A Parking Function Bijection Suggested by the Haglund-Morse-Zabrocki Conjecture

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University of California- San Diego

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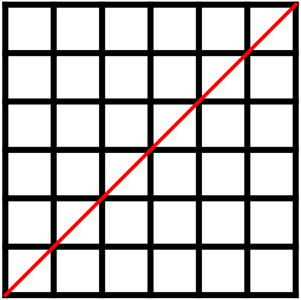
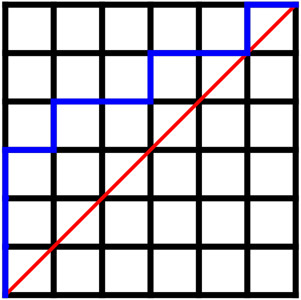
Background

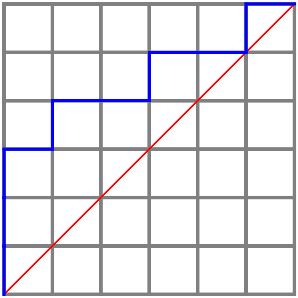
Dyck Paths



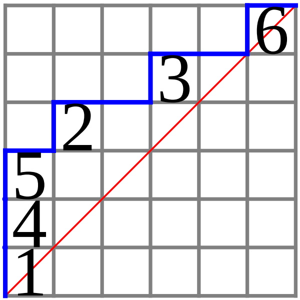
Dyck Paths



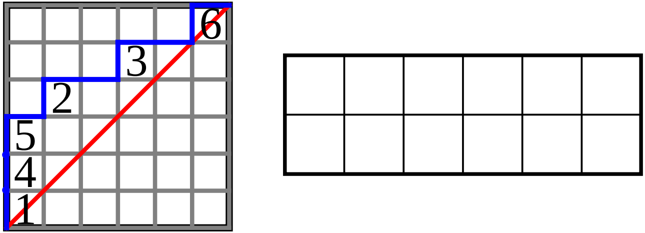
Parking Functions



Parking Functions



Parking Functions



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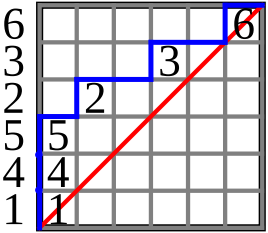
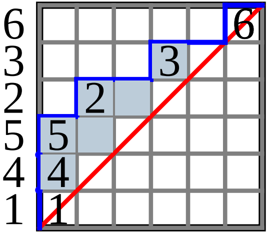


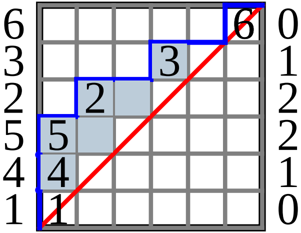
1	4	5	2	3	6

Parking Functions



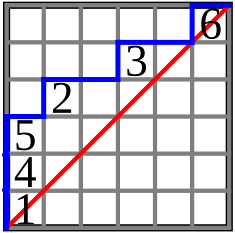
1	4	5	2	3	6

Parking Functions



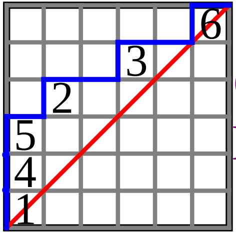
1	4	5	2	3	6
0	1	2	2	1	0

Parking Functions



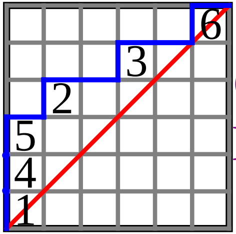
1	4	5	2	3	6
0	1	2	2	1	0

Parking Functions



C=	1	4	5	2	3	6
D=	0	1	2	2	1	0

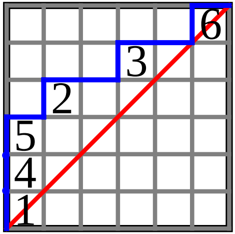
Parking Functions



C=	1	4	5	2	3	6
D=	0	1	2	2	1	0

- (Dyck Path Condition) $D_1 = 0$ and $0 \leq D_i \leq D_{i-1} + 1$.
- (Increasing Column Condition) If $D_i = D_{i-1} + 1$, $C_{i-1} < C_i$.

Parking Functions

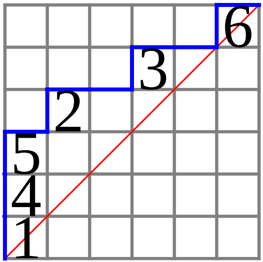


1	4	5	2	3	6
0	1	2	2	1	0

Parking Function Statistics

Definition

The *area* of a parking function is $\sum D_i$.



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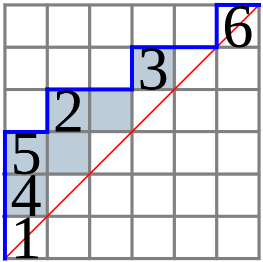


Figure: $\text{area}(PF) = 6$

Parking Function Statistics

Primary Dinv

When $s < b$,

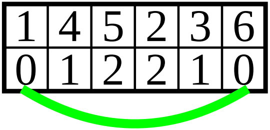
s	...	b
d	...	d

Parking Function Statistics

Primary Dinv

When $s < b$,

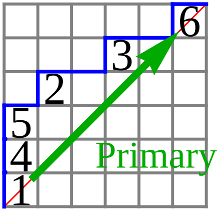
s	...	b
d	...	d

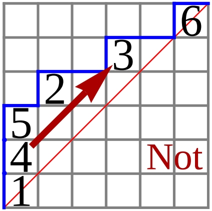


Parking Function Statistics

Primary Dinv

s	...	b
d	...	d





Parking Function Statistics

Secondary Dinv

When $s < b$,

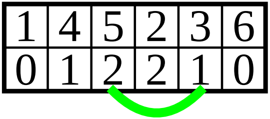
b	...	s
d+1	...	d

Parking Function Statistics

Secondary Dinv

When $s < b$,

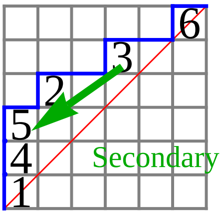
b	...	s
d+1	...	d



Parking Function Statistics

Secondary Dinv

b	...	s
d+1	...	d

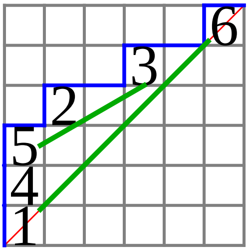


Parking Function Statistics

Definition

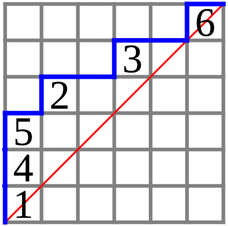
The *dinv* of a parking function is the number of primary and secondary diagonal inversions it contains.

$$\text{dinv}(PF) = 2$$



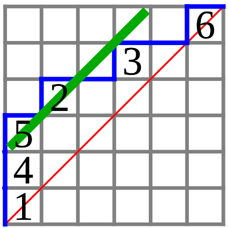
Parking Function Statistics

Word



Parking Function Statistics

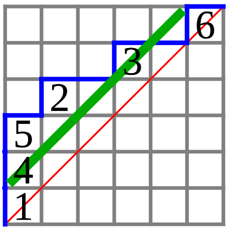
Word



[2,5]

Parking Function Statistics

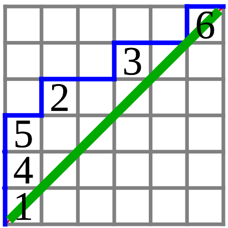
Word



[2,5,3,4]

Parking Function Statistics

Word



[2,5,3,4,6,1]

Parking Function Statistics

I-descents

Definition

The *i-descent set* of a permutation P, is

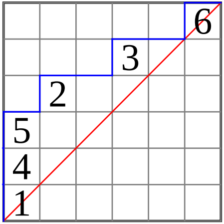
$$\text{ides}(P) = \{i : i \text{ occurs after } i+1 \text{ in } P\}.$$

Definition

Let $\text{ides}(PF) = \text{ides}(\text{word}(PF))$.

Parking Function Statistics

I-descents



$$\text{ides}(PF) = \text{ides}([2, 5, 3, 4, 6, 1]) = \{1, 4\}$$

Parking Function Statistics

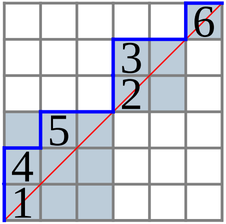
Definition

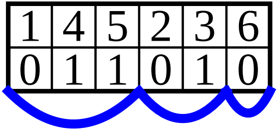
The weight of a parking function is defined as:

$$\mathrm{wt}(PF) = t^{\mathrm{area}(PF)} q^{\mathrm{dinv}(PF)} Q_{\mathrm{ides}(PF)}.$$

Parking Function Statistics

Composition





1	4	5	2	3	6
0	1	1	0	1	0

Figure: comp$(PF) = [3, 2, 1]$

Conjectures

Conjecture (Haglund, Haiman, Loehr, Remmel, Ulyanov.)

The "Shuffle Conjecture" states that

$$\nabla e_n = \sum_{PF \in PF_n} t^{area(PF)} q^{dinv(PF)} Q_{ides(PF)},$$

Conjecture (Haglund, Haiman, Loehr, Remmel, Ulyanov.)

