Noncommutative Schur Functions

K. Luoto*,
C. Bessenrodt, S. van Willigenburg

Department of Mathematics
University of British Columbia

Banff International Research Station
Workshop on Quasisymmetric Functions
Nov. 18th, 2010

Graded dual Hopf algebras

- Sym is self-dual
  - m_λ dual to h_λ (complete symmetric fcns)
  - s_λ dual to itself

- QSym is dual to NSym, the noncommutative symmetric fcns
  - M_α dual to $\mathbf{h}_\alpha$ (noncommutative complete symmetric fcns)
  - F_α dual to $\mathbf{r}_\alpha$ (noncommutative ribbon Schurs)

  - $\mathcal{S}_\alpha$ dual to $\mathbf{s}_\alpha$ (noncommutative Schurs)[†]

$$\Delta \mathcal{S}_\gamma = \sum_\beta \mathcal{S}_{\gamma /\!/ \beta} \otimes \mathcal{S}_\beta$$

$$\mathcal{S}_{\gamma /\!/ \beta} = \sum_\alpha C^\gamma_{\alpha,\beta} \mathcal{S}_\alpha \quad \Longleftrightarrow \quad \mathbf{s}_\alpha \mathbf{s}_\beta = \sum_\gamma C^\gamma_{\alpha,\beta} \mathbf{s}_\gamma$$

[†] not the same as Fomin & Greene's

Graded dual Hopf algebras

- Sym is self-dual
  - m_λ dual to h_λ (complete symmetric fcns)
  - s_λ dual to itself

- QSym is dual to NSym, the noncommutative symmetric fcns
  - M_α dual to $\mathbf{h}_\alpha$ (noncommutative complete symmetric fcns)
  - F_α dual to $\mathbf{r}_\alpha$ (noncommutative ribbon Schurs)
  - $\mathcal{S}_\alpha$ dual to $\mathbf{s}_\alpha$ (noncommutative Schurs)†

$$\Delta \mathcal{S}_\gamma = \sum_\beta \mathcal{S}_{\gamma /\!/ \beta} \otimes \mathcal{S}_\beta$$

$$\mathcal{S}_{\gamma /\!/ \beta} = \sum_\alpha C^\gamma_{\alpha,\beta} \mathcal{S}_\alpha \quad \Longleftrightarrow \quad \mathbf{s}_\alpha \mathbf{s}_\beta = \sum_\gamma C^\gamma_{\alpha,\beta} \mathbf{s}_\gamma$$

† not the same as Fomin & Greene's

Hopf algebra maps

$$\text{NSym} \xrightarrow{\ \chi\ } \text{Sym} \hookrightarrow \text{QSym}$$

$$m_\lambda = \sum_{\widetilde{\alpha}=\lambda} M_\alpha \quad \Longrightarrow \quad \chi(\mathbf{h}_\alpha) = h_{\widetilde{\alpha}} = m^*_{\widetilde{\alpha}}$$

$$s_\lambda = \sum_{\widetilde{\alpha}=\lambda} \mathcal{S}_\alpha \quad \Longrightarrow \quad \chi(\mathbf{s}_\alpha) = s_{\widetilde{\alpha}}$$

Littlewood-Richardson reverse tableaux

$$T = \begin{array}{|c|c|c|c|}\hline * & * & * & 6 \\ \hline * & * & 7 & 4 \\ \hline * & 8 & 5 & 2 \\ \hline 9 & 3 & 1 \\ \cline{1-3}\end{array} \qquad \widetilde{U}_{4221} = \begin{array}{|c|c|c|c|}\hline 9 & 8 & 7 & 6 \\ \hline 5 & 4 \\ \cline{1-2} 3 & 2 \\ \cline{1-2} 1 \\ \cline{1-1}\end{array}$$

$$w_{col}(T) = 9\,38\,157\,246 \qquad w_{col}(\widetilde{U}_{4221}) = 1359\,248\,7\,6$$

$$RSK: \quad \pi \quad \longleftrightarrow \quad (P(\pi), Q(\pi))$$

$$rect(T) := P(w_{col}(T))$$

T is a LR standard reverse tableau if

$$rect(T) = \widetilde{U}_{\lambda} \quad \text{for some } \lambda$$

Littlewood-Richardson reverse tableaux

$T =$

$*$	$*$	$*$	6
$*$	$*$	7	4
$*$	8	5	2
9	3	1	

$\widetilde{U}_{4221} =$

9	8	7	6
5	4		
3	2		
1			

$w_{col}(T) = 9\,38\,157\,246$ $\qquad$ $w_{col}(\widetilde{U}_{4221}) = 1359\,248\,7\,6$

$$RSK : \qquad \pi \quad \longleftrightarrow \quad (P(\pi), Q(\pi))$$

$$rect(T) := P(w_{col}(T))$$

T is a LR standard reverse tableau if

$$rect(T) = \widetilde{U}_\lambda \quad \text{for some } \lambda$$

Littlewood-Richardson reverse tableaux

$$T = \begin{array}{|c|c|c|c|}\hline * & * & * & 6 \\ \hline * & * & 7 & 4 \\ \hline * & 8 & 5 & 2 \\ \hline 9 & 3 & 1 & \\ \cline{1-3}\end{array} \qquad \widetilde{U}_{4221} = \begin{array}{|c|c|c|c|}\hline 9 & 8 & 7 & 6 \\ \hline 5 & 4 \\ \cline{1-2} 3 & 2 \\ \cline{1-2} 1 \\ \cline{1-1}\end{array}$$

