Mass conservation of the finite element immersed boundary method

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Nonstandard Discretizations for Fluid Flows

Contents

IBM – Immersed boundary method

Introduced by Peskin for the simulation of the blood flow in the heart.

<Peskin '72–'77>
<McQueen–Peskin '83–>
<Peskin '02>

Successfully applied to many biological problems, where a fluid interacts with a flexible structure.

The main feature is that the structure is considered as a part of the fluid by introducing suitable additional forces and masses. The Navier–Stokes equations are solved in the whole domain (fluid + solid) by *finite differences* and the interaction with the structure is obtained by means of singular force and mass terms defined by a Dirac delta function localized in the solid domain.

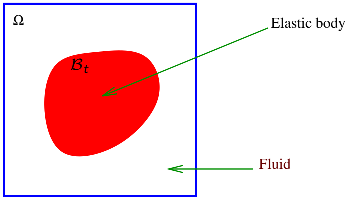
Finite elements for IBM

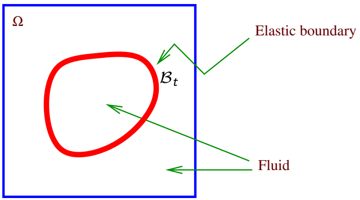
At the beginning, we used *finite elements* mainly because we thought this would simplify the mathematical analysis. Indeed, it turned out that this is a good choice also from the practical point of view.

<B.–Gastaldi '03>
<B.–Gastaldi–Heltai '04–'07>
<B.–Gastaldi–Heltai–Peskin '08>

- No need to approximating the Dirac delta functions, since the variational formulation takes care of it in a natural way
- Better interface approximation (less diffusion, sharp pressure jump)
- The fluid equations can be approximated with standard mixed schemes ($Q_2 - P_1$, Hood–Taylor, $P_1 \mathrm{iso} P_2 - P_1^c$, ...)

Immersed elastic bodies





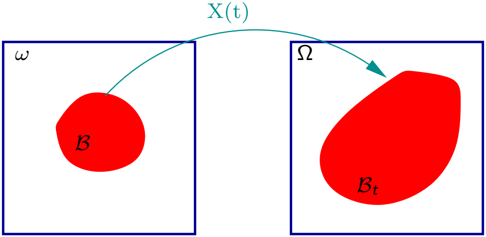
Immersed body of codimension 0

the fluid domain and the immersed body have the same dimension

Immersed body of codimension 1

the immersed body is either a curve in 2D or a surface in 3D

Notation



Ω fluid + solid
$\Omega \subset \mathbb{R}^d$, $d = 2, 3$
$\mathbf{x}$ Euler. var. in Ω

$\mathbf{u}(\mathbf{x}, t)$ fluid velocity

$p(\mathbf{x}, t)$ fluid pressure

$\mathcal{B}_t$ deformable structure domain
$\mathcal{B}_t \subset \mathbb{R}^m$, $m = d, d-1$
s Lagrangian var. in $\mathcal{B}$
$\mathcal{B}$ reference domain
$\mathbf{X}(\cdot, t) : \mathcal{B} \to \mathcal{B}_t$ position of the solid
$\mathbb{F} = \dfrac{\partial \mathbf{X}}{\partial s}$ deformation grad. $(\det \mathbb{F} > 0)$

$$\mathbf{u}(\mathbf{x}, t) = \frac{\partial \mathbf{X}}{\partial t}(s, t) \text{ where } \mathbf{x} = \mathbf{X}(s, t)$$

From conservation of momenta, in absence of external forces, it holds

$$\rho\dot{\mathbf{u}} = \rho\left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\,\mathbf{u}\right) = \nabla\cdot\boldsymbol{\sigma} \quad \text{in } \Omega$$

In our case the Cauchy stress tensor has the following form

$$\boldsymbol{\sigma} = \begin{cases} \boldsymbol{\sigma}_f & \text{in } \Omega\setminus\mathcal{B}_t \\ \boldsymbol{\sigma}_f + \boldsymbol{\sigma}_s & \text{in } \mathcal{B}_t \end{cases}$$

- Incompressible fluid: $\boldsymbol{\sigma} = \boldsymbol{\sigma}_f = -p\mathbb{I} + \mu(\nabla\,\mathbf{u} + (\nabla\,\mathbf{u})^T)$
- Visco-elastic material: $\boldsymbol{\sigma} = \boldsymbol{\sigma}_f + \boldsymbol{\sigma}_s$ with $\boldsymbol{\sigma}_s$ elastic part of the stress

Moreover, if the structural material has a density ρ_s different from the fluid density ρ_f, we have

$$\rho = \begin{cases} \rho_f & \text{in } \Omega\setminus\mathcal{B}_t \\ \rho_s & \text{in } \mathcal{B}_t \end{cases}$$

Virtual work principle ($\rho_s = \rho_f$)