The "Shuffle Conjecture" states that

$$\nabla e_n = \sum_{PF \in PF_n} t^{area(PF)} q^{dinv(PF)} Q_{ides(PF)},$$

Conjecture (Haglund, Morse, Zabrocki)

$$\nabla \mathcal{C}_{p_1} \mathcal{C}_{p_2} \dots \mathcal{C}_{p_k} 1 = \sum_{\text{comp}(PF)=[p_1,\cdots,p_k]} t^{area(PF)} q^{dinv(PF)} Q_{ides(PF)}$$

A Commutativity Relation

When $k < n - k$,

$$q(\mathcal{C}_k\mathcal{C}_{n-k} + \mathcal{C}_{n-k-1}\mathcal{C}_{k+1}) = \mathcal{C}_{n-k}\mathcal{C}_k + \mathcal{C}_{k+1}\mathcal{C}_{n-k-1}$$

Definition

$$\mathcal{F}_p = \{PF : \text{comp}(PF) = p\}$$

Definition

$$\mathcal{A}_p = \sum_{PF \in \mathcal{F}_p} t^{\text{area}(PF)} q^{\text{dinv}(PF)} Q_{\text{ides}(PF)}.$$

Conjecture

For $k < n - k$,

$$q(\mathcal{A}_{\{k,n-k\}} + \mathcal{A}_{\{n-k-1,k+1\}}) = \mathcal{A}_{\{n-k,k\}} + \mathcal{A}_{\{k+1,n-k-1\}}$$

Conjecture

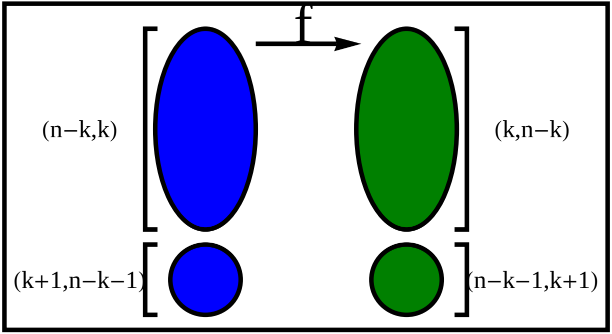
Then there exists a bijective map

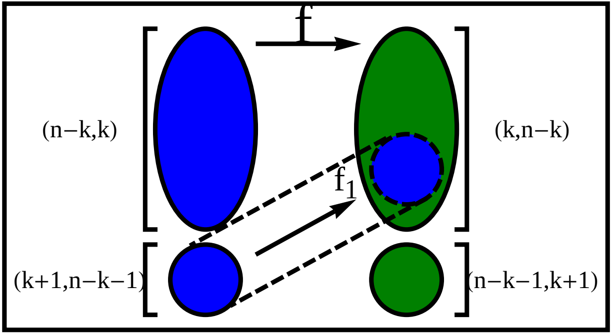
$$f : \mathcal{F}_{\{k,n-k\}} \cup \mathcal{F}_{\{n-k-1,k+1\}} \Leftrightarrow \mathcal{F}_{\{n-k-1,k+1\}} \cup \mathcal{F}_{\{n-k,k\}}$$

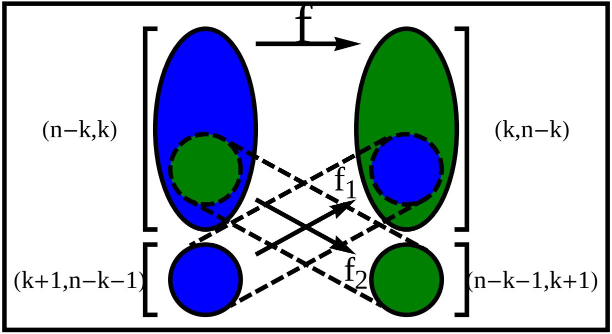
such that $q\,\mathrm{wt}(f(PF)) = \mathrm{wt}(PF)$.

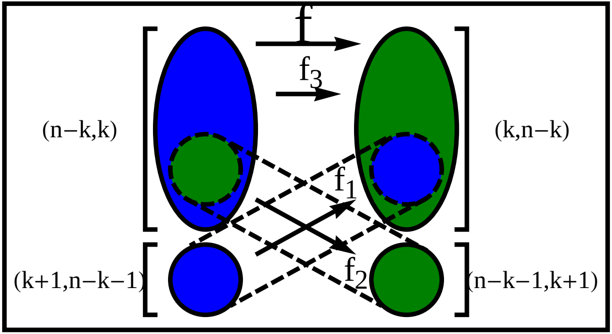
We'd like a map f such that:

- $\text{dinv}(f(PF)) = \text{dinv}(PF) - 1$
- $\text{ides}(f(PF)) = \text{ides}(PF)$
- $\text{area}(f(PF)) = \text{area}(PF)$
- f is "local"

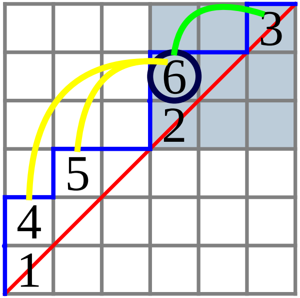

f
(n−k,k)
(k,n−k)
(k+1,n−k−1)
(n−k−1,k+1)


f
(n−k,k)
(k,n−k)
f1
(k+1,n−k−1)
(n−k−1,k+1)

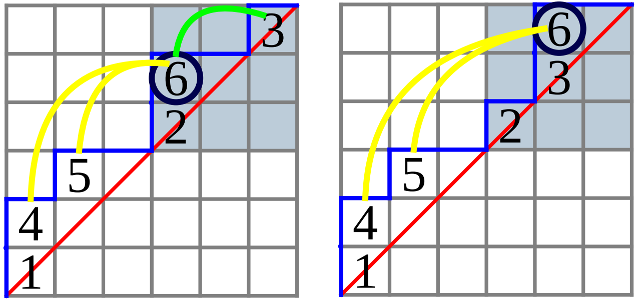

f
(n−k,k)
(k,n−k)
f1
f2
(k+1,n−k−1)
(n−k−1,k+1)

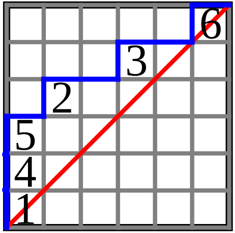

f
f_3
(n−k,k)
(k,n−k)
f_1
f_2
(k+1,n−k−1)
(n−k−1,k+1)

$$\text{diag}(PF) = \text{diag}(f(PF))$$



$$\text{diag}(PF) = \text{diag}(f(PF))$$





1	4	5	2	3	6
0	1	2	2	1	0

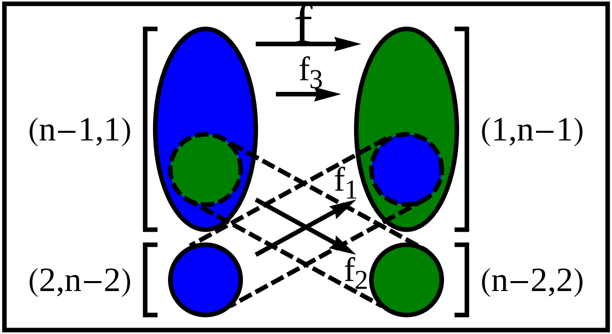
We'd like a map f such that:

(Now considering f which rearranges elements *within* a diagonal:)

- $\text{dinv}(f(PF)) = \text{dinv}(PF) - 1$
- $\text{ides}(f(PF)) = \text{ides}(PF)$
  - iff $\begin{bmatrix} c \\ d \end{bmatrix}$ does not move past $\begin{bmatrix} c+1 \\ d \end{bmatrix}$
- $\text{area}(f(PF)) = \text{area}(PF)$
  - True for all domino permutations
- f is "local"
  - True for all domino permutations

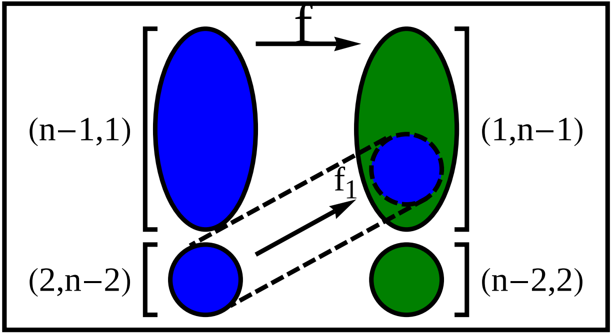
Theorem

$$q(\mathcal{A}_{\{1,n-1\}} + \mathcal{A}_{\{n-2,2\}}) = \mathcal{A}_{\{n-1,1\}} + \mathcal{A}_{\{2,n-2\}}$$

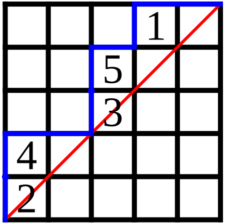


When $k = 1$

A First Map

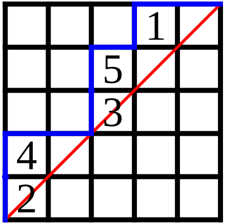

f
(n−1,1)
(1,n−1)
f_1
(2,n−2)
(n−2,2)

$$f_1 : \mathcal{A}_{\{2,n-2\}} \hookrightarrow \mathcal{A}_{\{1,n-1\}}$$



2	4	3	5	1
0	1	0	1	1

$$f_1 : \mathcal{A}_{\{2,n-2\}} \hookrightarrow \mathcal{A}_{\{1,n-1\}}$$



2	4	3	5	1
0	1	0	1	1

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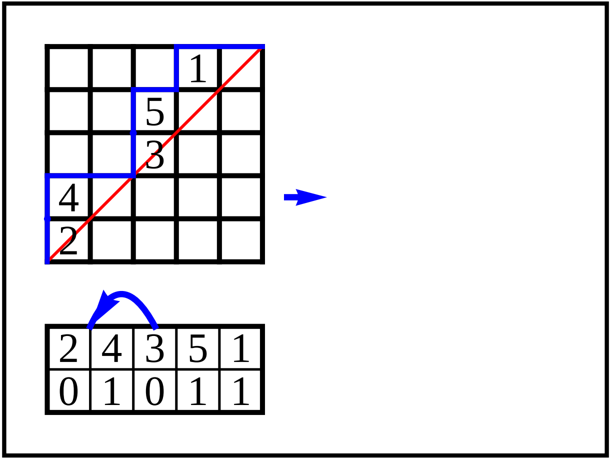
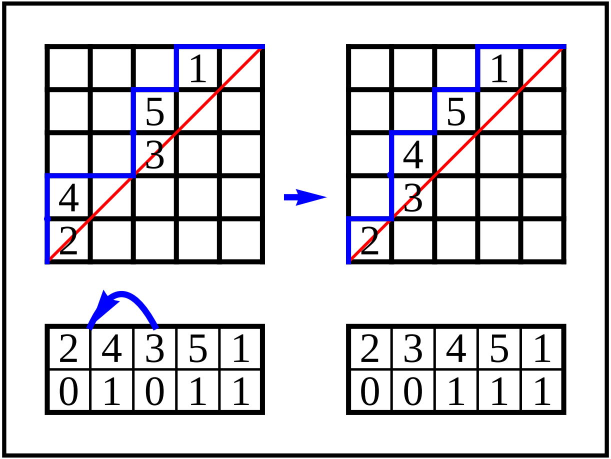