$$w_{col}(T) = 9\,38\,157\,246 \qquad w_{col}(\widetilde{U}_{4221}) = 1359\,248\,7\,6$$

$$RSK : \quad \pi \quad \longleftrightarrow \quad (P(\pi), Q(\pi))$$

$$rect(T) := P(w_{col}(T))$$

T is a LR standard reverse tableau if

$$rect(T) = \widetilde{U}_\lambda \quad \text{for some } \lambda$$

Classical Littlewood-Richardson rule

Littlewood-Richardson coefficients $c^{\nu}_{\lambda,\mu}$

$$
\begin{aligned}
s_{\nu/\mu} &= \sum_{\lambda} c^{\nu}_{\lambda,\mu} s_{\lambda} \\
s_{\lambda} s_{\mu} &= \sum_{\nu} c^{\nu}_{\lambda,\mu} s_{\nu}
\end{aligned}
$$

Theorem (Littlewood-Richardson rule)

In the above expansions, $c^{\nu}_{\lambda,\mu}$ *is the number of* $T \in SRT(\nu/\mu)$ *such that* $rect(T) = \widetilde{U}_{\lambda}$.

Posets $\mathcal{L}_Y$ and $\mathcal{L}_c$

- $\mathcal{L}_Y$: Partitions, partially ordered by containment:
  Cover by
  - appending 1
  - incrementing first (leftmost) $k \mapsto k+1$

  examples:
  - $(2,1,1) \lessdot_Y (2,1,1,1)$
  - $(2,1,1) \lessdot_Y (2,2,1)$
  - $(2,1,1) \lessdot_Y (3,1,1)$

- $\mathcal{L}_c$: Partial order on compositions:
  Cover by
  - *prepending* 1
  - incrementing first (leftmost) $k \mapsto k+1$

  examples:
  - $(1,2,1) \lessdot_c (1,1,2,1)$
  - $(1,2,1) \lessdot_c (2,2,1)$
  - $(1,2,1) \lessdot_c (1,3,1)$

Posets $\mathcal{L}_Y$ and $\mathcal{L}_c$

- $\mathcal{L}_Y$: Partitions, partially ordered by containment:
  Cover by
  - appending 1
  - incrementing first (leftmost) $k \mapsto k+1$

  examples:
  - $(2,1,1) \lessdot_Y (2,1,1,1)$
  - $(2,1,1) \lessdot_Y (2,2,1)$
  - $(2,1,1) \lessdot_Y (3,1,1)$

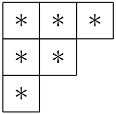
- $\mathcal{L}_c$: Partial order on compositions:
  Cover by
  - *prepending* 1
  - incrementing first (leftmost) $k \mapsto k+1$

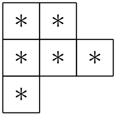
  examples:
  - $(1,2,1) \lessdot_c (1,1,2,1)$
  - $(1,2,1) \lessdot_c (2,2,1)$
  - $(1,2,1) \lessdot_c (1,3,1)$

Mason's bijection ρ

Fix $\mu = \widetilde{\beta}$.

$$SRT(-/\mu) \quad \overset{\rho}{\longleftrightarrow} \quad SCT(-/\!\!/\beta)$$





$\mu = (3, 2, 1) \qquad \leftarrow \text{base} \rightarrow \qquad \beta = (2, 3, 1)$

Mason's bijection ρ

Fix $\mu = \widetilde{\beta}$.

$$SRT(-/\mu) \quad \overset{\rho}{\longleftrightarrow} \quad SCT(-/\!\!/\beta)$$

*	*	*
*	*	
*	9	

*	*	
*	*	*
*	9	

$\mu = (3, 2, 1)$ $\leftarrow$ base $\rightarrow$ $\beta = (2, 3, 1)$

Mason's bijection ρ

Fix $\mu = \widetilde{\beta}$.

$$SRT(-/\mu) \quad \overset{\rho}{\longleftrightarrow} \quad SCT(-/\!\!/\beta)$$

$*$	$*$	$*$
$*$	$*$	8
$*$	9	

$*$	$*$	8
$*$	$*$	$*$
$*$	9	

$\mu = (3, 2, 1) \qquad \leftarrow$ base $\rightarrow \qquad \beta = (2, 3, 1)$

Mason's bijection ρ

Fix $\mu = \widetilde{\beta}$.

$$SRT(-/\mu) \quad \overset{\rho}{\longleftrightarrow} \quad SCT(-/\!\!/\beta)$$

$*$	$*$	$*$	7
$*$	$*$	8	
$*$	9		

$*$	$*$	8	7
$*$	$*$	$*$	
$*$	9		

$\mu = (3, 2, 1)$ $\leftarrow$ base $\rightarrow$ $\beta = (2, 3, 1)$

Mason's bijection ρ

Fix $\mu = \widetilde{\beta}$.

$$SRT(-/\mu) \quad \stackrel{\rho}{\longleftrightarrow} \quad SCT(-/\!\!/\beta)$$

$*$	$*$	$*$	7
$*$	$*$	8	
$*$	9		
6			

6			
$*$	$*$	8	7
$*$	$*$	$*$	
$*$	9		

$\mu = (3, 2, 1)$ $\quad \leftarrow$ base $\rightarrow \quad$ $\beta = (2, 3, 1)$

Mason's bijection ρ

Fix $\mu = \widetilde{\beta}$.