Assume for simplicity that $\rho_s = \rho_f = \rho$, then

$$\int_\Omega \rho \dot{\mathbf{u}} \mathbf{v} dx + \int_\Omega \boldsymbol{\sigma}_f : \nabla \mathbf{v} \, dx = - \int_{\mathcal{B}_t} \boldsymbol{\sigma}_s : \nabla \mathbf{v} \, dx \quad \forall \mathbf{v}$$

The elastic stress $\boldsymbol{\sigma}_s$ can be expressed in Lagrangian variables by means of the Piola-Kirchhoff stress tensor by:

$$\mathbb{P}(s,t) = |\mathbb{F}(s,t)| \boldsymbol{\sigma}_s(\mathbf{X}(s,t),t) \mathbb{F}^{-T}(s,t), \quad s \in \mathcal{B}$$

So that

$$\int_\Omega \rho \dot{\mathbf{u}} \mathbf{v} dx + \int_\Omega \boldsymbol{\sigma}_f : \nabla \mathbf{v} \, dx = - \int_{\mathcal{B}} \mathbb{P} : \nabla_s \mathbf{v}(\mathbf{X}(s,t)) \, ds$$

Strong formulation

Navier–Stokes

$$\rho\left(\frac{\partial \mathbf{u}}{\partial t}+\mathbf{u}\cdot\nabla\,\mathbf{u}\right)-\mu\Delta\mathbf{u}+\nabla\,p=\mathbf{g}+\mathbf{t}\quad \text{in } \Omega\times\,]0,T[$$

$$\operatorname{div}\mathbf{u}=0\qquad \text{in } \Omega\times\,]0,T[$$

Force density in $\Omega\times]0,T[$

$$\mathbf{g}(\mathbf{x},t)=\int_{\mathcal{B}}\nabla_s\cdot\mathbb{P}(s,t)\delta(\mathbf{x}-\mathbf{X}(s,t))ds$$

$$\mathbf{t}(\mathbf{x},t)=-\int_{\partial\mathcal{B}}\mathbb{P}(s,t)\mathbf{N}(s)\delta(\mathbf{x}-\mathbf{X}(s,t))ds$$

Immersed structure motion

$$\frac{\partial \mathbf{X}}{\partial t}(s,t)=\mathbf{u}(\mathbf{X}(s,t),t)\quad \text{in } \mathcal{B}\times\,]0,T[$$

Initial and boundary condition

$$\mathbf{u}(\mathbf{x},0)=\mathbf{u}_0(\mathbf{x}) \text{ in } \Omega\qquad \mathbf{X}(s,0)=\mathbf{X}_0(s) \text{ in } \mathcal{B}$$

$$\mathbf{u}(\mathbf{x},t)=0\qquad \text{su } \partial\Omega\times]0,T[$$

Variational formulation

Source term

$$\begin{aligned}\int_\Omega (\mathbf{g}+\mathbf{t})\cdot\mathbf{v}\,d\mathbf{x} &= \int_\mathcal{B} (\nabla_s\cdot\mathbb{P})\cdot\mathbf{v}(\mathbf{X}(s,t))\,ds \\ &\quad - \int_{\partial\mathcal{B}} \mathbb{P}\mathbf{N}\cdot\mathbf{v}(\mathbf{X}(s,t))\,dA \\ &= -\int_\mathcal{B} \mathbb{P} : \nabla_s\mathbf{v}(\mathbf{X}(s,t))\,ds\end{aligned}$$

Lemma

For any $t\in[0,T]$, let $\partial\mathcal{B}_t$ be C^1 and $\mathbb{P}$ be $W^{1,\infty}$. Then, for any $t\in]0,T[$, the force density $\mathbf{F}=\mathbf{g}+\mathbf{t}$ is a distribution belonging to $H^{-1}(\Omega)^d$ defined as follows: for any $\mathbf{v}\in H^1_0(\Omega)^d$

$${}_{H^{-1}}\langle\mathbf{F}(t),\mathbf{v}\rangle_{H^1_0} = -\int_\mathcal{B} \mathbb{P}(\mathbb{F}(s,t)) : \nabla_s\mathbf{v}(\mathbf{X}(s,t))\,ds \quad \forall t\in]0,T[$$

Final form of the variational formulation

- Navier–Stokes

$$\rho\frac{d}{dt}(\mathbf{u}(t),\mathbf{v}) + a(\mathbf{u}(t),\mathbf{v}) + b(\mathbf{u}(t),\mathbf{u}(t),\mathbf{v}) - (\operatorname{div}\mathbf{v}, p(t)) = <\mathbf{F}(t),\mathbf{v}> \quad \forall\mathbf{v}\in H_0^1(\Omega)^d$$

$$(\operatorname{div}\mathbf{u}(t), q) = 0 \qquad \forall q\in L_0^2(\Omega)$$

$$a(\mathbf{u},\mathbf{v}) = \mu(\nabla\mathbf{u},\nabla\mathbf{v})$$
$$b(\mathbf{u},\mathbf{v},\mathbf{w}) = \frac{\rho}{2}\left((\mathbf{u}\cdot\nabla\mathbf{v},\mathbf{w}) - (\mathbf{u}\cdot\nabla\mathbf{w},\mathbf{v})\right)$$