$$f_1 : \mathcal{A}_{\{2,n-2\}} \hookrightarrow \mathcal{A}_{\{1,n-1\}}$$



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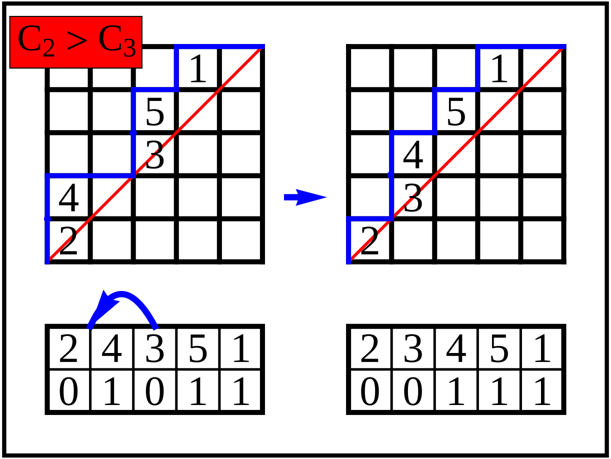
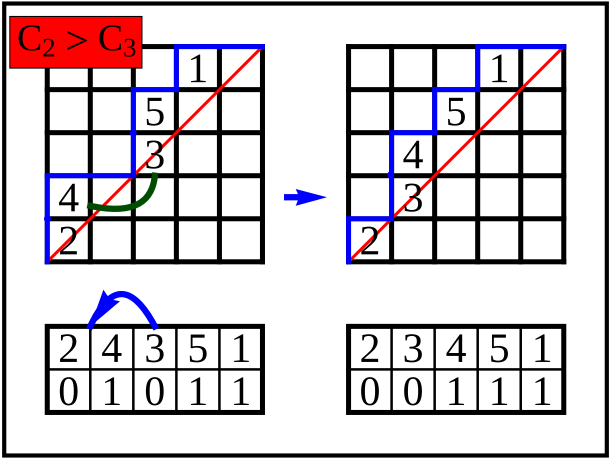
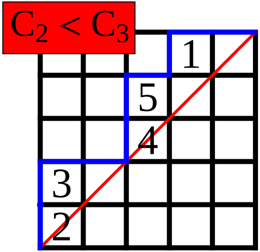


$$f_1 : \mathcal{A}_{\{2,n-2\}} \hookrightarrow \mathcal{A}_{\{1,n-1\}}$$

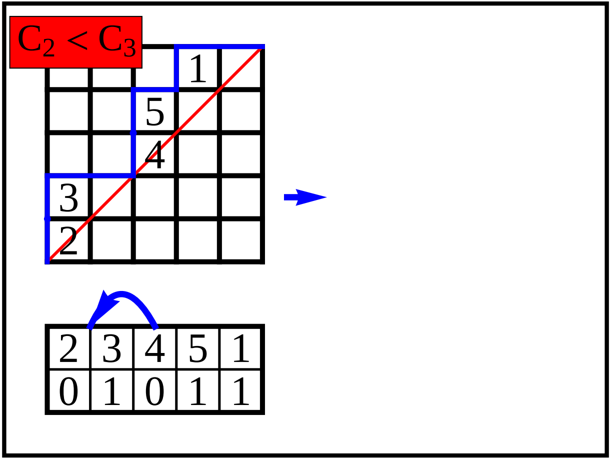


$$f_1 : \mathcal{A}_{\{2,n-2\}} \hookrightarrow \mathcal{A}_{\{1,n-1\}}$$

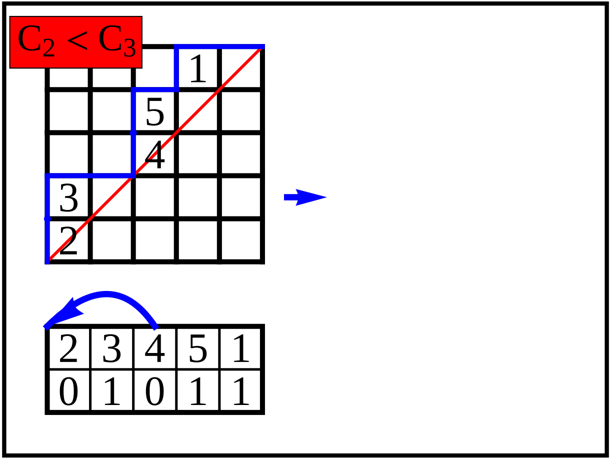


2	3	4	5	1
0	1	0	1	1

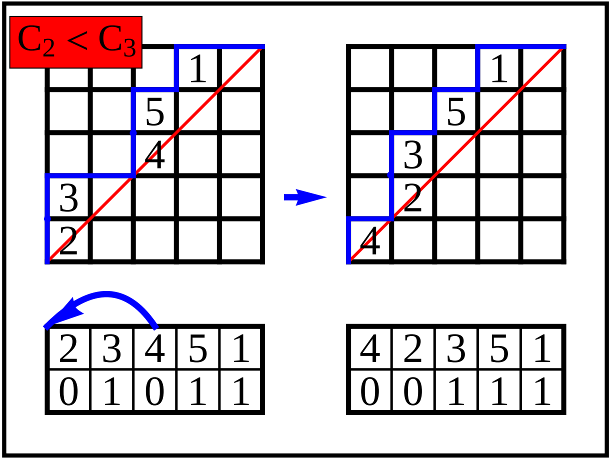
$$f_1 : \mathcal{A}_{\{2,n-2\}} \hookrightarrow \mathcal{A}_{\{1,n-1\}}$$



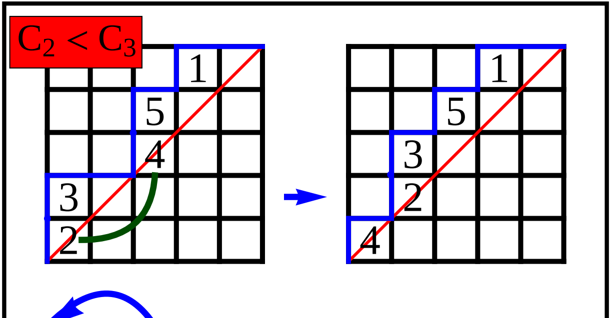
$$f_1 : \mathcal{A}_{\{2,n-2\}} \hookrightarrow \mathcal{A}_{\{1,n-1\}}$$



$$f_1 : \mathcal{A}_{\{2,n-2\}} \hookrightarrow \mathcal{A}_{\{1,n-1\}}$$



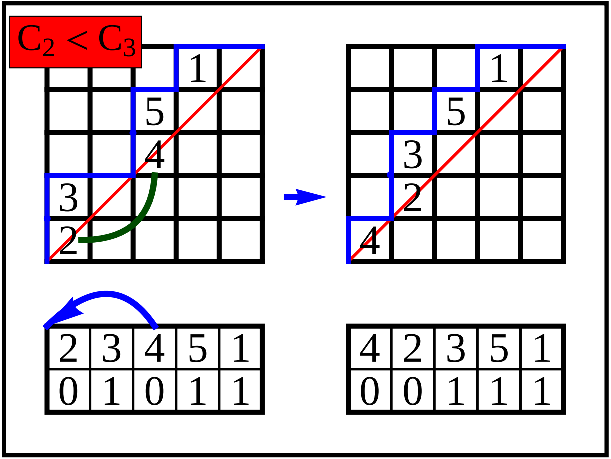
$$f_1 : \mathcal{A}_{\{2,n-2\}} \hookrightarrow \mathcal{A}_{\{1,n-1\}}$$



2	3	4	5	1
0	1	0	1	1

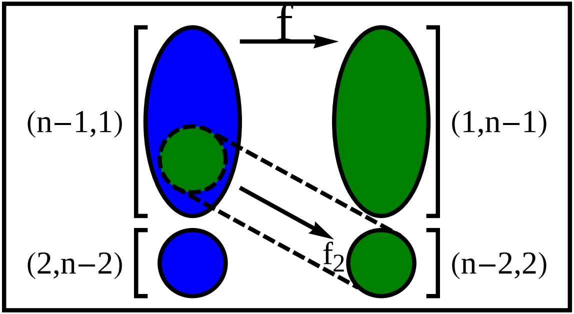
4	2	3	5	1
0	0	1	1	1

$$f_1 : \mathcal{A}_{\{2,n-2\}} \hookrightarrow \mathcal{A}_{\{1,n-1\}}$$

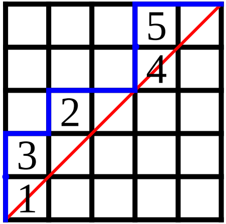


When $k = 1$

An Easy Second Map

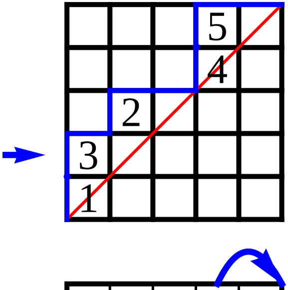

f
(n−1,1)
(1,n−1)
f2
(2,n−2)
(n−2,2)