$$SRT(-/\mu) \quad \overset{\rho}{\longleftrightarrow} \quad SCT(-/\!\!/\beta)$$

*	*	*	7
*	*	8	
*	9		
6			
5			

5			
6			
*	*	8	7
*	*	*	
*	9		

$\mu = (3, 2, 1) \quad \leftarrow$ base $\rightarrow \quad \beta = (2, 3, 1)$

Mason's bijection ρ

Fix $\mu = \widetilde{\beta}$.

$$SRT(-/\mu) \quad \overset{\rho}{\longleftrightarrow} \quad SCT(-/\!\!/\beta)$$

*	*	*	7
*	*	8	
*	9	4	
6			
5			

5			
6			
*	*	8	7
*	*	*	
*	9	4	

$\mu = (3, 2, 1) \qquad \leftarrow$ base $\rightarrow \qquad \beta = (2, 3, 1)$

Mason's bijection ρ

Fix $\mu = \widetilde{\beta}$.

$$SRT(-/\mu) \quad \overset{\rho}{\longleftrightarrow} \quad SCT(-/\!\!/\beta)$$

$*$	$*$	$*$	7
$*$	$*$	8	3
$*$	9	4	
6			
5			

5			
6			
$*$	$*$	8	7
$*$	$*$	$*$	3
$*$	9	4	

$\mu = (3, 2, 1) \qquad \leftarrow \text{base} \rightarrow \qquad \beta = (2, 3, 1)$

Mason's bijection ρ

Fix $\mu = \widetilde{\beta}$.

$$SRT(-/\mu) \quad \overset{\rho}{\longleftrightarrow} \quad SCT(-/\!\!/\beta)$$

*	*	*	7
*	*	8	3
*	9	4	
6	2		
5			

5	2		
6			
*	*	8	7
*	*	*	3
*	9	4	

$\mu = (3, 2, 1)$ $\quad \leftarrow$ base $\rightarrow \quad$ $\beta = (2, 3, 1)$

Mason's bijection ρ

Fix $\mu = \widetilde{\beta}$.

$$SRT(-/\mu) \quad \overset{\rho}{\longleftrightarrow} \quad SCT(-/\!\!/\beta)$$

$*$	$*$	$*$	7
$*$	$*$	8	3
$*$	9	4	
6	2	1	
5			

5	2	1	
6			
$*$	$*$	8	7
$*$	$*$	$*$	3
$*$	9	4	

$\mu = (3, 2, 1)$ $\quad \leftarrow$ base $\rightarrow \quad$ $\beta = (2, 3, 1)$

Canonical and LR SCT

$T =$

6	5	4	3
$*$	$*$	9	8
$*$	$*$	$*$	7
$*$	2	1	

$U_{2412} =$

2	1		
6	5	4	3
7			
9	8		

$w_{col}(T) = 6\,25\,149\,378$ $\quad$ $w_{col}(U_{2412}) = 2679\,158\,4\,3$

$$rect(T) := P(w_{col}(T))$$

T is a LR SCT if

$$rect(T) = U_\alpha \quad \text{for some } \alpha$$

Noncommutative Littlewood-Richardson rule (new)

Noncommutative Littlewood-Richardson coefficients $C^{\gamma}_{\alpha,\beta}$

$$\mathcal{S}_{\gamma/\!/\beta} = \sum_{\alpha} C^{\gamma}_{\alpha,\beta}\, \mathcal{S}_{\alpha}$$

$$\mathbf{s}_{\alpha}\, \mathbf{s}_{\beta} = \sum_{\gamma} C^{\gamma}_{\alpha,\beta}\, \mathbf{s}_{\gamma}$$

Theorem (Noncommutative Littlewood-Richardson rule)

In the above expansions, $C^{\gamma}_{\alpha,\beta}$ is the number of $T \in SCT(\gamma /\!/ \beta)$ such that $rect(T) = U_{\alpha}$.

Note: If $\lambda = \widetilde{\alpha}$, and $\mu = \widetilde{\beta}$, then $c^{\nu}_{\lambda,\mu} = \sum_{\widetilde{\gamma}=\nu} C^{\gamma}_{\alpha,\beta}$.

Noncommutative Littlewood-Richardson rule (new)

Noncommutative Littlewood-Richardson coefficients $C^{\gamma}_{\alpha,\beta}$

$$\mathcal{S}_{\gamma/\!\!/\beta} = \sum_{\alpha} C^{\gamma}_{\alpha,\beta}\, \mathcal{S}_{\alpha}$$

$$\mathbf{s}_{\alpha}\, \mathbf{s}_{\beta} = \sum_{\gamma} C^{\gamma}_{\alpha,\beta}\, \mathbf{s}_{\gamma}$$

Theorem (Noncommutative Littlewood-Richardson rule)

In the above expansions, $C^{\gamma}_{\alpha,\beta}$ *is the number of* $T \in SCT(\gamma /\!\!/ \beta)$ *such that* $rect(T) = U_{\alpha}$.

Note: If $\lambda = \widetilde{\alpha}$, and $\mu = \widetilde{\beta}$, then $c^{\nu}_{\lambda,\mu} = \sum_{\widetilde{\gamma}=\nu} C^{\gamma}_{\alpha,\beta}$.