- $\langle\mathbf{F}(t),\mathbf{v}\rangle = -\int_{\mathcal{B}}\mathbb{P}(\mathbb{F}(s,t)) : \nabla_s\mathbf{v}(\mathbf{X}(s,t))\,ds \;\forall\mathbf{v}\in H_0^1(\Omega)^d$
- $\dfrac{\partial\mathbf{X}}{\partial t}(s,t) = \mathbf{u}(\mathbf{X}(s,t),t) \quad \forall s\in\mathcal{B}$
- $\mathbf{u}(\mathbf{x},0) = \mathbf{u}_0(\mathbf{x}) \;\forall\mathbf{x}\in\Omega, \quad \mathbf{X}(s,0) = \mathbf{X}_0(s) \;\forall s\in\mathcal{B}.$

Virtual work principle ($\rho_s \neq \rho_f$)

Excess Lagrangian mass density: we assume $\rho = \rho_s$ in $\mathcal{B}_t$ and $\rho = \rho_f$ in $\Omega - \mathcal{B}_t$, with $\rho_s - \rho_f \geq 0$ (might be relaxed)

$$\int_\Omega \rho_f \dot{\mathbf{u}} \mathbf{v} \, d\mathbf{x} + \int_\Omega \boldsymbol{\sigma}_f : \nabla \mathbf{v} \, d\mathbf{x} \\ = - \int_{\mathcal{B}_t} (\rho_s - \rho_f) \dot{\mathbf{u}} \mathbf{v} \, d\mathbf{x} - \int_{\mathcal{B}_t} \boldsymbol{\sigma}_s : \nabla \mathbf{v} \, d\mathbf{x} \quad \forall \mathbf{v}$$

Using the Lagrangian description in the solid domain, there is no need for convective terms and the material derivative is the same as the time derivative, hence $\dot{\mathbf{u}} = \partial^2 \mathbf{X} / \partial t^2$, and we get

$$\int_\Omega \rho_f \dot{\mathbf{u}} \mathbf{v} \, d\mathbf{x} + \int_\Omega \boldsymbol{\sigma}_f : \nabla \mathbf{v} \, d\mathbf{x} \\ = - \int_{\mathcal{B}} (\rho_s - \rho_f) \frac{\partial^2 \mathbf{X}}{\partial t^2} \mathbf{v}(\mathbf{X}(s,t)) ds - \int_{\mathcal{B}} \mathbb{P} : \nabla_s \mathbf{v}(\mathbf{X}(s,t)) \, ds$$

Then the variational formulation reads (with the same definition as above):

- Navier–Stokes

$$\rho_f \frac{d}{dt}(\mathbf{u}(t),\mathbf{v}) + a(\mathbf{u}(t),\mathbf{v}) + b(\mathbf{u}(t),\mathbf{u}(t),\mathbf{v}) - (\operatorname{div}\mathbf{v}, p(t)) = -\int_{\mathcal{B}} (\rho_s - \rho_f)\frac{\partial^2 \mathbf{X}}{\partial t^2}\mathbf{v}(\mathbf{X}(s,t))ds + < \mathbf{F}(t),\mathbf{v} > \forall \mathbf{v} \in H_0^1(\Omega)^d$$

$$(\operatorname{div}\mathbf{u}(t), q) = 0 \qquad \forall q \in L_0^2(\Omega)$$

- $\langle \mathbf{F}(t),\mathbf{v}\rangle = -\int_{\mathcal{B}} \mathbb{P}(\mathbb{F}(s,t)) : \nabla_s \mathbf{v}(\mathbf{X}(s,t))\, ds \;\forall \mathbf{v} \in H_0^1(\Omega)^d$
- $\frac{\partial \mathbf{X}}{\partial t}(s,t) = \mathbf{u}(\mathbf{X}(s,t),t) \quad \forall s \in \mathcal{B}$
- $\mathbf{u}(\mathbf{x},0) = \mathbf{u}_0(\mathbf{x}) \;\forall \mathbf{x} \in \Omega, \quad \mathbf{X}(s,0) = \mathbf{X}_0(s) \;\forall s \in \mathcal{B}.$

Stability

<B.–Cavallini–Gastaldi '10>

Recalling that

$$\frac{\partial \mathbf{X}}{\partial t}(s,t) = \mathbf{u}(\mathbf{X}(s,t),t) \quad \forall s \in \mathcal{B}$$

it holds

$$\frac{\rho_f}{2}\frac{d}{dt}||\mathbf{u}(t)||_0^2 + \mu||\nabla \mathbf{u}(t)||_0^2 + \frac{d}{dt}E(\mathbf{X}(t)) + \frac{1}{2}(\rho_s - \rho_f)\frac{d}{dt}\left\|\frac{\partial \mathbf{X}}{\partial t}\right\|_B^2 = 0$$

