P. Constantin, B. Perthame and M. del Pino (all of whom had confirmed their interest), and more specialised shorter talks (20-30 minutes) with a lot of time left for interaction between the participants. This format seemed to work very well in 2006 and was generously appreciated by the participants. However, to keep the present workshop fresh, the list of main speakers has been completely changed, actually, only Constantin was even a participant in 2006. Moreover, although it seemed to us indispensable to invite some of the 2006 speakers we have changed at least half of the 2006 short time speakers.

The goal of this workshop was to present the state of the art of current research in front propagation for reaction-diffusion equations, and our aim was to cover a large spectrum of problems and methods in the main lectures. The plan was for P. Constantin to review the problems involving the coupling of fronts and fluids, while L. Caffarelli would speak on various problems outlined above. The emergence of new biological models was reflected in the invitation of B. Perthame; whereas the solution of the de Giorgi conjecture made the invitation of M. del Pino natural. A special emphasis was put on models posed in heterogeneous media; this led, in a natural way, to the study of stochastic aspects of front propagation and the inclusion of this topic in the programme. A lot is still to be understood qualitatively here, and much should be learnt from the physical side: hence the invitation of E. Brunet. Priority was given to participation of young researchers as we thought that the workshop would be very beneficial for them.

3.2 Highlights from the talks

Most of the invited participants were able to attend, to the unfortunate exception of L. Caffarelli and P. Constantin who canceled at a late moment. Their slots were not reallocated to the remaining talks, we preferred to favour informal interactions or discussions. This was very much appreciated by the participants.
Global solutions, qualitative properties, links with geometry. M. del Pino gave a 2 hour course on his result (with M. Kowalczyk and J. Wei) about existence of nontrivial global solutions of the Allen-Cahn equation inR^N, with $N \geq 9$. The level sets of the solutions are asymptotically close (up to a logarithmic factor) to

surfaces parallel to the Bombieri-de Giorgi-Giusti minimal cone. The solution itself is sought for under the form $\phi_0(d(x)/\varepsilon)$, where $d(x)$ is (up to some correction) the distance function to the minimal cone. The key point of the construction is the use of a new super-solution (later re-used by the team of D. Jerison for the Bernouilli free boundary problem) that controls very precisely the behaviour of the cone et infinity, and which allows the setting up of a Lyapunov-Schmidt procedure. This raises the question of construction of solutions of the Allen-Cahn equation whose level sets resemble an arbitrary global minimal surface at infinity; using the del Pino-Kowalczyk-Wei procedure entails the verification of some nondegeneracy properties that may turn to be quite hard to check. In the same sprit, J. Wei reported on a series of results on multi-dimensional travelling waves (with very small speed) to the Allen-Cahn equation. In this problem, the level sets evolve - at least, formally - like the mean curvature motion at infinity; the constructed solution connects an arbitrary number of paraboloids evolving by this motion. Still in this context, C. Gui presented advances on qualitative properties (axial symmetry, monotonicity) in the 2D Allen-Cahn equation (which are made difficult by the fast growth of the level sets). Finally, the talk of R. Jerrard showed surprising links between the singularly perturbed nonlinear wave equation and the usual singularly perturbed Allen-Cahn equation.
A natural extension of this class of problems is to replace the Laplacian by the fractional Laplacian, thus investiating models of the form $(-\Delta)^\alpha u = f(u)$; f for instance of the Allen-Cahn type and $0 < \alpha < 1$. Interesting questions arise, because the associated functional

$$J(u) = \frac{\varepsilon}{2}\int \frac{(u(x)-u(y))^2}{|x-y|^{N+2\alpha}}dxdy - \frac{1}{\varepsilon}\int F(u))\,dx,$$