Classical LR rule viewpoint

- Dual equivalent SRT: $T \sim T'$ if

  T, T' same skew shape and $w_{col}(T) \overset{Q}{\sim} w_{col}(T')$

- (Haiman '92) Equivalence classes are *complete*:

  bijection $w_{col} : [T] \to [w_{col}(T)]_Q$

  $$shape(rect(T)) = \lambda \implies s_\lambda = \sum_{T' \sim T} F_{Des(T')}$$

- LR (skew) tableaux $\{T \in SRT : rect(T) = \widetilde{U}_\lambda \text{ for some } \lambda\}$ is a transversal of $\sim$

Classical LR rule viewpoint

- Dual equivalent SRT: $T \sim T'$ if

  $$T, T' \text{ same skew shape} \quad \text{and} \quad w_{col}(T) \overset{Q}{\sim} w_{col}(T')$$

- (Haiman '92) Equivalence classes are *complete*:

  $$\text{bijection} \qquad w_{col} : [T] \to [w_{col}(T)]_Q$$

  $$shape(rect(T)) = \lambda \quad \Longrightarrow \quad s_\lambda = \sum_{T' \sim T} F_{Des(T')}$$

- LR (skew) tableaux $\{T \in SRT : rect(T) = \widetilde{U}_\lambda \text{ for some } \lambda\}$ is a transversal of $\sim$

Classical LR rule viewpoint

- Dual equivalent SRT: $T \sim T'$ if
  $$T, T' \text{ same skew shape} \quad \text{and} \quad w_{col}(T) \overset{Q}{\sim} w_{col}(T')$$

- (Haiman '92) Equivalence classes are *complete*:
  $$\text{bijection} \qquad w_{col} : [T] \to [w_{col}(T)]_Q$$
  $$shape(rect(T)) = \lambda \quad \Longrightarrow \quad s_\lambda = \sum_{T' \sim T} F_{Des(T')}$$

- LR (skew) tableaux $\{T \in SRT : rect(T) = \widetilde{U}_\lambda \text{ for some } \lambda\}$ is a transversal of $\sim$

Classical LR rule viewpoint

- Dual equivalent SRT: $T \sim T'$ if
  $$T, T' \text{ same skew shape} \quad \text{and} \quad w_{col}(T) \overset{Q}{\sim} w_{col}(T')$$
- (Haiman '92) Equivalence classes are *complete*:
  $$\text{bijection} \qquad w_{col} : [T] \to [w_{col}(T)]_Q$$
  $$shape(rect(T)) = \lambda \quad \Longrightarrow \quad s_\lambda = \sum_{T' \sim T} F_{Des(T')}$$
- LR (skew) tableaux $\{T \in SRT : rect(T) = \widetilde{U}_\lambda \text{ for some } \lambda\}$ is a transversal of $\sim$

Noncommutative LR rule viewpoint

- C-equivalent permutations: $\pi \overset{C}{\sim} \sigma$ if

$$\pi \overset{Q}{\sim} \sigma \quad \text{and} \quad C\text{-shape}(P(\pi)) = C\text{-shape}(P(\sigma))$$

- C-equivalent SCT: $T \overset{C}{\sim} T'$ if

$$T, T' \text{ same skew shape} \quad \text{and} \quad w_{col}(T) \overset{C}{\sim} w_{col}(T')$$

- (BLvW '10) Equivalence classes are *complete* :

$$\text{bijection} \qquad w_{col} : [T]_C \to [w_{col}(T)]_C$$

$$C\text{-}shape(rect(T)) = \alpha \quad \Longrightarrow \quad \mathcal{S}_\alpha = \sum_{T' \overset{C}{\sim} T} F_{Des(T')}$$

- LR SCT $\{T \in SCT : rect(T) = U_\alpha \text{ for some } \alpha\}$ is a transversal of $\overset{C}{\sim}$

Noncommutative LR rule viewpoint

- C-equivalent permutations: $\pi \overset{c}{\sim} \sigma$ if

$$\pi \overset{Q}{\sim} \sigma \quad \text{and} \quad C\text{-shape}(P(\pi)) = C\text{-shape}(P(\sigma))$$

- C-equivalent SCT: $T \overset{c}{\sim} T'$ if

$$T, T' \text{ same skew shape} \quad \text{and} \quad w_{col}(T) \overset{c}{\sim} w_{col}(T')$$

- (BLvW '10) Equivalence classes are *complete* :

$$\text{bijection} \qquad w_{col} : [T]_c \to [w_{col}(T)]_c$$

$$C\text{-}shape(rect(T)) = \alpha \implies \mathcal{S}_\alpha = \sum_{T' \overset{c}{\sim} T} F_{Des(T')}$$

- LR SCT $\{T \in SCT : rect(T) = U_\alpha \text{ for some } \alpha\}$ is a transversal of $\overset{C}{\sim}$

Noncommutative LR rule viewpoint

- C-equivalent permutations: $\pi \overset{C}{\sim} \sigma$ if

$$\pi \overset{Q}{\sim} \sigma \quad \text{and} \quad C\text{-shape}(P(\pi)) = C\text{-shape}(P(\sigma))$$

- C-equivalent SCT: $T \overset{C}{\sim} T'$ if

$$T, T' \text{ same skew shape} \quad \text{and} \quad w_{col}(T) \overset{C}{\sim} w_{col}(T')$$

- (BLvW '10) Equivalence classes are *complete* :

$$\text{bijection} \qquad w_{col} : [T]_C \to [w_{col}(T)]_C$$

$$C\text{-}shape(rect(T)) = \alpha \implies \mathcal{S}_\alpha = \sum_{T' \overset{C}{\sim} T} F_{Des(T')}$$

- LR SCT $\{T \in SCT : rect(T) = U_\alpha \text{ for some } \alpha\}$ is a transversal of $\overset{C}{\sim}$

Noncommutative LR rule viewpoint

- C-equivalent permutations: $\pi \overset{C}{\sim} \sigma$ if

  $$\pi \overset{Q}{\sim} \sigma \quad \text{and} \quad C\text{-shape}(P(\pi)) = C\text{-shape}(P(\sigma))$$