$$f_2 : S \twoheadrightarrow \mathcal{A}_{\{n-2,2\}}$$



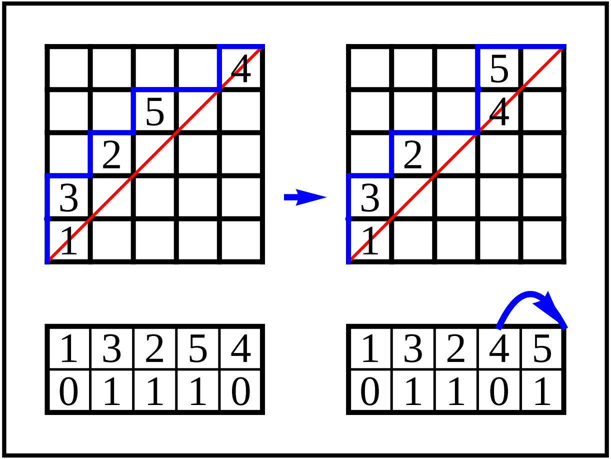
1	3	2	4	5
0	1	1	0	1

$$f_2 : S \twoheadrightarrow \mathcal{A}_{\{n-2,2\}}$$

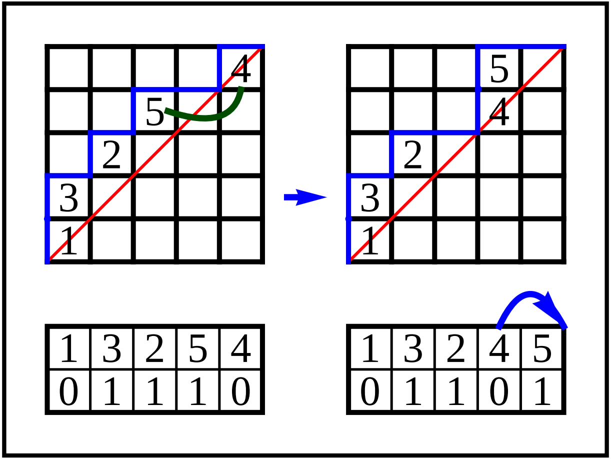


1	3	2	4	5
0	1	1	0	1

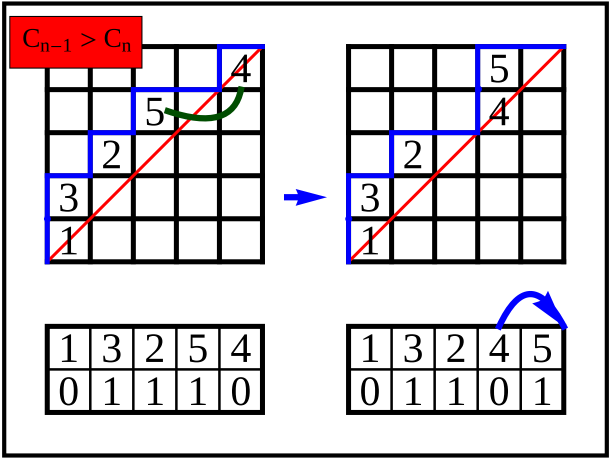
$$f_2 : S \twoheadrightarrow \mathcal{A}_{\{n-2,2\}}$$



$$f_2 : S \twoheadrightarrow \mathcal{A}_{\{n-2,2\}}$$



$$f_2 : S \twoheadrightarrow \mathcal{A}_{\{n-2,2\}}$$

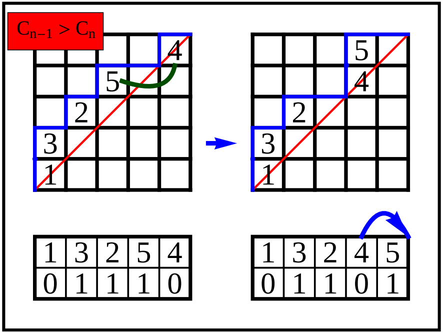


Notation

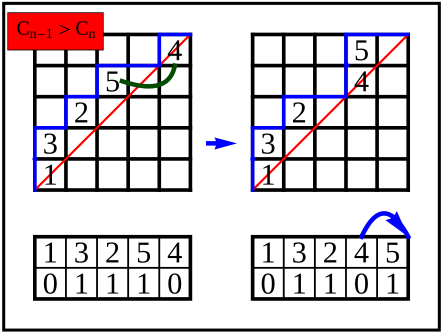
Let C_L be the last car in the main diagonal.

Notation

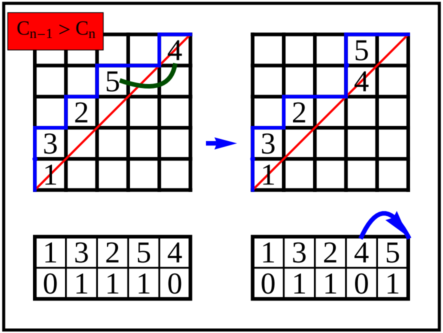
Say a car is “big” (“small”) if it is bigger (smaller) than C_L.


C_{n−1} > C_n
1 3 2 5 4
0 1 1 1 0
1 3 2 4 5
0 1 1 0 1

The last car before C_L is big and in the first diagonal.



The last car before C_L is big and in the first diagonal.

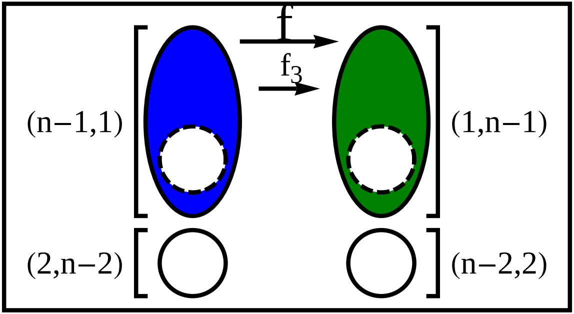


Recursive Condition

The last car before C_L is either small or not in the first diagonal.

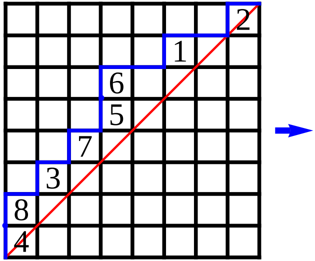
When $k = 1$

The Remaining Map


f
f3
(n−1,1)
(1,n−1)
(2,n−2)
(n−2,2)

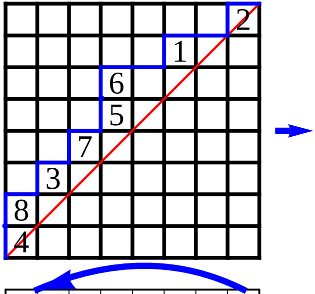
$$f_3 : \mathcal{A}_{\{n-1,1\}} \to \mathcal{A}_{\{1,n-1\}}$$

$$f_3 : \mathcal{A}_{\{n-1,1\}} \to \mathcal{A}_{\{1,n-1\}}$$



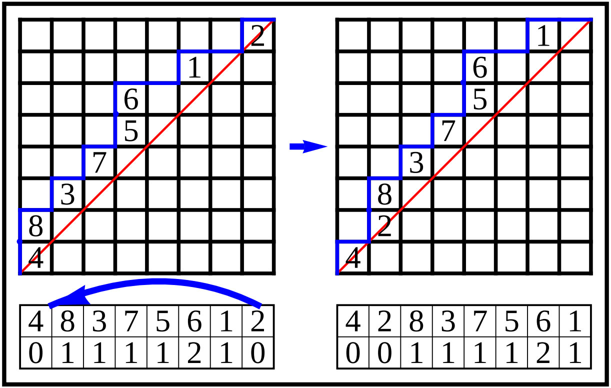
4	8	3	7	5	6	1	2
0	1	1	1	1	2	1	0

$$f_3 : \mathcal{A}_{\{n-1,1\}} \to \mathcal{A}_{\{1,n-1\}}$$

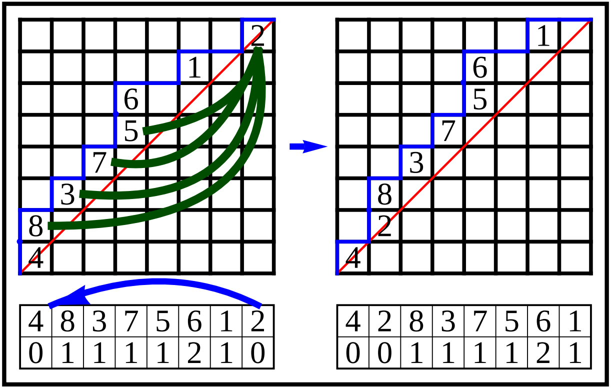


4	8	3	7	5	6	1	2
0	1	1	1	1	2	1	0

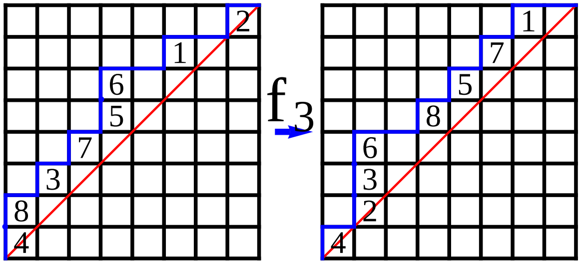
$$f_3 : \mathcal{A}_{\{n-1,1\}} \to \mathcal{A}_{\{1,n-1\}}$$



$$f_3 : \mathcal{A}_{\{n-1,1\}} \rightarrow \mathcal{A}_{\{1,n-1\}}$$



$$f_3 : \mathcal{A}_{\{n-1,1\}} \to \mathcal{A}_{\{1,n-1\}}$$

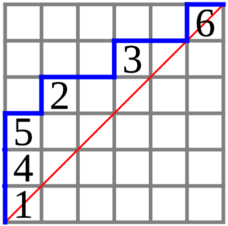


4	8	3	7	5	6	1	2
0	1	1	1	1	2	1	0

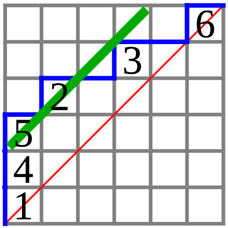
4	2	3	6	8	5	7	1
0	0	1	2	1	1	1	1

Diagonal Words

Diagonal Word

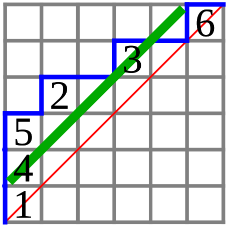


Diagonal Word



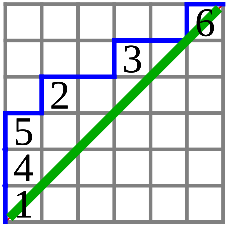
[2,5]

Diagonal Word

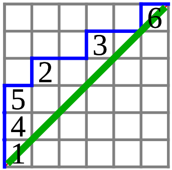


[2,5,3,4]

Diagonal Word

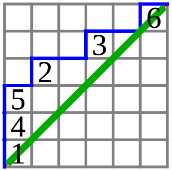


[2,5,3,4,1,6]



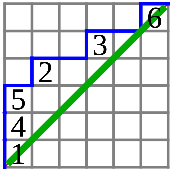
[2,5,3,4,1,6]

2 5 3 4 1 6



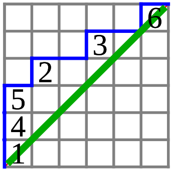
[2,5,3,4,1,6]

2 5 3 4 1 6



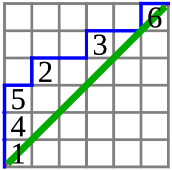
[2,5,3,4,1,6]

2 5	3 4	1 6



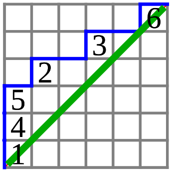
[2,5,3,4,1,6]

2	5	3	4	1	6
2	2	1	1	0	0



[2,5,3,4,1,6]

2	5	3	4	1	6
2	2	1	1	0	0



[2,5,3,4,1,6]

2	5	3	4	1	6
2	2	1	1	0	0

Theorem (Haglund and Loehr)

$$\sum_{\text{diag}(PF)=\tau} t^{\text{area}(PF)} q^{\text{dinv}(PF)} = t^{\text{maj}(\tau)} \prod_{i=1}^{n} [w_i^{\tau}]_q$$

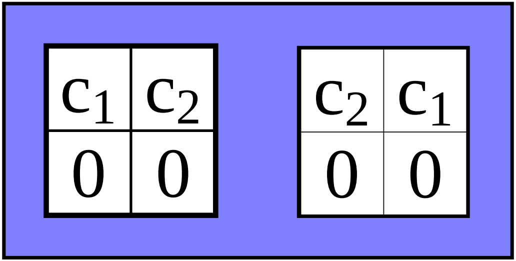
Diagonal Words

A Recursive Construction

2	4	1	3
1	1	0	0

Place the cars in a parking function recursively, working from the end of the diagonal word forward.

