Spherical Unitary dual for quasisplit real groups

Dan Barbasch

(joint work with Dan Ciubotaru)

Banff

July 2010

Notation

NOTATION

- G is the real points of a linear connected reductive group.

- $\mathfrak{g}_0 := Lie(G)$, θ Cartan involution, $\mathfrak{g}_0 = \mathfrak{k}_0 + \mathfrak{s}_0$, $\mathfrak{g} := (\mathfrak{g}_0)_{\mathbb{C}}$, K maximal compact subgroup, $\mathfrak{g} = \mathfrak{k} + \mathfrak{s}$,

- $P = MN$ minimal parabolic subgroup, $\theta(M) = M$, and A is the split part of the center of M; then $M \cap K := C_K(A)$, and $M = M \cap K \cdot A$.

- $W := N_K(A)/M \cap K$ the Weyl group.

- $\lambda \in \widehat{K}$ a $K-$type, then W acts on $V_\lambda^{M \cap K}$.

Problem

Compute the representation of W on $V_\lambda^{M\cap K}$

More generally if $\chi \in \widehat{M \cap K}$, compute the representation of W_χ (the centralizer of χ in W) on $\mathrm{Hom}_{M\cap K}[\chi, V_\lambda]$.

Motivation

(1) For $G = GL(n, \mathbb{C})$, $K = U(n)$ and M is the diagonal torus, and $W = S_n$. Kostka-Foulkes polynomials encode information about V_λ^M.

(2) Spherical unitary dual.

Spherical unitary representations

Let $\chi \in \widehat{M}$. The spherical principal series is

$$X(\chi) := Ind_P^G(\chi \otimes \delta_P^{-1/2} \otimes \mathbb{1}), \tag{1}$$

where χ is an unramified character, ($i.e.$ $\chi\,|_{M\cap K} = triv$), and δ_P is the modulus function of P.

- $\mathrm{Hom}_K[Triv : X(\chi)] = 1$, $L(\chi)$ the unique irreducible subquotient containing the trivial $K-$type.
- Every spherical irreducible module is an $L(\chi)$ for some χ.
- $L(\chi) \cong L(\chi')$ if and only if there exists $w \in W$ such that $w\chi = \chi'$.
- $L(\chi)$ is hermitian if and only if there is $w \in W$ such that $w\chi = \overline{\chi^{-1}}$.

- For every $w \in W$ there is an intertwining operator $A_w(\chi) : X(\chi) \longrightarrow X(w\chi)$.
- A_w gives rise to

  $a_w(\chi, \lambda)$:

  $$\mathrm{Hom}_K[V_\lambda, X(\chi)] \cong V_\lambda^{M\cap K} \longrightarrow \mathrm{Hom}_K[V_\lambda : X(w\chi)] \cong V_\lambda^{M\cap K},$$

- A_w is normalized so that $a_w(\chi, triv) = id$; this makes A_w analytic for the region for which $\langle Re\chi, \alpha\rangle \geq 0$ for all roots of N,
- in the hermitian case $a_w(\chi, \lambda)$ gives rise to a hermitian form. $L(\chi)$ is unitary iff $a_w(\chi, \lambda)$ positive semidefinite for all λ.
- if $w = s_1 \dots s_k$ is a reduced decomposition,

  $$a_w = a_{s_1} \cdot \dots \cdot a_{s_k},$$

  and each a_{s_i} is induced from a corresponding operator on a real rank one group.

- A $K-$type will be called **single petaled**, if $a_w(\chi, \lambda)$ only depends on the Weyl group representation V_λ^M. More precisely this is a condition on the $a_{s_i}(\chi, \lambda)$ so that they are as simple as possible. For example when a_{s_i} comes from $SL(2, \mathbb{R})$, it has the form

$$a_{s_\alpha}(2m, \chi) = \begin{cases} Id & \text{if } m = 0, \\ \prod_{0<j\leq m} \frac{2j-1-<\nu,\check{\alpha}>}{2j-1+<\nu,\check{\alpha}>} \; Id & \text{if } m \neq 0, \end{cases}$$

($2m$ parametrize the spherical $K-$types of $SO(2)$). For other real rank one groups there are similar formulas by [JW]. We require that $m = 0, 1$ only,

$$a_{s\alpha}(\lambda, \chi)v = \begin{cases} v & \text{if } s_\alpha v = v, \\ \frac{q_\alpha - <\nu,\check{\alpha}>}{q_\alpha + <\nu,\check{\alpha}>} & \text{if } s_\alpha v = -v \end{cases}$$

for $v \in V_\lambda^{M\cap K}$. The q_α are (positive) scalars that only depend on the $W-$orbit of α.