- C-equivalent SCT: $T \overset{C}{\sim} T'$ if

  T, T' same skew shape and $w_{col}(T) \overset{C}{\sim} w_{col}(T')$

- (BLvW '10) Equivalence classes are *complete* :

  bijection $\quad w_{col} : [T]_C \to [w_{col}(T)]_C$

  $$C\text{-}shape(rect(T)) = \alpha \implies \mathcal{S}_\alpha = \sum_{T' \overset{C}{\sim} T} F_{Des(T')}$$

- LR SCT $\{T \in SCT : rect(T) = U_\alpha \text{ for some } \alpha\}$ is a transversal of $\overset{C}{\sim}$

Noncommutative LR rule viewpoint

- C-equivalent permutations: $\pi \overset{C}{\sim} \sigma$ if
$$\pi \overset{Q}{\sim} \sigma \quad \text{and} \quad C\text{-shape}(P(\pi)) = C\text{-shape}(P(\sigma))$$
- C-equivalent SCT: $T \overset{C}{\sim} T'$ if
$$T, T' \text{ same skew shape} \quad \text{and} \quad w_{col}(T) \overset{C}{\sim} w_{col}(T')$$
- (BLvW '10) Equivalence classes are *complete* :
$$\text{bijection} \qquad w_{col} : [T]_C \to [w_{col}(T)]_C$$
$$C\text{-}shape(rect(T)) = \alpha \quad \Longrightarrow \quad \mathcal{S}_\alpha = \sum_{T' \overset{C}{\sim} T} F_{Des(T')}$$
- LR SCT $\{T \in SCT : rect(T) = U_\alpha \text{ for some } \alpha\}$
is a transversal of $\overset{C}{\sim}$

Symmetric skew quasisymmetric Schur fcns

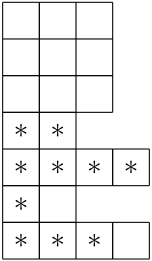
$*$	$*$	3
$*$	4	1
2		

$\rightsquigarrow$

$*$	$*$	$*$	3
$*$	$*$	4	1
$*$	2		

$$s_{(3,3,1)/(2,1)} = \mathcal{S}_{(4,4,2)/\!\!/(3,2,1)}$$

Symmetric skew quasisymmetric Schur fcns



$(3, 3, 3, 2, 4, 2, 4) /\!/ (2, 4, 1, 3)$

Conjecture

$\mathcal{S}_{\gamma/\!/\beta}$ is symmetric if and only if $\gamma /\!/ \beta$ is "uniform".

Poirier-Reutenauer tableau algebra

$$
\begin{array}{lll} 4 & 2 & 1 \\ 3 & & \end{array}
\;*\;
\begin{array}{ll} 3 & 1 \\ 2 & \end{array}
\;=\;
$$

$$
\begin{array}{lll} 7 & 5 & 4 \\ 6 & 1 & \\ 3 & & \\ 2 & & \end{array}
+
\begin{array}{llll} 7 & 5 & 4 & 1 \\ 6 & & & \\ 3 & & & \\ 2 & & & \end{array}
+
\begin{array}{lll} 7 & 5 & 4 \\ 6 & 3 & \\ 2 & 1 & \end{array}
+
\begin{array}{lll} 7 & 5 & 4 \\ 6 & 3 & 1 \\ 2 & & \end{array}
$$

$$
+
\begin{array}{llll} 7 & 5 & 4 & 1 \\ 6 & 3 & & \\ 2 & & & \end{array}
+
\begin{array}{llll} 7 & 5 & 4 & 3 \\ 6 & 1 & & \\ 2 & & & \end{array}
+
\begin{array}{lllll} 7 & 5 & 4 & 3 & 1 \\ 6 & & & & \\ 2 & & & & \end{array}
$$

$$
+
\begin{array}{llll} 7 & 5 & 4 & 3 \\ 6 & 2 & 1 & \end{array}
+
\begin{array}{lllll} 7 & 5 & 4 & 3 & 1 \\ 6 & 2 & & & \end{array}
$$

Poirier-Reutenauer tableau algebra

$$\begin{array}{lll} 4 & 2 & 1 \\ 3 & & \end{array} \quad * \quad \begin{array}{ll} 3 & 1 \\ 2 & \end{array} \quad =$$

$$\begin{array}{lll} 7 & 5 & 4 \\ 6 & 1 & \\ 3 & & \\ 2 & & \end{array} + \begin{array}{llll} 7 & 5 & 4 & 1 \\ 6 & & & \\ 3 & & & \\ 2 & & & \end{array} + \begin{array}{lll} 7 & 5 & 4 \\ 6 & 3 & \\ 2 & 1 & \end{array} + \begin{array}{lll} 7 & 5 & 4 \\ 6 & 3 & 1 \\ 2 & & \end{array}$$

$$+ \begin{array}{llll} 7 & 5 & 4 & 1 \\ 6 & 3 & & \\ 2 & & & \end{array} + \begin{array}{llll} 7 & 5 & 4 & 3 \\ 6 & 1 & & \\ 2 & & & \end{array} + \begin{array}{lllll} 7 & 5 & 4 & 3 & 1 \\ 6 & & & & \\ 2 & & & & \end{array}$$

$$+ \begin{array}{llll} 7 & 5 & 4 & 3 \\ 6 & 2 & 1 & \end{array} + \begin{array}{lllll} 7 & 5 & 4 & 3 & 1 \\ 6 & 2 & & & \end{array}$$