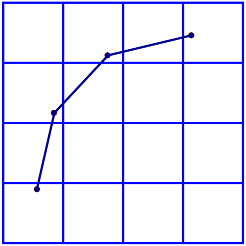
where E is the total elastic potential energy

$$E\left(\mathbf{X}(t)\right) = \int_{\mathcal{B}} W(\mathbb{F}(s,t))\,ds$$

Finite element approximation

- Uniform background grid $\mathcal{T}_h$ for the domain Ω (meshsize h_x)
- Inf-sup stable finite element pair

$$V_h \subset H_0^1(\Omega)^d$$
$$Q_h \subset L_0^2(\Omega)$$



- Grid $\mathcal{S}_h$ for $\mathcal{B}$ (meshsize h_s)
- Piecewise linear finite element space for $\mathbf{X}$
  $S_h = \{\mathbf{Y} \in C^0(\mathcal{B};\Omega) : \mathbf{Y} \in P1\}$

Notation

- T_k, $k = 1, \dots, M_e$ elements of $\mathcal{S}_h$
- $\mathbf{s}_j$, $j = 1, \dots, M$ vertices of $\mathcal{S}_h$
- $\mathcal{E}_h$ set of the edges e of $\mathcal{S}_h$

Discrete source term

Source term:

$$\langle \mathbf{F}(t), \mathbf{v} \rangle = -\int_{\mathcal{B}} \mathbb{P}(\mathbb{F}_h(s,t)) : \nabla_s \mathbf{v}(\mathbf{X}_h(s,t))\, ds \quad \forall \mathbf{v} \in V_h$$

$\mathbf{X}_h$ p.w. linear $\Rightarrow \mathbb{F}_h$, $\mathbb{P}_h$ p.w. constant

By integration by parts

$$\begin{aligned} \langle \mathbf{F}_h(t), \mathbf{v} \rangle_h &= -\sum_{k=1}^{M_e} \int_{T_k} \mathbb{P}_h : \nabla_s \mathbf{v}(\mathbf{X}(s,t))\, ds \\ &= -\sum_{k=1}^{M_e} \int_{\partial T_k} \mathbb{P}_h \mathbf{N} \mathbf{v}(\mathbf{X}(s,t))\, dA \end{aligned}$$

that is

$$\langle \mathbf{F}_h(t), \mathbf{v} \rangle_h = -\sum_{e \in \mathcal{E}_h} \int_e [\![\mathbb{P}_h]\!] \cdot \mathbf{v}(\mathbf{X}(s,t))\, dA$$

$[\![\mathbb{P}]\!] = \mathbb{P}^+ \mathbf{N}^+ + \mathbb{P}^- \mathbf{N}^-$ jump of $\mathbb{P}$ across e for internal edges

$[\![\mathbb{P}]\!] = \mathbb{P}\mathbf{N}$ *jump* when $e \subset \partial\mathcal{B}$

The *semidiscrete* problem becomes: find $(\mathbf{u}_h, p_h) :]0, T[\to V_h \times Q_h$ and $\mathbf{X}_h : [0, T] \to S_h$ such that

$$
\begin{cases}
\rho_f \dfrac{d}{dt}(\mathbf{u}_h(t), \mathbf{v}) + a(\mathbf{u}_h(t), \mathbf{v}) + b(\mathbf{u}_h(t), \mathbf{u}_h(t), \mathbf{v}) \\
\qquad -(\operatorname{div} \mathbf{v}, p_h(t)) = -\displaystyle\int_{\mathcal{B}} (\rho_s - \rho_f) \frac{\partial^2 \mathbf{X}_h}{\partial t^2} \mathbf{v}(\mathbf{X}_h(s,t)) ds \\
\qquad\quad - \displaystyle\sum_{e \in \mathcal{E}_h} \int_e [\![\mathbb{P}_h]\!] \cdot \mathbf{v}(\mathbf{X}_h(s,t)) dA & \forall \mathbf{v} \in V_h \\
(\operatorname{div} \mathbf{u}_h(t), q) = 0 & \forall q \in Q_h
\end{cases}
$$

$$\frac{d\mathbf{X}_{hi}}{dt}(t) = \mathbf{u}_h(\mathbf{X}_{hi}(t), t) \quad \forall i = 1, \dots, M$$

$$\mathbf{u}_h(0) = \mathbf{u}_{0h} \text{ in } \Omega$$

$$\mathbf{X}_{hi}(0) = \mathbf{X}_0(s_i) \quad \forall i = 1, \dots, M$$

Fully discrete problem

Backward Euler – **BE**

Find $(\mathbf{u}_h^{n+1}, p_h^{n+1}) \in V_h \times Q_h$ e $\mathbf{X}_h^{n+1} \in S_h$ such that