- Decide the order of the elements in the main diagonal.



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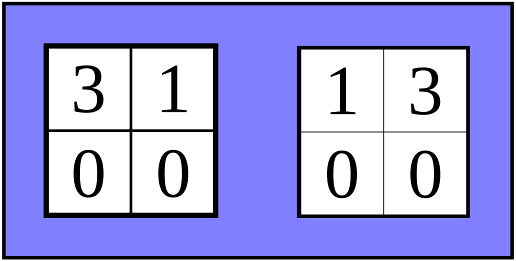
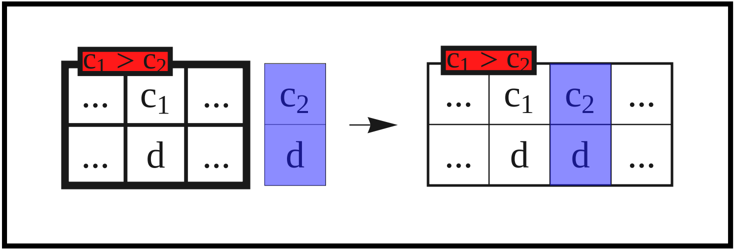
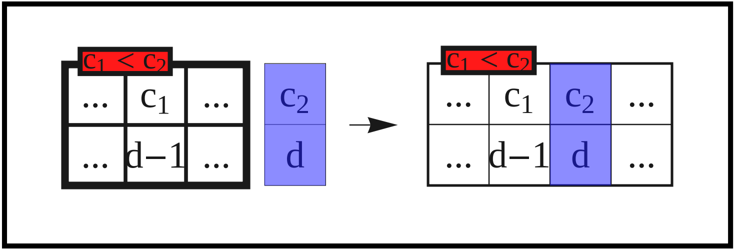
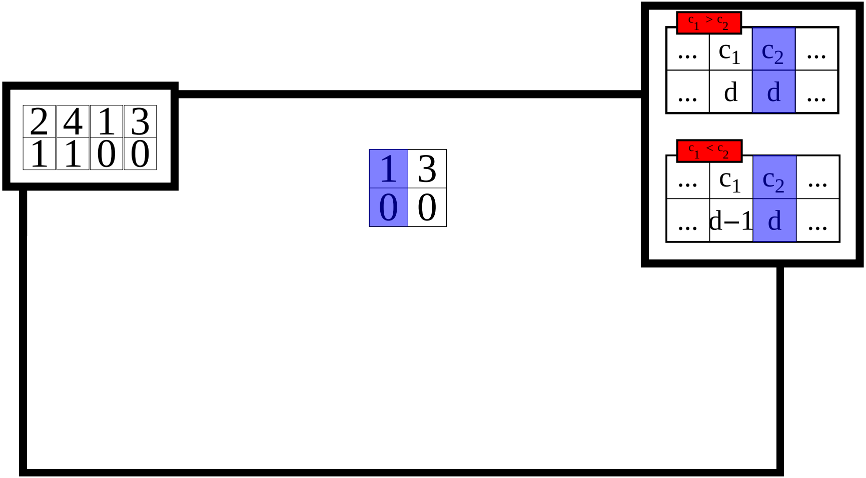


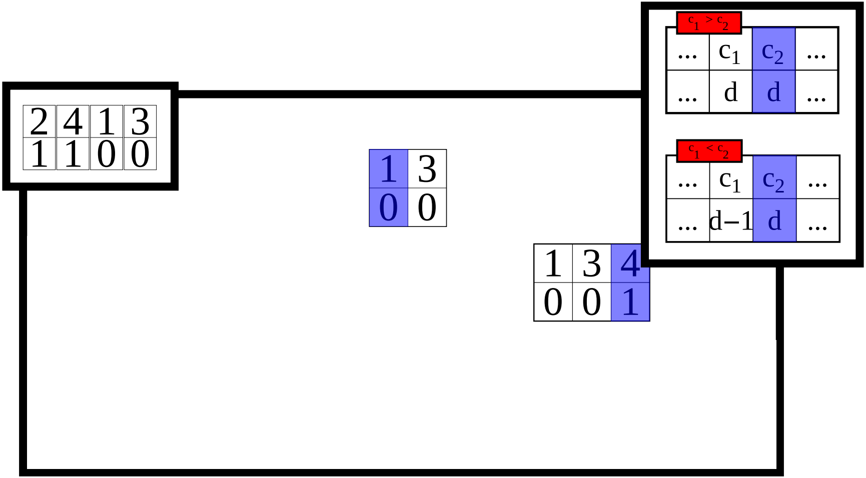
- Insert the remaining elements recursively in one of two ways:

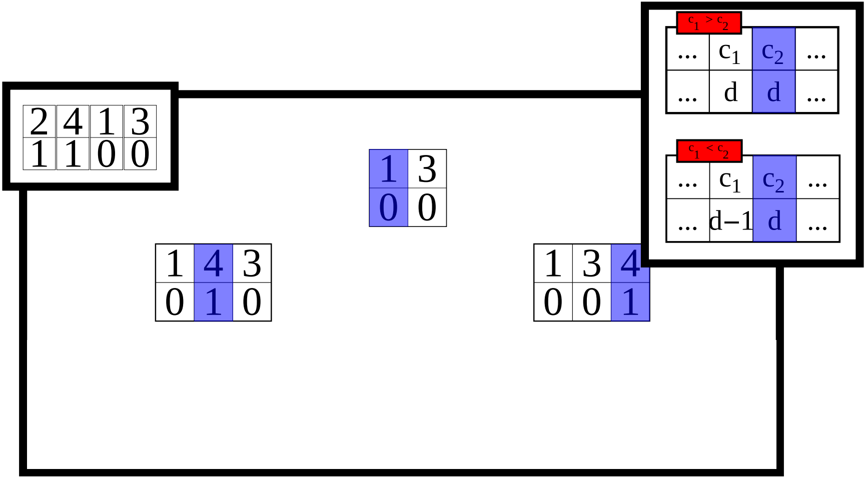


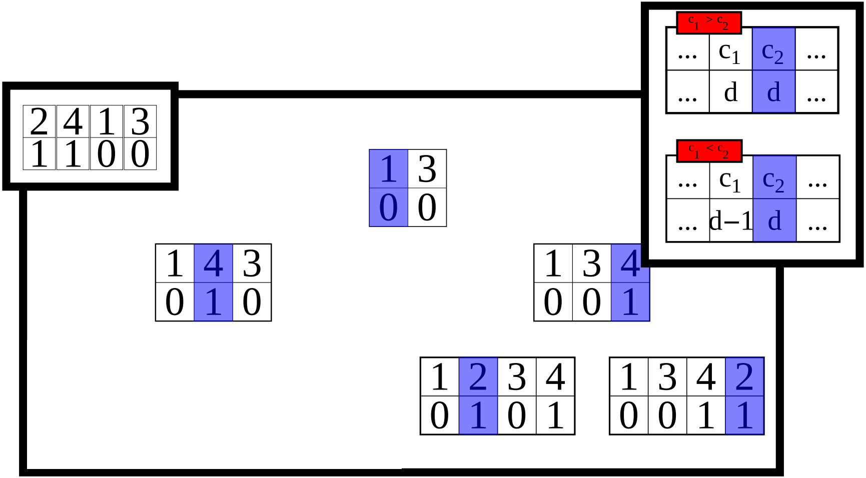
- Insert the remaining elements recursively in one of two ways:

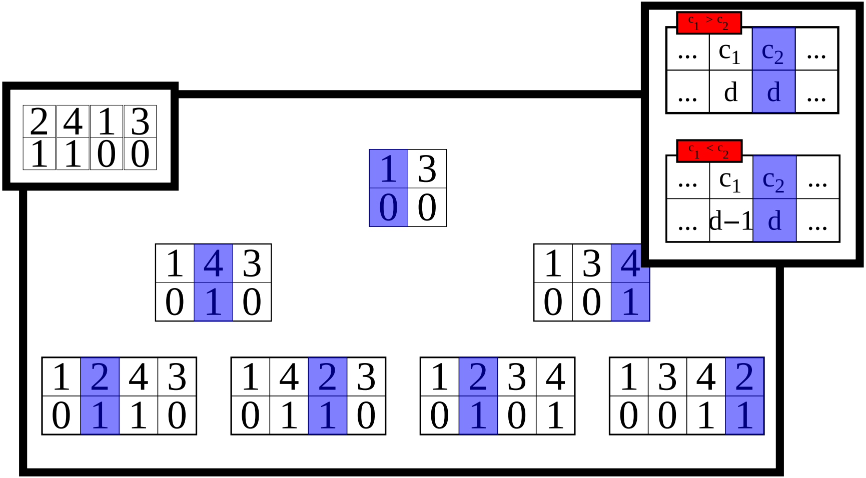


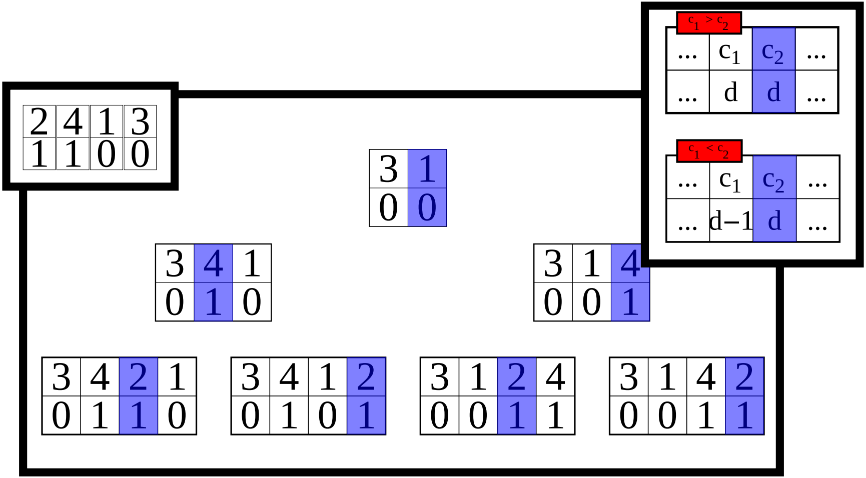

2 4 1 3
1 1 0 0
1 3
0 0
c1 > c2
… c1 c2 …
… d d …
c1 < c2
… c1 c2 …
… d−1 d …


2 4 1 3
1 1 0 0
1 3
0 0
1 3 4
0 0 1
c1 > c2
... c1 c2 ...
... d d ...
c1 < c2
... c1 c2 ...
... d−1 d ...