Poirier-Reutenauer tableau algebra

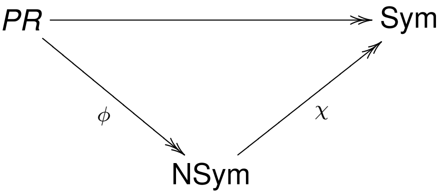
$$
\begin{array}{lll} 3 & 2 & 1 \\ 4 & & \end{array} \; * \; \begin{array}{ll} 2 & 1 \\ 3 & \end{array} \; =
$$

$$
\begin{array}{lll} 2 & 1 & \\ 3 & & \\ 6 & 5 & 4 \\ 7 & & \end{array} + \begin{array}{llll} 2 & & & \\ 3 & & & \\ 6 & 5 & 4 & 1 \\ 7 & & & \end{array} + \begin{array}{lll} 2 & 1 & \\ 6 & 5 & 4 \\ 7 & 3 & \end{array} + \begin{array}{lll} 2 & & \\ 6 & 5 & 4 \\ 7 & 3 & 1 \end{array}
$$

$$
+ \begin{array}{llll} 2 & & & \\ 6 & 5 & 4 & 1 \\ 7 & 3 & & \end{array} + \begin{array}{llll} 2 & 1 & & \\ 6 & 5 & 4 & 3 \\ 7 & & & \end{array} + \begin{array}{lllll} 2 & & & & \\ 6 & 5 & 4 & 3 & 1 \\ 7 & & & & \end{array}
$$

$$
+ \begin{array}{llll} 6 & 5 & 4 & 3 \\ 7 & 2 & 1 & \end{array} + \begin{array}{lllll} 6 & 5 & 4 & 3 & 1 \\ 7 & 2 & & & \end{array}
$$

Poirier-Reutenauer tableau algebra



$$T \overset{\phi}{\longmapsto} \mathbf{s}_\alpha \overset{\chi}{\longmapsto} s_{\widetilde{\alpha}}$$

where composition shape of $T = \alpha$

$$\phi(U * V) \; = \; \phi(V)\phi(U)$$

(Note: ϕ is not a Hopf morphism)

Colored / wreath product symmetric functions

- set of *colors* (*palette?*) $B = \{1, 2, \ldots, N\}$
- alphabet $\mathcal{A} = \mathbb{Z}_+ \times B$, lex ordered; $\mathcal{A}^b = \mathbb{Z}_+ \times \{b\}$
- $X^b = \{x_{1,b}, x_{2,b}, x_{3,b}, \ldots\}$, $X = \bigcup_{b \in B} X^b$
- $\bar{k} = (k, 2)$, $\bar{\bar{k}} = (k, 3)$, $\bar{x}_k = x_{k,2}$, $\bar{\bar{x}}_k = x_{k,3}$

$$Sym^{(B)} := Sym^{\otimes N} \cong Sym(X^1) \cdots Sym(X^N)$$

- colored partitions $\lambda = (\lambda^1, \ldots, \lambda^N)$ (multiset in $\mathcal{A}$)
- (Specht) colored / wreath product Schur functions

$$s_\lambda = s_{\lambda^1}(X^1) \cdots s_{\lambda^N}(X^N)$$

Colored / wreath product symmetric functions

- set of *colors* (*palette?*) $B = \{1, 2, \ldots, N\}$
- alphabet $\mathcal{A} = \mathbb{Z}_+ \times B$, lex ordered; $\mathcal{A}^b = \mathbb{Z}_+ \times \{b\}$
- $X^b = \{x_{1,b}, x_{2,b}, x_{3,b}, \ldots\}$, $X = \bigcup_{b \in B} X^b$
- $\bar{k} = (k, 2)$, $\bar{\bar{k}} = (k, 3)$, $\bar{x}_k = x_{k,2}$, $\bar{\bar{x}}_k = x_{k,3}$

$$Sym^{(B)} := Sym^{\otimes N} \cong Sym(X^1) \cdots Sym(X^N)$$

- colored partitions $\lambda = (\lambda^1, \ldots, \lambda^N)$ (multiset in $\mathcal{A}$)
- (Specht) colored / wreath product Schur functions

$$s_\lambda = s_{\lambda^1}(X^1) \cdots s_{\lambda^N}(X^N)$$

Colored / wreath product symmetric functions

- set of *colors* (*palette?*) $B = \{1, 2, \ldots, N\}$
- alphabet $\mathcal{A} = \mathbb{Z}_+ \times B$, lex ordered; $\mathcal{A}^b = \mathbb{Z}_+ \times \{b\}$
- $X^b = \{x_{1,b}, x_{2,b}, x_{3,b}, \ldots\}$, $X = \bigcup_{b \in B} X^b$
- $\bar{k} = (k, 2)$, $\bar{\bar{k}} = (k, 3)$, $\bar{x}_k = x_{k,2}$, $\bar{\bar{x}}_k = x_{k,3}$

$$Sym^{(B)} := Sym^{\otimes N} \cong Sym(X^1) \cdots Sym(X^N)$$

- colored partitions $\lambda = (\lambda^1, \ldots, \lambda^N)$ (multiset in $\mathcal{A}$)
- (Specht) colored / wreath product Schur functions