F: primitive of f, has different behaviours according to α: when $\alpha > 1/2$ it Γ-converges to the perimeter functional (hence a behaviour analogous to the case $\alpha = 1$); when $\alpha < 1/2$ it Γ-converges to the α-perimeter functional (i.e. the H^α norm of the indicator of the set $\{u = 1\}$ introduced by Caffarelli, Roquejoffre, Savin). In this context, E. Cinti presented energy estimates in view of proving de Giorgi type results in low dimensions. The talk of Y. Sire described Poincaré-type inequalities for Dirichlet forms with non-Gaussian measures, as well as a new class of Paneitz operators of arbitrary order that could prove useful in the study of the singularities of manifolds. Paneitz-type operators also appeared in N. Ghoussoub's talk on 4th order nonlinear elliptic equations.
Another important geometric equation: the eikonal equation was at the background of several talks. V. Roussier presented the study of a large class of solutions of the forced mean curvature motion in 3 space dimensions, based on the Caffarelli-Littman theorem on the positive solutions of $-\Delta u + u = 0$. These solutions are asymptotic to those of the eikonal equation $|\nabla\phi| = 1$, but they are more numerous. P. Souganidis reported on the homogenisation of the eikonal equation $u_t + V(x/\varepsilon)\cdot Du = |Du|$ and the corresponding enhancement in the speed of the motion of the level sets; this, in the absence of coercivity, requires de Giorgi type estimates on the level sets. The talk of G. Iyer was concerned with the maximisation, over all incompressible flows, of the exit time of a particule in a flow.
Reaction-diffusion fronts, interactions between PDE's and probability theory. Reaction-diffusion equations appear in various fields of science, from combustion to ecology. Front propagation and the formation of spatio-temporal complex patterns are central features in many such models. Traveling waves play an essential role, as they often describe the motion of phase transitions between different patterns. Thus, the analytic study of reaction-diffusion systems is fundamental in the understanding of related spreading phenomena and their underlying processes. These issues have been a focus of intensive research in the past few decades in both physical or biological models. Traditionally, the main mathematical tools come from linear and nonlinear analysis, the theory of partial and ordinary differential equations and the dynamical systems. Here we have mixed here probability theory: stochastic processes, large deviation theory.
E. Brunet gave a 2 hour course on several topics concerning the KPP equation: the logarithmic correction for the homogeneous model $h_t = h_{xx} + h(1-h)$, and the perturbation of the model by a small white noise or by a cutoff in the nonlinear term. He finished with considerations on the branching Brownian motion. Recall that, as far as the logarithmic correction is concerned, the most general result is due to Bramson, with probabilistic techniques - whereas the model is deterministic in essence. New notions of transition fronts and waves were recently introduced for general evolution equations in heterogeneous media, encompassing the classical notions of traveling waves and allowing for the investigation of more complex situations. In this framework, A. Zlatoš reported on the existence and non-existence of such generalized reaction-diffusion waves in media with nonnegative monostable reactions. In some cases, the existence of fronts depends on