2 4 1 3
1 1 0 0
1 3
0 0
1 4 3
0 1 0
1 3 4
0 0 1
$c_1 > c_2$
… c_1 c_2 …
… d d …
$c_1 < c_2$
… c_1 c_2 …
… d−1 d …


2 4 1 3
1 1 0 0
1 3
0 0
1 4 3
0 1 0
1 3 4
0 0 1
1 2 3 4
0 1 0 1
1 3 4 2
0 0 1 1
$c_1 > c_2$
... c_1 c_2 ...
... d d ...
$c_1 < c_2$
... c_1 c_2 ...
... d−1 d ...


2 4 1 3
1 1 0 0
1 3
0 0
1 4 3
0 1 0
1 3 4
0 0 1
1 2 4 3
0 1 1 0
1 4 2 3
0 1 1 0
1 2 3 4
0 1 0 1
1 3 4 2
0 0 1 1
$c_1 > c_2$
... c_1 c_2 ...
... d d ...
$c_1 < c_2$
... c_1 c_2 ...
... d−1 d ...


2 4 1 3
1 1 0 0
3 1
0 0
3 4 1
0 1 0
3 1 4
0 0 1
3 4 2 1
0 1 1 0
3 4 1 2
0 1 0 1
3 1 2 4
0 0 1 1
3 1 4 2
0 0 1 1
c1 > c2
... c1 c2 ...
... d d ...
c1 < c2
... c1 c2 ...
... d−1 d ...

Theorem (Haglund and Loehr)

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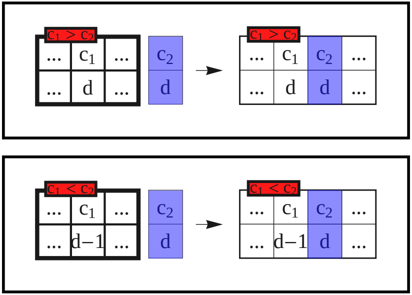
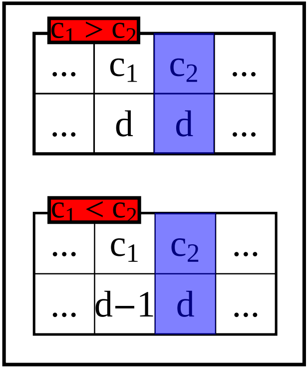
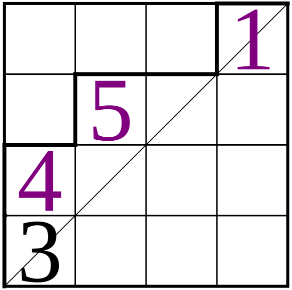
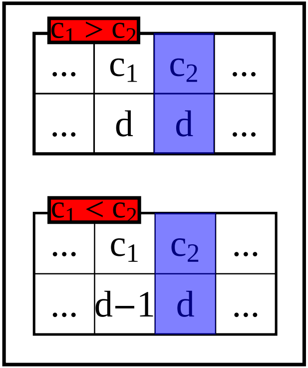


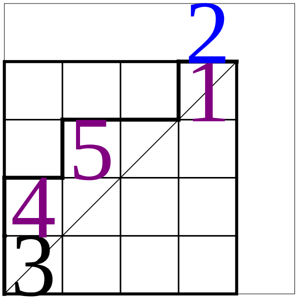
Diagonal Word [2, 4, 5, 1, 3]



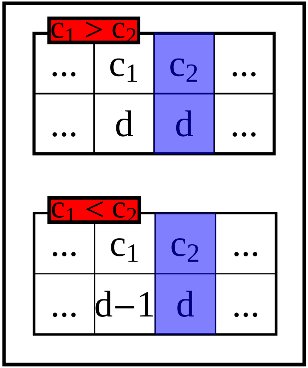


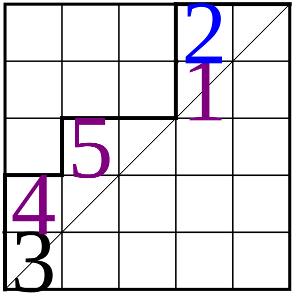
Diagonal Word [2, 4, 5, 1, 3]



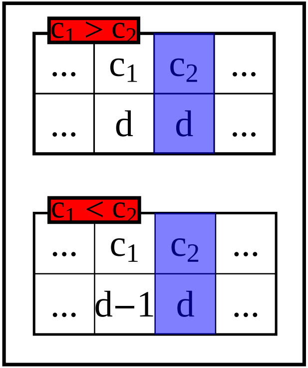


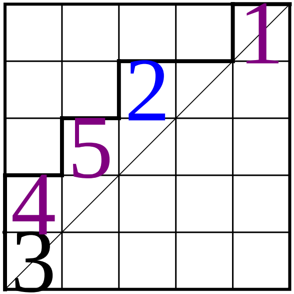
Diagonal Word [2, 4, 5, 1, 3]



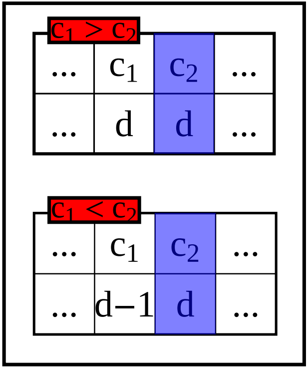


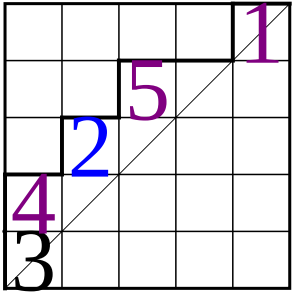
Diagonal Word [2, 4, 5, 1, 3]



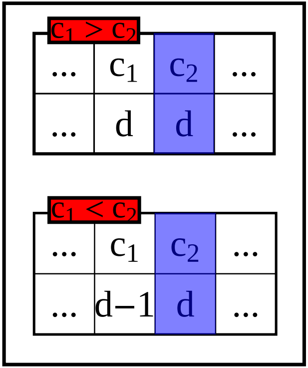


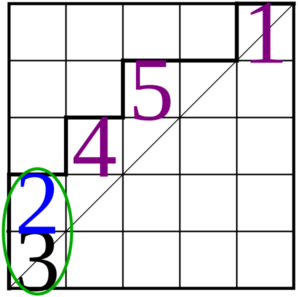
Diagonal Word [2, 4, 5, 1, 3]

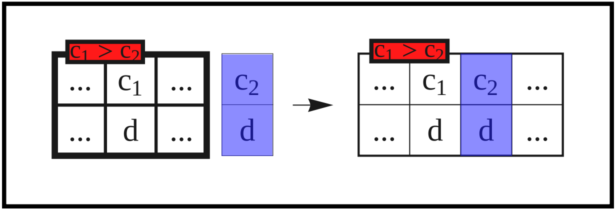


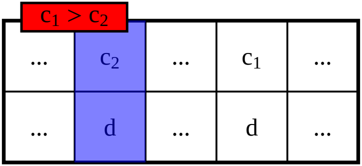


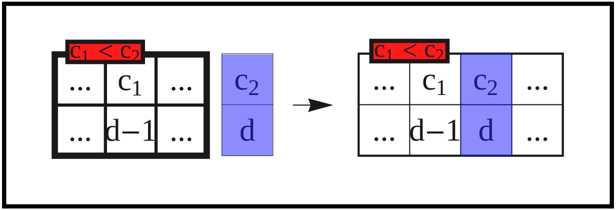
Diagonal Word [2, 4, 5, 1, 3]

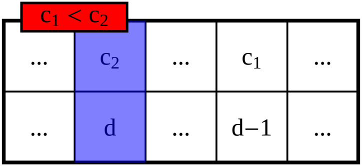




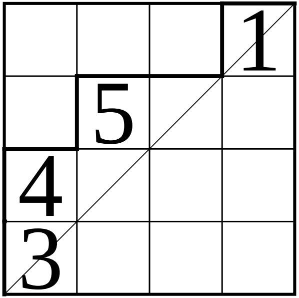

$c_1 > c_2$
... c_1 ...
... d ...
c_2
d
$c_1 > c_2$
... c_1 c_2 ...
... d d ...


$c_1 > c_2$
... c_2 ... c_1 ...
... d ... d ...

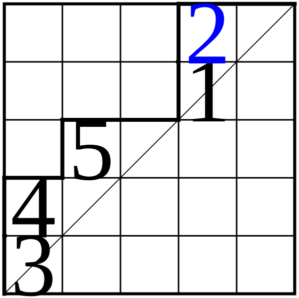

c1 < c2
... c1 ...
... d−1 ...
c2
d
c1 < c2
... c1 c2 ...
... d−1 d ...


c1 < c2
... c2 ... c1 ...
... d ... d−1 ...