There are analogous results when we replace χ by an arbitrary character, or $\mathbb{R}$ by a p-adic field. In the case of a split adjoint $p-$adic group, [BM1] and [BM2] replace the group by an affine graded Hecke algebra. The V_λ are replaced by Weyl group representations, and the formulas above are exact; they are the formulas for the intertwining operators.

The guiding principle is that for these $K-$types we can do the calculation in the affine graded Hecke algebra with parameters q_α, and V_λ is replaced by a Weyl group representation.

The p-adic case

- G split, $B = AN$ a Borel subgroup, $\mathbb{F} \supset \mathcal{R} \supset \mathcal{P}$, $K = G(\mathcal{R})$,
- $\mathcal{I}$ an Iwahori subgroup.
- $\chi\mid_{A \cap K} = triv$, *i.e.* unramified.
- ${}^{\vee}G$ be the complex dual group.

Then

$$\{L(\chi) \text{ spherical }\} \longleftrightarrow \{s \in {}^{\vee}G \ \ \text{semisimple}\}/{}^{\vee}G.$$

s decomposes into an elliptic and a hyperbolic part $s = s_e s_h$.

$$Unit_{sph}(G) = \bigsqcup Unit_{sph,s_e}(G)$$

[BM1] and [BM2] show that

1. $Unit_{\mathcal{I}-sph}(G) \cong Unit(\mathcal{H})$ where $\mathcal{H}$ is the Iwahori-Hecke algebra,
2. $Unit(\mathcal{H}_{s_e}) \cong Unit(\mathbb{H}(s_e))$, where $\mathbb{H}(s_e)$ is the affine graded I-Hecke algebra at s_e.

In particular,

$$Unit_{sph,s_e}(G) \cong Unit_{sph,1}(G(s_e)),$$

where $G(s_e)$ is the split group dual to ${}^{\vee}G(s_e)$.

We will assume that $s_e = 1$.

Main Result

Joint with Dan Ciubotaru we have extended the results for $\mathcal{I}-$spherical representations to groups other than adjoint type and

- arbitrary χ for split groups of any kind, (using results of Roche)
- blocks (in the sense of Bernstein) when there are types, *e.g.* unipotent representations for p-adic groups studied by Lusztig,
- blocks associated to unramified characters of quasisplit groups.

Main topic of this talk

Let G be quasisplit, *"but with no factor which is a complex group viewed as a real group"*.

Associated to G there is an (outer) automorphism ${}^\vee\tau$ of ${}^\vee G$. Then form ${}^LG := {}^\vee G \ltimes \{{}^\vee\tau\}$, and let ${}^\vee G{}^\vee\tau$ be the connected component of ${}^\vee\tau$. In this case,

$$\{L(\chi) \text{ unramified }\} \leftrightarrow \{s \in {}^\vee G{}^\vee\tau \text{ semisimple }\}/{}^\vee G.$$

A semisimple element decomposes $s = s_h s_e$ with $s_e \in {}^\vee G{}^\vee\tau$. Let $G(s_e)$ be as before (split real group). Then there is an inclusion

$$Unit_{sph,s_e}(G) \subset Unit_{sph,1}(G(s_e)).$$

Here are the groups for real infinitesimal character, *i.e.* $s_e = {}^\vee\tau$:

G	${}^\vee G_\tau$	$G(\tau)$	G_τ
$U(n,n)$	$Sp(2n)$	$So(n+1,n)$	$Sp(2n,\mathbb{R}), So(n,n)$
$U(n+1,n)$	$So(2n+1)$	$Sp(2n,\mathbb{R})$	$So(n+1,n)$
$So(n+2,n)$	$So(2n+1)$	$Sp(2n,\mathbb{R})$	$So(n+1,n)$
E_6	$F_4(\mathbb{C})$	$F_4(\mathbb{R}, split)$	$F_4(\mathbb{R}, split)$

By [B3], this is an equality for $U(n+1,n)$, $U(n,n)$. For type E_6 the inclusion is into the spherical unitary dual for split $p-$adic F_4 which is known by [C1].

Split groups, p-adic case

$$Sph(G) = \bigsqcup_{{}^\vee\mathcal{O}\subset{}^\vee\mathfrak{g}} Sph(G)_{{}^\vee\mathcal{O}}$$

where ${}^\vee\mathcal{O}$ is a nilpotent orbit. Let $A({}^\vee\mathcal{O})$ be the reductive part of the centralizer of ${}^\vee e \in {}^\vee\mathcal{O}$. Then

$$Sph(G)_{{}^\vee\mathcal{O},u} = Sph(A({}^\vee\mathcal{O}))_{0,u}.$$

The spherical unitary dual only depends on the adjoint group, not the isogeny classes. So we only need to specify $Sph(G)_{0,u}$ for G simple. This is a union of simplices in the dominant chamber, explicity determined in [B1] for classical types, [C1] for F_4, [BC] for E_6, E_7, E_8. G_2 and small rank cases were known before.