$$s_\lambda = s_{\lambda^1}(X^1) \cdots s_{\lambda^N}(X^N)$$

Colored / wreath product symmetric functions

- set of *colors* (*palette?*) $B = \{1, 2, \ldots, N\}$
- alphabet $\mathcal{A} = \mathbb{Z}_+ \times B$, lex ordered; $\mathcal{A}^b = \mathbb{Z}_+ \times \{b\}$
- $X^b = \{x_{1,b}, x_{2,b}, x_{3,b}, \ldots\}$, $X = \bigcup_{b \in B} X^b$
- $\bar{k} = (k, 2)$, $\bar{\bar{k}} = (k, 3)$, $\bar{x}_k = x_{k,2}$, $\bar{\bar{x}}_k = x_{k,3}$

$$Sym^{(B)} := Sym^{\otimes N} \cong Sym(X^1) \cdots Sym(X^N)$$

- colored partitions $\lambda = (\lambda^1, \ldots, \lambda^N)$ (multiset in $\mathcal{A}$)
- (Specht) colored / wreath product Schur functions

$$s_\lambda = s_{\lambda^1}(X^1) \cdots s_{\lambda^N}(X^N)$$

Colored / wreath product symmetric functions

- set of *colors* (*palette?*) $B = \{1, 2, \ldots, N\}$
- alphabet $\mathcal{A} = \mathbb{Z}_+ \times B$, lex ordered; $\mathcal{A}^b = \mathbb{Z}_+ \times \{b\}$
- $X^b = \{x_{1,b}, x_{2,b}, x_{3,b}, \ldots\}$, $X = \bigcup_{b \in B} X^b$
- $\bar{k} = (k, 2)$, $\bar{\bar{k}} = (k, 3)$, $\bar{x}_k = x_{k,2}$, $\bar{\bar{x}}_k = x_{k,3}$

$$Sym^{(B)} := Sym^{\otimes N} \cong Sym(X^1) \cdots Sym(X^N)$$

- colored partitions $\lambda = (\lambda^1, \ldots, \lambda^N)$ (multiset in $\mathcal{A}$)
- (Specht) colored / wreath product Schur functions

$$s_\lambda = s_{\lambda^1}(X^1) \cdots s_{\lambda^N}(X^N)$$

Colored quasisymmetric functions

- cf. Poirier, Hsiao, Petersen, Baumann, Hohlweg, et al.
- colored composition = finite sequence in $\mathcal{A}$.

$$\alpha = \big((a_1, b_1), \ldots, (a_k, b_k)\big)$$

- colored monomial quasisymmetric functions:

$$M_\alpha := \sum_{(i_1,b_1)<\cdots<(i_k,b_k)} x_{i_1,b_1}^{a_1} \cdots x_{i_k,b_k}^{a_k}$$

- E.g. $\alpha = \bar{1}21, \quad M_\alpha = \bar{x}_1 x_2^2 x_3 + \bar{x}_1 x_2^2 x_4 + \bar{x}_2 x_3^2 x_4 + \cdots$
- $QSym^{(B)} = \text{span}\{M_\alpha\}$

Mantaci-Reutenauer algebra

- $NSym^{(B)} =$ graded Hopf dual of $QSym^{(B)}$
- Isomorphic to Mantaci-Reutenauer algebra
- Colored noncommutative symmetric functions (?)
- Freely generated by $\{\mathbf{h}_{(n,b)}\}_{(n,b)\in\mathcal{A}}$; $\deg h_{(n,b)} = n$

Colored analogs (known)

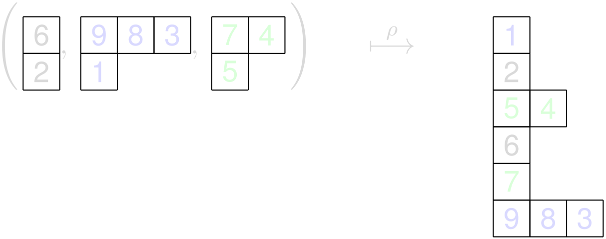
- colored words, permutations, standardizations, descents
- refinement of colored compositions
- colored Young tableaux (CSSRT, CSRT) $T = (T^1, \ldots, T^N)$, descents
- Knuth and dual Knuth equivalence
- RSK correspondence

$$\bar{1}\bar{6}2\bar{3}\bar{4}5 \quad \mapsto \quad \left[P = \left(\begin{array}{|c|}\hline 5 \\ \hline 2 \\ \hline \end{array}, \begin{array}{|c|c|}\hline 6 & 4 \\ \hline 3 \\ \cline{1-1} 1 \\ \cline{1-1} \end{array} \right), Q = \left(\begin{array}{|c|}\hline 4 \\ \hline 1 \\ \hline \end{array}, \begin{array}{|c|c|}\hline 6 & 3 \\ \hline 5 \\ \cline{1-1} 2 \\ \cline{1-1} \end{array} \right) \right]$$

- $$s_\lambda = \sum_{T \in CSSRT(\lambda)} \mathbf{x}^T = \sum_{T \in CSRT(\lambda)} F_{Des(T)}$$

The Change

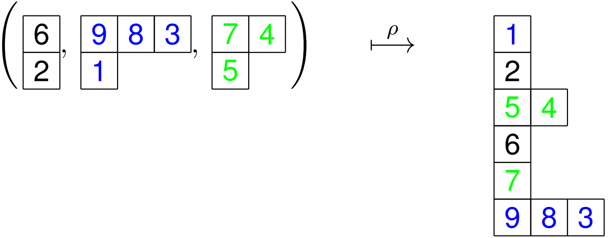
- Poset of colored compositions: cover by
  - prepending $(1, b)$ for any $b \in B$
  - incrementing first (leftmost) $(k, b) \mapsto (k+1, b)$
- colored composition tableaux (CSSCT, CSRT)