$$\langle \mathbf{F}_h^{n+1}, \mathbf{v} \rangle_h = -\sum_{e\in\mathcal{E}_h} \int_e [\![\mathbb{P}_h]\!]^{n+1} \cdot \mathbf{v}(\mathbf{X}_h^{n+1}(s))dA \qquad \forall \mathbf{v} \in V_h$$

$$\textbf{NS} \begin{cases} \rho_f \left(\dfrac{\mathbf{u}_h^{n+1} - \mathbf{u}_h^n}{\Delta t}, \mathbf{v} \right) + a(\mathbf{u}_h^{n+1}, \mathbf{v}) + b(\mathbf{u}_h^{n+1}, \mathbf{u}_h^{n+1}, \mathbf{v}) \\ -(\operatorname{div} \mathbf{v}, p_h^{n+1}) = \\ -\displaystyle\int_{\mathcal{B}} (\rho_s - \rho_f) \frac{\mathbf{X}_h^{n+1} - 2\mathbf{X}_h^n + \mathbf{X}_h^{n-1}}{\Delta t^2} \cdot \mathbf{v}(\mathbf{X}_h^{n+1}(s))ds \\ \quad + < \mathbf{F}_h^{n+1}, \mathbf{v} >_h \qquad \forall \mathbf{v} \in V_h \\ (\operatorname{div} \mathbf{u}_h^{n+1}, q) = 0 \qquad \forall q \in Q_h; \end{cases}$$

$$\frac{\mathbf{X}_{hi}^{n+1} - \mathbf{X}_{hi}^n}{\Delta t} = \mathbf{u}_h^{n+1}(\mathbf{X}_{hi}^{n+1}) \quad \forall i = 1, \dots, M.$$

Fully discrete problem

Modified bckward Euler – **MBE**

Step 1. $\langle \mathbf{F}_h^n, \mathbf{v}\rangle_h = -\sum_{e\in\mathcal{E}_h}\int_e [\![\mathbb{P}_h]\!]^n \cdot \mathbf{v}(\mathbf{X}_h^n(s,t))\,dA \qquad \forall \mathbf{v}\in V_h$

Step 2. find $(\mathbf{u}_h^{n+1}, p_h^{n+1}) \in V_h \times Q_h$ such that

$$\textbf{NS}\begin{cases}\rho_f\left(\dfrac{\mathbf{u}_h^{n+1}-\mathbf{u}_h^n}{\Delta t},\mathbf{v}\right)+a(\mathbf{u}_h^{n+1},\mathbf{v})+b(\mathbf{u}_h^{n+1},\mathbf{u}_h^{n+1},\mathbf{v})\\ \quad -(\operatorname{div}\mathbf{v},p_h^{n+1}) = \\ \quad -\displaystyle\int_{\mathcal{B}}(\rho_s-\rho_f)\frac{\mathbf{X}_h^{n+1}-2\mathbf{X}_h^n+\mathbf{X}_h^{n-1}}{\Delta t^2}\cdot\mathbf{v}(\mathbf{X}_h^n(s))ds \\ \qquad +\langle \mathbf{F}_h^n,\mathbf{v}\rangle_h \qquad\qquad \forall \mathbf{v}\in V_h \\ (\operatorname{div}\mathbf{u}_h^{n+1},q)=0 \qquad\qquad \forall q\in Q_h;\end{cases}$$

Step 3. $\dfrac{\mathbf{X}_{hi}^{n+1}-\mathbf{X}_{hi}^n}{\Delta t} = \mathbf{u}_h^{n+1}(\mathbf{X}_{hi}^n) \quad \forall i = 1,\dots,M.$

Using **Step 3** in **Step 2** we get:

Step 1. $\langle \mathbf{F}_h^n, \mathbf{v} \rangle_h = -\sum_{e \in \mathcal{E}_h} \int_e [\![\mathbb{P}_h]\!]^n \cdot \mathbf{v}(\mathbf{X}_h^n(s,t))\, dA \qquad \forall \mathbf{v} \in V_h$

Step 2. find $(\mathbf{u}_h^{n+1}, p_h^{n+1}) \in V_h \times Q_h$ such that

$$
\textbf{NS}\begin{cases}
\rho_f \left(\dfrac{\mathbf{u}_h^{n+1} - \mathbf{u}_h^n}{\Delta t}, \mathbf{v} \right) + a(\mathbf{u}_h^{n+1}, \mathbf{v}) + b(\mathbf{u}_h^{n+1}, \mathbf{u}_h^{n+1}, \mathbf{v}) \\
\quad -(\operatorname{div} \mathbf{v}, p_h^{n+1}) = \\
\quad -\displaystyle\int_{\mathcal{B}} (\rho_s - \rho_f) \frac{\mathbf{u}_h^{n+1}(\mathbf{X}_h^n(s)) - \mathbf{u}_h^n(\mathbf{X}_h^{n-1}(s))}{\Delta t} \cdot \mathbf{v}(\mathbf{X}_h^n(s)) ds \\
\qquad + \langle \mathbf{F}_h^n, \mathbf{v} \rangle_h & \forall \mathbf{v} \in V_h \\
(\operatorname{div} \mathbf{u}_h^{n+1}, q) = 0 & \forall q \in Q_h;
\end{cases}
$$