There are some exceptions, where the answer has to be given case by case:

$$\{\underbrace{A_2 + 3A_1}_{\text{in } E_7}, \underbrace{A_4A_2A_1, A_4A_2, D_4(a_1)A_2, A_3 + 2A_1, A_2 + 2A_1, 4A_1}_{\text{in } E_8}\}. \tag{2}$$

See [C1] for F_4.

Sketch of some proofs

Type $\mathbf{F_4}$. The maximal compact subgroup (actually of the double cover of F_4) is $Sp(2) \times Sp(6)$. There is a matchup $\sigma \longleftrightarrow V_{\mu(\sigma)}$ with the proerty that V_μ is petite, and the representation of W on $V_\mu^{M \cap K}$ is σ :

$K - type$	$W - type$
$(0 \mid 0,0,0)$	$1_1,$
$(0 \mid 1,1,0)$	$2_1,$
$(4 \mid 0,0,0)$	$2_3,$
$(1 \mid 2,1,0)$	$8_1,$
$(1 \mid 1,1,1)$	$4_2,$
$(2 \mid 2,0,0)$	$9_1.$

These $W-$ types are called *relevant*; a spherical irreducible representation is unitary if and only if it is positive definite on these $W-$types. This implies an embedding of the spherical unitary dual of the split real F_4 into the spherical unitary dual of the split p-adic F_4. Similar results are proved for all split groups, [B1], [B2], [C1], [BC].

We consider the case of quasisplit E_6. For each $\sigma \in \widehat{W}$ on the list, we need a $V_{\lambda(\sigma)}$ which is petite, and such that $V^M_{\lambda(\sigma)}$ contains σ. Let $\tau \in Aut(G)$ satisfy

- τ and θ commute,
- G_τ is split type F_4.

Then $K_\tau = C_1C_3 \subset K = A_1A_5$ with C_1 identified with the A_1, and $C_3 \subset A_5$ the usual inclusion. Let $H = MT$ be a Cartan subgroup of K with T a Cartan subgroup of K_τ. We can ignore the $C_1 \cong A_1$.

In coordinates

$$\mathfrak{t} = \{(a, b, c, -c, -b, -a)\}$$
$$\mathfrak{m} = \{(a_1, a_2, -a_1 - a_2, -a_1 - a_2, a_2, a_1)\}.$$

Suppose we want to match 8_1 with a petite representation of A_5. We choose a λ as small as possible so that $V_\lambda\mid_{C_3}$ contains the representation $(2, 1, 0)$ of C_3. The best choice would be a λ such that $\lambda\mid_{\mathfrak{t}}= (2, 1, 0)$ and $\lambda\mid_{\mathfrak{m}}= 0$. This does not work. It turns out that the good choice is $\lambda = (2, 1, 0, 0, 0, 0)$. It is easy to see that $\dim V^{\mathfrak{m}} = 16$, and $\dim V^M = 8$. Since also

$$(2, 1, 0, 0, 0, 0)\mid_{C_3}= (2, 1, 0) + (1, 0, 0),$$

and the second factor does not contain any M_τ fixed vectors, the claim follows from the F_4 computation.

References

References

[B1] D. Barbasch *The spherical unitary dual of split real and p-adic groups,* preprint, to appear in Journal of Institute Jussieu

[B2] D. Barbasch *Relevant and petite K-types for split groups*, Functional analysis VIII, 35–71, Various Publ. Ser. (Aarhus), 47, Aarhus Univ., Aarhus, 2004

[B3] D. Barbasch *A reduction theorem for the unitary dual of* $U(p,q)$ in volume in honor of Carmona

[BC] D. Barbasch, D. Ciubotaru *Whittaker unitary dual of affine graded Hecke algebras of type E*, preprint

[BM1] D. Barbasch, A. Moy *A unitarity criterion for p-adic groups* Invent. Math., vol 98, 1989, 19-38

[BM2] ______ *Reduction to real infinitesimal character in affine Hecke algebras*, Journal of the AMS, vol 6, Number 3, 1993, 611-635

[C1] D. Ciubotaru *The Iwahori spherical unitary dual of the split group of type F4*, Represent. Theory, vol 9, 2005, 94–137

[JW] K. Johnson, N. Wallach, *Composition series and intertwining operators for spherical principal series I*, Transactions of AMS, vol. 229, (1977), 137-173