$\lambda = 11\bar{3}\bar{1}\bar{\bar{2}}\bar{\bar{1}}$ $\alpha = \bar{1}1\bar{\bar{2}}1\bar{\bar{1}}\bar{3}$

The Change

- Poset of colored compositions: cover by
  - prepending $(1, b)$ for any $b \in B$
  - incrementing first (leftmost) $(k, b) \mapsto (k+1, b)$
- colored composition tableaux (CSSCT, CSRT)



$\lambda = 11\bar{3}\bar{1}\bar{\bar{2}}\bar{\bar{1}}$

$\alpha = \bar{1}1\bar{\bar{2}}1\bar{\bar{1}}\bar{3}$

The Hope

$$\mathcal{S}_{\gamma/\!\!/\beta} = \sum_{T \in CSCRT(\gamma/\!\!/\beta)} \mathbf{x}^T = \sum_{T \in CSCT(\gamma/\!\!/\beta)} F_{Des(T)}$$

$$\Delta \mathcal{S}_\gamma = \sum_\beta \mathcal{S}_{\gamma/\!\!/\beta} \otimes \mathcal{S}_\beta$$

$$s_\lambda = \sum_{\widetilde{\alpha}=\lambda} \mathcal{S}_\alpha \qquad \Longrightarrow \qquad \chi(\mathbf{s}_\alpha) = s_{\widetilde{\alpha}}$$

Conjecture

In the expansion

$$\mathcal{S}_{\gamma/\!\!/\beta} = \sum_\alpha C^\gamma_{\alpha,\beta} \mathcal{S}_\alpha,$$

$C^\gamma_{\alpha,\beta}$ *is the number of* $T \in SCT(\gamma /\!\!/ \beta)$ *such that* $rect(T) = U_\alpha$.

Noncommutative character theory

$$\begin{array}{ccc} \Sigma & \hookrightarrow & \mathbb{C}W \\ {\scriptstyle \theta}\downarrow & & \\ Cl(W) & & \end{array}$$

Noncommutative character theory

$$\begin{array}{ccc} \Sigma & \hookrightarrow & \mathbb{C}W \\ \downarrow{\scriptstyle \theta} & & \\ Cl(W) & & \end{array}$$

Noncommutative characters are
pre-images of characters under θ.

Noncommutative character theory

$$
\begin{array}{ccc}
\Sigma & \hookrightarrow & \mathbb{C}S_n \\
\theta \big\downarrow & & \\
Cl(S_n) & &
\end{array}
$$

Noncommutative character theory

$$
\begin{array}{ccc}
\bigoplus_{n\geq 0} \Sigma \cong NSym & \hookrightarrow & \bigoplus_{n\geq 0} \mathbb{C}S_n \\
\Big\downarrow \chi & & \\
\bigoplus_{n\geq 0} Cl(S_n) \cong Sym & &
\end{array}
$$

Noncommutative character theory

$$\begin{array}{ccc} \bigoplus_{n\geq 0} \Sigma \cong NSym & \hookrightarrow & \bigoplus_{n\geq 0} \mathbb{C}S_n \\ \downarrow \chi & & \\ \bigoplus_{n\geq 0} Cl(S_n) \cong Sym & & \end{array}$$

The $\{\mathbf{s}_\alpha\}$ are irreducible noncommutative characters.

Noncommutative character theory

$$
\begin{array}{ccc}
\bigoplus_{n\geq 0} \Sigma \;\cong\; \mathit{NSym} & \hookrightarrow & \bigoplus_{n\geq 0} \mathbb{C}S_n \\
\downarrow \chi & & \downarrow \\
\bigoplus_{n\geq 0} Cl(S_n) \cong \mathit{Sym} & \hookrightarrow & \mathit{QSym}
\end{array}
$$

The $\{\mathbf{s}_\alpha\}$ are irreducible noncommutative characters.

Noncommutative character theory

$$\begin{array}{ccc} \Sigma & \hookrightarrow & \mathbb{C}G \wr S_n \\ \downarrow & & \\ Cl(G \wr S_n) & & \end{array}$$

Noncommutative character theory

$$
\begin{array}{ccc}
\bigoplus_{n\geq 0} \Sigma \;\cong\; NSym^{(G)} & \hookrightarrow & \bigoplus_{n\geq 0} \mathbb{C}G \wr S_n \\
\Big\downarrow \chi & & \Big\downarrow \\
\bigoplus_{n\geq 0} Cl(G \wr S_n) \cong Sym^{(G)} & \hookrightarrow & QSym^{(G)}
\end{array}
$$

The colored $\{\mathbf{s}_\alpha\}$ are irreducible noncommutative characters.

Further directions

- Other representation theoretical interpretations?
- Geometric interpretations?
- Properties of skew QS Schurs that are symmetric?
- Extension of Sami's machinery for QS positivity?
- Analogous bases for other algebras?

For Further Reading I

J. Haglund, K. Luoto, S. Mason, S. van Willigenburg
Quasisymmetric Schur functions
J. of Comb. Theory, Series A, *to appear*

C. Bessenrodt, K. Luoto, S. van Willigenburg
Skew quasisymmetric Schur functions and noncommutative Schur functions
Adv. Math, *accepted*

D. Blessenohl, M. Schocker
Noncommutative character theory of the symmetric groups
Imperial College Press, (2005)

P. Baumann, C. Hohlweg
A Solomon descent theory for the wreath products $G \wr S_n$
TAMS, 360(3):1475-1538(2008)