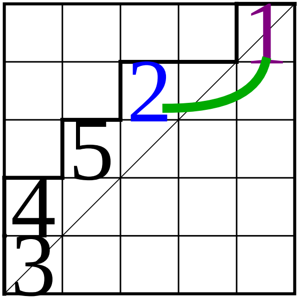
$$t^{\text{maj}([2,4,5,1,3])}\overbrace{(1)}^{3}\overbrace{(1+q)}^{1}\overbrace{(1+q)}^{5}\overbrace{(1+q+q^2)}^{4}\overbrace{(1+q+q^2)}^{2}$$



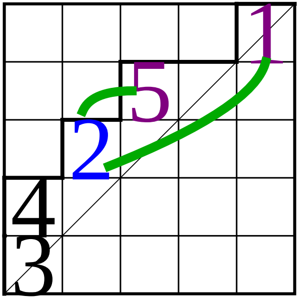
$$t^{\mathrm{maj}([2,4,5,1,3])}\overbrace{(1)}^{3}\overbrace{(1+q)}^{1}\overbrace{(1+q)}^{5}\overbrace{(1+q+q^2)}^{4}\overbrace{({\color{blue}1}+q+q^2)}^{2}$$



$$t^{\text{maj}([2,4,5,1,3])} \overbrace{(1)}^{3} \overbrace{(1+q)}^{1} \overbrace{(1+q)}^{5} \overbrace{(1+q+q^2)}^{4} \overbrace{(1+{\color{blue}q}+q^2)}^{2}$$



$$t^{\mathrm{maj}([2,4,5,1,3])} \overbrace{(1)}^{3}\overbrace{(1+q)}^{1}\overbrace{(1+q)}^{5}\overbrace{(1+q+q^2)}^{4}\overbrace{(1+q+{\color{blue}q^2})}^{2}$$



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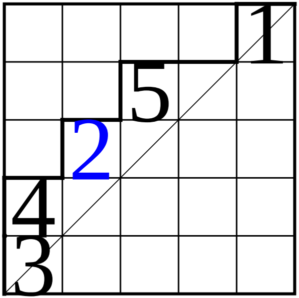
Diagonal Words

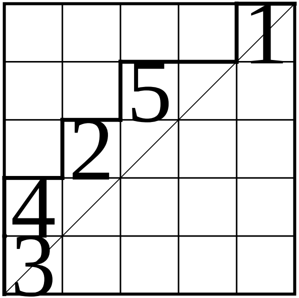
The Dinv of a Car

$$t^{\mathrm{maj}([2,4,5,1,3])} \overbrace{(1)}^{3} \overbrace{(1+q)}^{1} \overbrace{(1+q)}^{5} \overbrace{(1+q+q^2)}^{4} \overbrace{(1+q+q^2)}^{2}$$

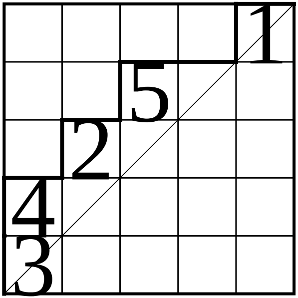
$\mathrm{dinv}(2) = 2$



$$t^{\text{maj}([2,4,5,1,3])}\overbrace{(1)}^{3}\overbrace{(1+q)}^{1}\overbrace{(1+q)}^{5}\overbrace{(1+q+q^2)}^{4}\overbrace{(1+q+q^2)}^{2}$$



$$t^{\mathrm{maj}([2,4,5,1,3])}\overbrace{(1)}^{3}\overbrace{(1+q)}^{1}\overbrace{(1+q)}^{5}\overbrace{(1+q+q^2)}^{4}\overbrace{(1+q+q^2)}^{2}$$

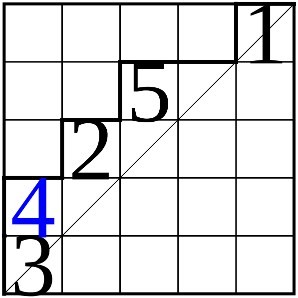


car	dinv
3	0
1	0
5	1
4	2
2	2

Diagonal Words

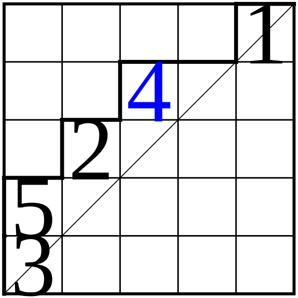
Changing the Dinv

$$t^{\mathrm{maj}([2,4,5,1,3])} \overbrace{(1)}^{3} \overbrace{(1+q)}^{1} \overbrace{(1+q)}^{5} \overbrace{(1+q+q^2)}^{4} \overbrace{(1+q+q^2)}^{2}$$



car	dinv
3	0
1	0
5	1
4	2
2	2

$$t^{\text{maj}([2,4,5,1,3])} \overbrace{(1)}^{3} \overbrace{(1+q)}^{1} \overbrace{(1+q)}^{5} \overbrace{(1+q+q^2)}^{4} \overbrace{(1+q+q^2)}^{2}$$



car	dinv
3	0
1	0
5	1
4	1
2	2

Definition

Say $\mathrm{Dec}(C_i, PF)$ is the unique parking function PF' such that:

1. $\mathrm{diag}(PF) = \mathrm{diag}(PF')$.
2. For $j \neq i$, $\mathrm{dinv}(C_j, PF) = \mathrm{dinv}(C_j, PF')$.
3. $\mathrm{dinv}(C_i, PF') = \mathrm{dinv}(C_i, PF) - 1$.

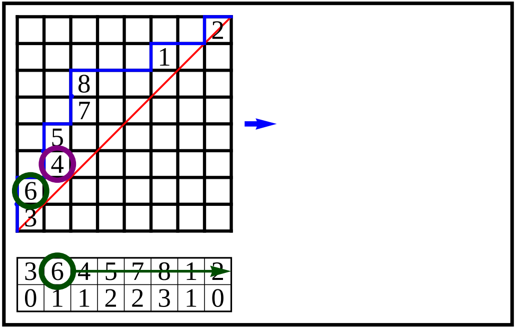
Define $(\mathrm{Inc}(C_i, PF))$ analogously.

Definition

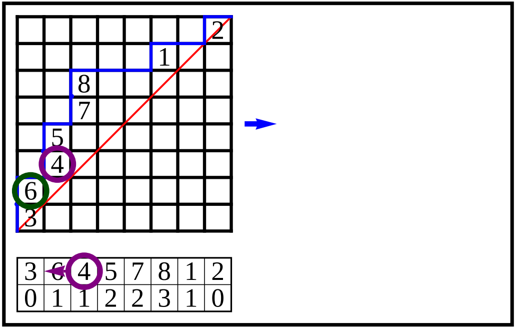
Let

$$\mathrm{Change}(C_j, C_k, PF) = \mathrm{Dec}(C_k, \mathrm{Inc}(C_j, PF)).$$

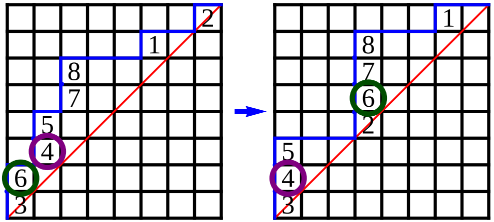
Change(4, 6, PF)



Change(4, 6, *PF*)



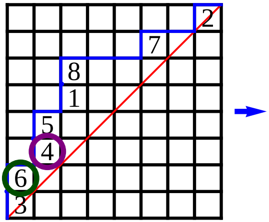
Change$(4, 6, PF)$



3	6	4	5	7	8	1	2
0	1	1	2	2	3	1	0

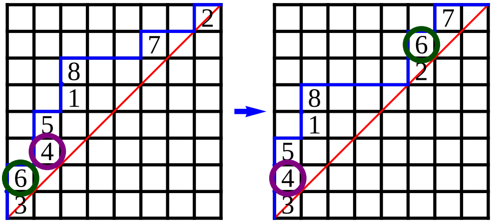
3	4	5	2	6	7	8	1
0	1	2	0	1	2	3	1

Change(4, 6, PF)



3	6	4	5	1	8	7	2
0	1	1	2	2	3	1	0

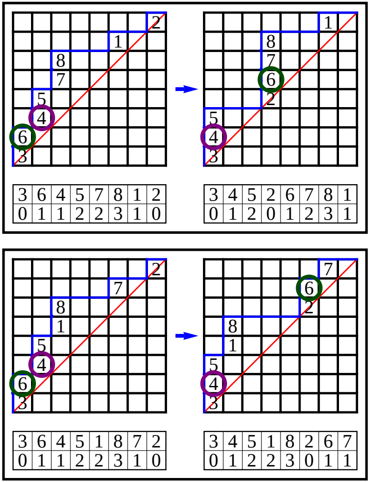
Change(4, 6, *PF*)



3	6	4	5	1	8	7	2
0	1	1	2	2	3	1	0

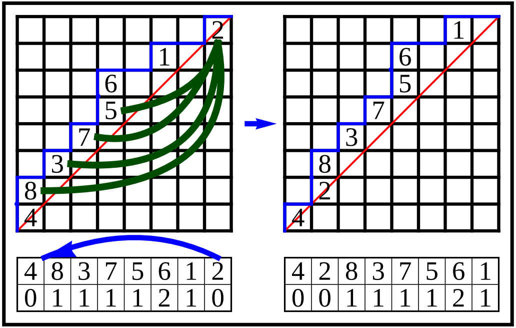
3	4	5	1	8	2	6	7
0	1	2	2	3	0	1	1

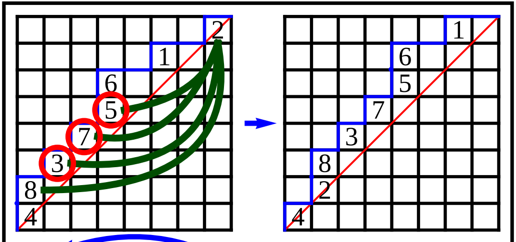
Change$(4, 6, PF)$


3 6 4 5 7 8 1 2
0 1 1 2 2 3 1 0
3 4 5 2 6 7 8 1
0 1 2 0 1 2 3 1
3 6 4 5 1 8 7 2
0 1 1 2 2 3 1 0
3 4 5 1 8 2 6 7
0 1 2 2 3 0 1 1

When $k = 1$

The Remaining Map (returned)


2
1
6
5
7
3
8
4
4 8 3 7 5 6 1 2
0 1 1 1 1 2 1 0
1
6
5
7
3
8
2
4
4 2 8 3 7 5 6 1
0 0 1 1 1 1 2 1



4	8	3	7	5	6	1	2
0	1	1	1	1	2	1	0

4	2	8	3	7	5	6	1
0	0	1	1	1	1	2	1

Definition (Troublesome Set)

$$T(PF) = \{C_j : 2 < j < L,\ D_j = 1, \text{ and } C_j \text{ is big.}\}.$$

Big Idea

1. Recursively use a series of dinv changes to reduce the size of $T(PF)$.
2. Apply f_1 when $T(PF)$ is empty.

Notation

Let $M(PF)$ be the last element in $T(PF)$ and say $M(PF) = C_{m(PF)}$.