spectral properties of some linear operators. As a matter of fact, the notions of principal eigenvalues of elliptic or parabolic operators are one of the key tools to describe the large time behavior of many models arising in population dynamics or ecology. Various generalizations of the principal eigenvalues in unbounded domains have been given recently. H. Berestycki reported on some theoretical properties and comparisons of these eigenvalues, as well as on some applications to nonlinear problems. G. Nadin referred to these notions and results in his talk, in order to get quantitative estimates of the spreading speeds of solutions of some reaction-diffusion equations in completely heterogeneous space-time dependent environments. E. Kosygina discussed large deviations in homogenization of random Hamilton-Jacobi equations. On the other hand, M. El Smaily reported on speed-up rates of propagation speeds for reaction-diffusion equations in the presence of large advection, and in particular on a geometrical necessary and sufficient condition for linear speed-up in the two-dimensional periodic case. F. Comets presented a phase transition phenomenon in the spreading velocity of a randomly time dependent KPP equation. J. Berard reported on a rigorous proof of the Brunet-Derrida velocity shift for branching-selection particle systems; this can be viewed as a counterpart of the Bramson correction for KPP PDEs.
Modelling in biology and medicine, nonlocal effects. P. Perthame gave a 2 hour course on the dynamics of pulses and branching patterns, motivated by experiments in microbiology (growth of colonies of bacteria) or evolutionnary dynamics. Contrary to physics, where modelling from first principles may often be done, modelling biplogical situations rather involves the (numerical or mathematical) study of various reaction-diffusion systems, the one exhibiting the closest behaviour to the situation under study being declared the most relevant. Although the study of multi-dimensional patterns of reaction-diffusion systems is not new, most known results concern situations where the parameters are small; the results presented do not - due to the nature of the phenomena - rely on any smallness assumptions. Among the models studied in the course were the Mimura model, the Gray-Scott model, and cross-diffusion systems (Keller-Segel and hyperbolic Keller Segel, the latter exhibiting branching patters). The talk of V. Calvez gave more mathematical details on the Keller-Segel model for the movement of bacteria, which he obtained from a diffusion limit of a kinetic equation. L. Roques reported on some reaction-diffusion recolonization models arising in the study of plant dynamics after the last age of ice. These models correspond to very slowly decaying initial conditions and qualitative and quantitative properties of the asymptotic rates of spread have been given, some of them being analogue to the ones obtained with some (nonlocal) fractional-diffusion models. G. Chapuisat reported on some qualitative results for the travelling waves in some mathematical models for the ischemic cerebral stroke. The existence of such waves typically depend on the size of different components (gray and white matters) in the brain, and give raise to new kinds of generalised waves. It is to be noted that the existence of waves in the mechanism of ischemic strokes is the matter of heated debated within the neurologist community.

4 Outcome of the Meeting

Similarly to the 2006 meeting, this one was very much appreciated by the participants (as some testimonials attest it). It seems that we managed to strike the right balance between talks and informal discussions, theory and modelling, talks of general interest and more specialised talks. The mixture of geometry, analysis, probability was also very much appreciated; it is to be expected that collaborations between the members of these communities will be enhanced. At the end of the meeting, it became clear future research directions mixing these three fields were ripe for interesting developments. Among them we can list the following ones.
- *Global solutions of reaction-diffusion equations beyond geometric motions.* Although geometric motions may exhibit quite rich dynamics, perturbing them by a diffusion term or approximating them by a reaction-diffusion equation generates even richer dynamical properties. The results presented in this spirit show the existence of deeper phenomena than what intuition could suggest, and this should lead to very interesting findings in the near future. The investigation of the dynamics of geometric motions, and related topics such as homogenisation, or inclusion of random effects, should be pushed even further.
- *Dynamics of KPP type equations.* Although it has long been known that there are rich links between the large time dynamics of the KPP equation and the theory of Markov processes (representation of solutions, Feynman-Kac formulae...), this meeting has made it clear that these links should be pushed much further. The subject is all the more timely as new models are emerging, both from PDEs (KPP equations in heterogeneous or random media, questions of spreading velocities, quantitative estimates) and from stochastic processes

(branching random walks, existence of travelling waves). Among some of the hot questions are the corrections to the linear approximation in propagation estimates, the best known being the logarithmic Bramson correction.
- *Dynamics of models from biology.* The first conclusion that can be drawn from the talks in this particular topic is that new types of mathematical problems arise, and that fresh ideas and methods are needed to treat novel, and unconventional types of PDE's, that cannot really be thrown in any existing category. Among the challenges that have been set forth in this meeting were the necessity of understanding situations away from small parameters, and the need to tackle nonlocal effects, that arise in a large number of models from ecology and medicine.
These fascinating research lines, where not only questions are posed, but a wide range of possible tools and solutions are explained, is one of the main outcome of this meeting. And, as a conclusion to this report, the exceptional working conditions of BIRS, both from the point of view of the quality of the infrastructure, and the helpfulness of the suporting staff, have to be emphasised and acknowledged.