Step 3. $\dfrac{\mathbf{X}_{hi}^{n+1} - \mathbf{X}_{hi}^n}{\Delta t} = \mathbf{u}_h^{n+1}(\mathbf{X}_{hi}^n) \quad \forall i = 1, \dots, M.$

Discrete Energy Estimate

<B.–Cavallini–Gastaldi '10>

Artificial Viscosity Theorem

Let $\mathbf{u}_h^n$, p_h^n and $\mathbf{X}_h^n$ be a solution to the FE-IBM, then

$$\frac{\rho_f}{2\Delta t}\left(\|\mathbf{u}_h^{n+1}\|_0^2 - \|\mathbf{u}_h^n\|_0^2\right) + (\mu + \mu_a)\|\nabla \mathbf{u}_h^{n+1}\|_0^2 + \frac{1}{\Delta t}\left(E\left[\mathbf{X}_h^{n+1}\right] - E\left[\mathbf{X}_h^n\right]\right) + \frac{1}{2\Delta t}(\rho_s - \rho_f)\left(\|\mathbf{u}_h^{n+1}(\mathbf{X}_h^n)\|_{0,\mathcal{B}}^2 - \|\mathbf{u}_h^n(\mathbf{X}_h^{n-1}\|_{0,\mathcal{B}}^2\right) \le 0$$

CFL Conditions: $\mu + \mu_a \ge 0$, $\rho_s \ge \rho_f$ (might be relaxed)

CFL condition

BE is unconditionally stable, while MBE requires the term μ_a to be not too large

$$\mu_a = -\kappa_{max} C \frac{h_s^{(m-2)} \Delta t}{h_x^{(d-1)}} L^n$$

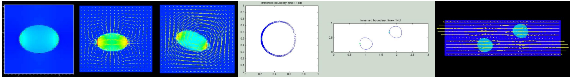
space dim.	solid dim.	CFL condition
2	1	$L^n \Delta t \leq C h_x h_s$
2	2	$L^n \Delta t \leq C h_x$
3	2	$L^n \Delta t \leq C h_x^2$
3	3	$L^n \Delta t \leq C h_x^2 / h_s$

Some numerical results

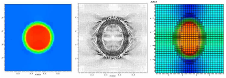
Original 2D code in Fortran 77, ported to DEAL.II (c++) (www.dealii.org) by L. Heltai ($Q_2 - P_1$)

2D

Codimension 1

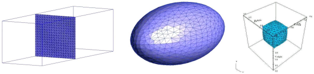


Codimension 0



3D

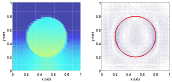
Codimension 1



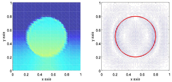
More numerical results

Fortran 90 code written by N. Cavallini (P_1iso$P_2 - P_1^c$)

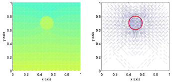
Densities: $\rho_s = 21$ and $\rho_f = 1$



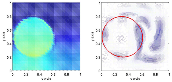
$\kappa = 1$



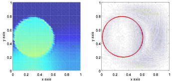
$\kappa = 0.1$



$\kappa = 0.1$



$\kappa = 1$



$\kappa = 0.1$

Mass conservation of the IBM

<B.–Cavallini–Gardini–Gastaldi '10>

Well-known and studied problem

The discrete divergence free condition is imposed in a weak sense

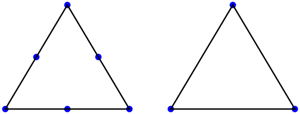
$$\int_\Omega \operatorname{div} \mathbf{u}_h q_h \, d\mathbf{x} = 0 \quad \forall q_h \in Q_h$$

which is not exact unless $\operatorname{div}(V_h) \subset Q_h$

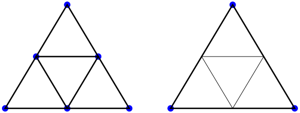
Basic remark

Discontinuous pressure schemes enjoy *local* mass conservation properties (average of divergence is zero element by element)

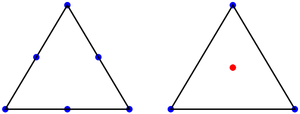
Our elements



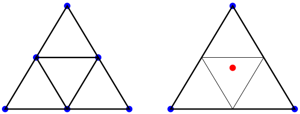
Hood–Taylor



$P_1 \text{iso} P_2 - P_1^c$



Enhanced Hood–Taylor



Enhanced $P_1 \text{iso} P_2 - P_1^c$

We actually considered generalized Hood–Taylor in two and three dimensions $P_{k+1} - P_k^c$ $(k \geq 1)$

Not a new idea

Local mass conservation is guaranteed by extra degree of freedom: add piecewise constant pressures

Analysis of our elements

Known facts

Hood–Taylor

- Introduced in 1973 <Hood–Taylor '73>
- First analysis <Bercovier–Pironneau '79, Verfürth '84>
- Full analysis with some restrictions on boundary elements <Scott–Vogelius '85, Brezzi–Falk '91>
- General analysis for the $P_{k+1} - P_k^c$ element with no restrictions (mesh contains at least 3 elements) <B. '94>

P_1 iso $P_2 - P_1^c$

- Same analysis as for the Hood-Taylor element can be carried on <Bercovier–Pironneau '79, Brezzi–Fortin '91>
- Error estimates are suboptimal (unbalanced spaces); ease of implementation makes it appealing, in particular in 3D

Analysis of our elements (cont'ed)

Pressure enhancement

- Numerical evidence for lowest order Hood-Taylor (triangles and squares)

<Gresho–Lee–Chan–Leone '80>
<Griffiths '82>
<Tidd–Thatcher–Kaye '88>

- Proof of inf-sup for lowest order Hood-Taylor (triangles and squares)

<Thatcher '90>
<Pierre '94>
<Quin–Zhang '05>

Analysis of our elements (cont'ed)

Theorem (B.–Cavallini–Gardini–Gastaldi '10)

The generalized enhanced Hood-Taylor scheme

$$P_{k+1} - (P_k^c + P_0)$$

in two ($k \geq 1$) and three ($k \geq 2$) dimensions and the enhanced

$$P_1 iso P_2 - (P_1^c + P_0)$$

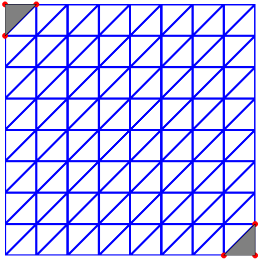
in two dimensions satisfy the inf-sup condition

Minimal restriction on the mesh: each element has at least one internal vertex.

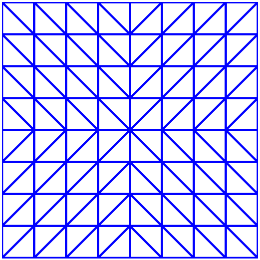
Mesh restrictions

2D: let us understand the restrictions

- Standard schemes: the mesh needs at least three elements
- Enhanced schemes: each element needs at least an internal vertex



Uniform mesh



Symmetric mesh

Numerical results

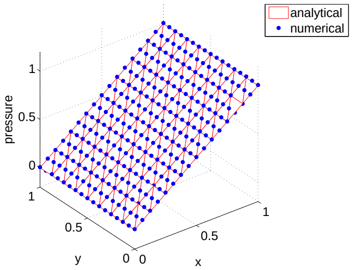
$\Omega =]0,1[\times]0,1[$

f chosen such that exact solution is

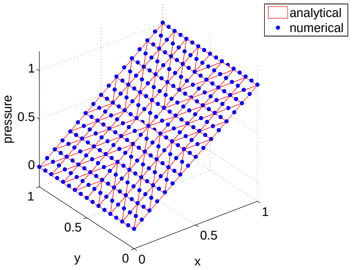
$$\mathbf{u}(x,y) = \mathbf{rot}\,\varphi(x,y)$$
$$\varphi(x,y) = x^2(x-1)^2y^2(1-y)^2$$
$$p(x,y) = x$$

Solution computed with the four different schemes on uniform and symmetric meshes, successively refined

Hood-Taylor

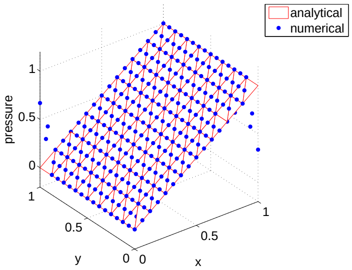


Uniform mesh

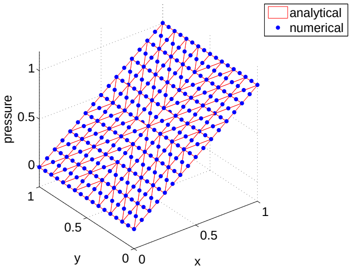


Symmetric mesh

Enhanced Hood-Taylor

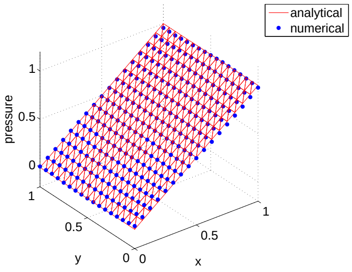


Uniform mesh

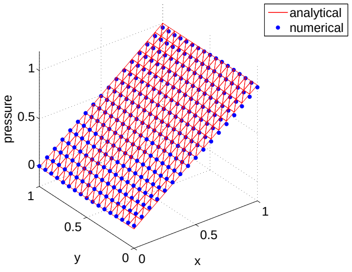


Symmetric mesh

$P_1 \text{iso} P_2 - P_1^c$

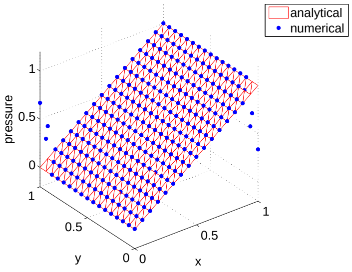


Uniform mesh

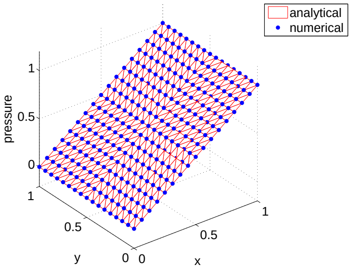


Symmetric mesh

Enhanced $P_1 \mathrm{iso} P_2 - (P_1^c + P_0)$

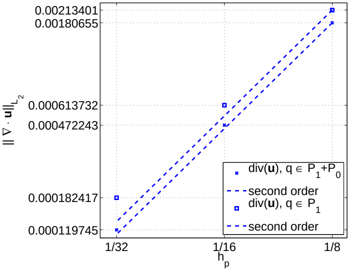


Uniform mesh

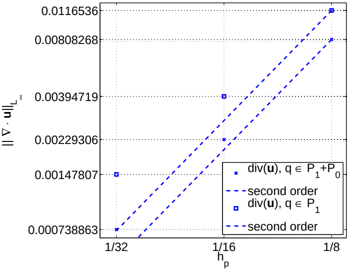


Symmetric mesh

Divergence error: Hood-Taylor

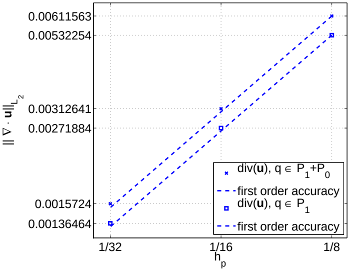


$L^2(\Omega)$-error

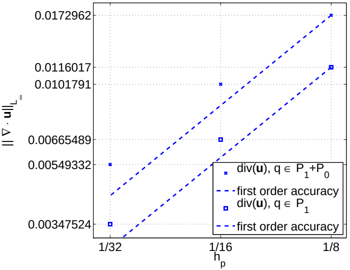


$L^\infty(\Omega)$-error

Divergence error: $P_1 \mathrm{iso} P_2$

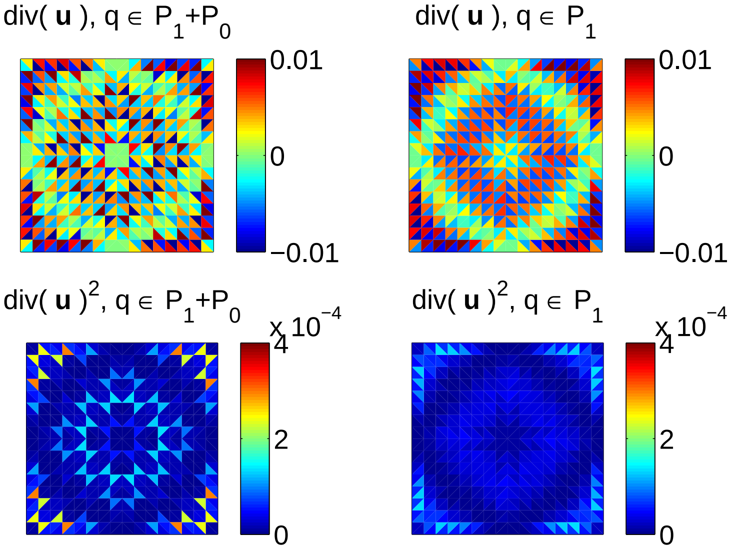


$L^2(\Omega)$-error



$L^\infty(\Omega)$-error

Enhanced $P_1 \mathrm{iso} P_2 - (P_1^c + P_0)$



Iterative solver

Number of iterations needed to reach convergence when using conjugate gradient à la Glowinski

Element type	Iterations		
	$h_p = 1/8$	$h_p = 1/16$	$h_p = 1/32$
$P_2 - P_1^c$	130	169	172
$P_2 - (P_1^c + P_0)$	25	29	29
$P_1\mathrm{iso}P_2 - P_1^c$	19	24	24
$P_1\mathrm{iso}P_2 - (P_1^c + P_0)$	30	35	35

Conclusions

1. The finite element Immersed Boundary Method provides interesting results for the approximation of fluid-structure interaction problems. Rigorous proof of a CFL condition shows that modified BE scheme can be successfully used in this framework
2. We performed a rigorous analysis of locally mass preserving Stokes element in a general setting