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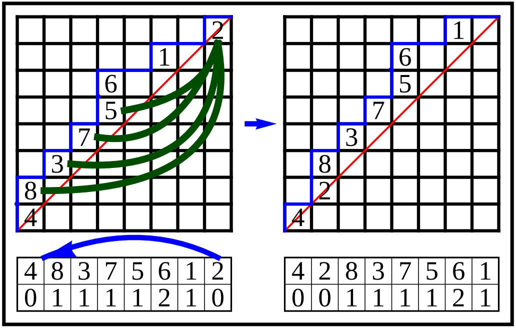

Notation

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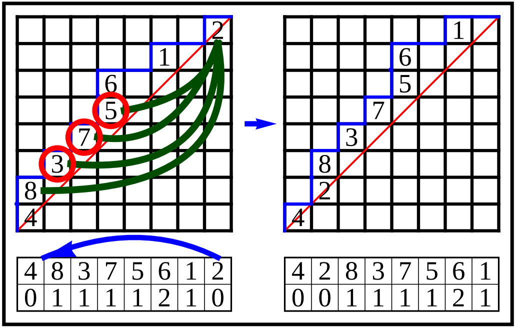
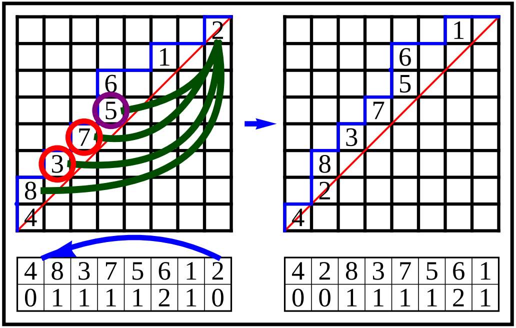


Notation

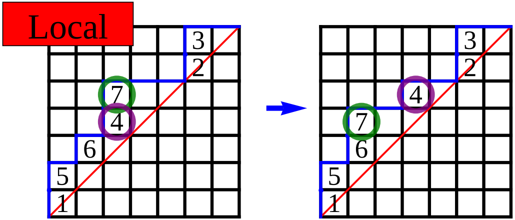
Let $M(PF)$ be the last element in $T(PF)$ and say $M(PF) = C_{m(PF)}$.



Definition

Let $PF' = \text{Inc}(c, PF)$. Say PF' is a **local** increase of PF if any car to the left of $M(PF)$ in PF is to the left of c in PF'.

$$Inc(7, PF)$$



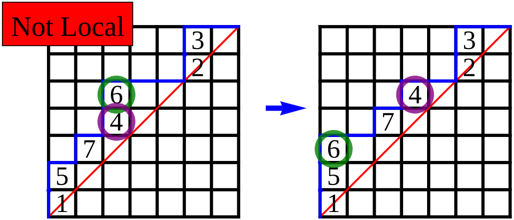
1	5	6	4	7	2	3
0	1	1	1	2	0	1

1	5	6	7	4	2	3
0	1	1	2	1	0	1

Definition

Let $PF' = \text{Inc}(c, PF)$. Say PF' is a **local** increase of PF if any car to the left of $M(PF)$ in PF is to the left of c in PF'.

$$Inc(6, PF)$$



1	5	7	4	6	2	3
0	1	1	1	2	0	1

1	5	6	7	4	2	3
0	1	2	1	1	0	1

Definition

Say a dinv change is **local** if its dinv increase is local.

Procedure

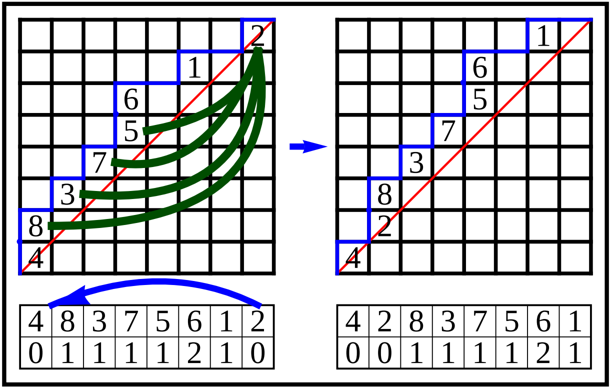
Beginning with a parking function PF, we can construct $f_3(PF)$ by forming a sequence

$$PF = PF^1, PF^2, \cdots, PF^s = f_3(PF)$$

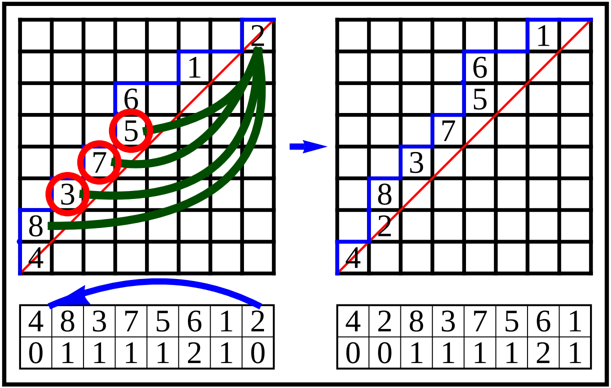
by repeatedly applying the following:

1. *If $T(PF^i) = \emptyset$, $f_3(PF) = f_1(PF^i)$.*
2. *Otherwise, if $PF' = \text{Change}(C_{m(PF^i)+1}, C_{m(PF^i)}, PF^i)$ is a parking function and is local, then let $PF^{i+1} = PF'$.*
3. *Otherwise, if $PF' = \text{Change}(C_{m(PF^i)}, C_{m(PF^i)-1}, PF^i)$ is a parking function, then let $PF^{i+1} = PF'$.*
4. *Otherwise, let $PF^{i+1} = \text{Change}(C_L, C_2, PF^i)$.*

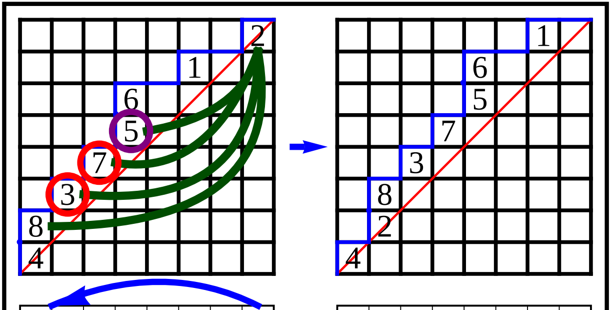
$$f_3 : \mathcal{A}_{\{n-1,1\}} \to \mathcal{A}_{\{1,n-1\}}$$



$$f_3 : \mathcal{A}_{\{n-1,1\}} \to \mathcal{A}_{\{1,n-1\}}$$



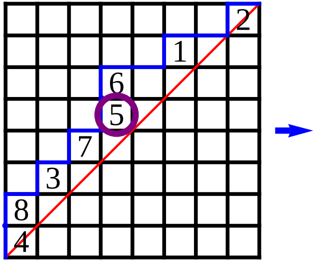
$$f_3 : \mathcal{A}_{\{n-1,1\}} \to \mathcal{A}_{\{1,n-1\}}$$



4	8	3	7	5	6	1	2
0	1	1	1	1	2	1	0

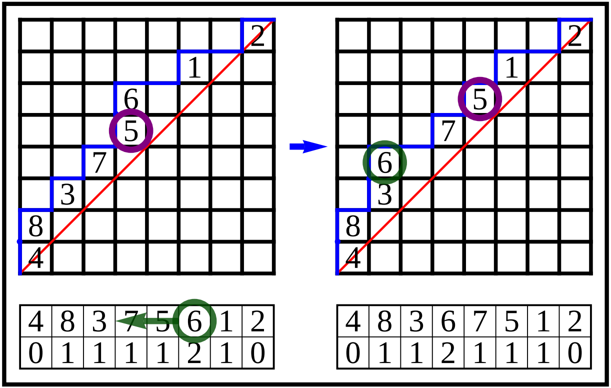
4	2	8	3	7	5	6	1
0	0	1	1	1	1	2	1

$$f_3 : \mathcal{A}_{\{n-1,1\}} \to \mathcal{A}_{\{1,n-1\}}$$

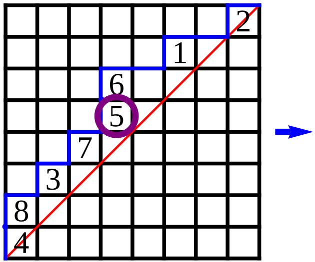


4	8	3	7	5	6	1	2
0	1	1	1	1	2	1	0

$$f_3 : \mathcal{A}_{\{n-1,1\}} \to \mathcal{A}_{\{1,n-1\}}$$

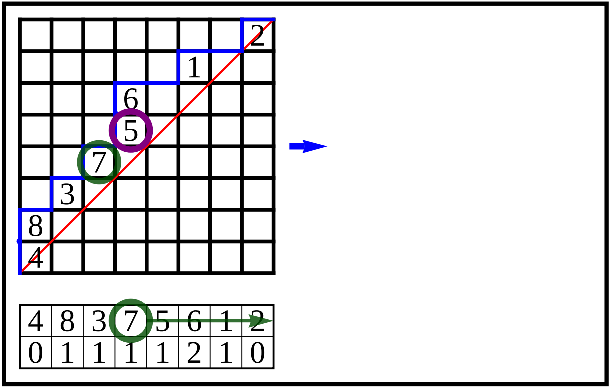


$$f_3 : \mathcal{A}_{\{n-1,1\}} \to \mathcal{A}_{\{1,n-1\}}$$

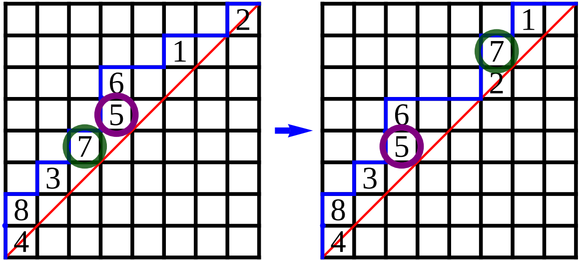


4	8	3	7	5	6	1	2
0	1	1	1	1	2	1	0

$$f_3 : \mathcal{A}_{\{n-1,1\}} \to \mathcal{A}_{\{1,n-1\}}$$



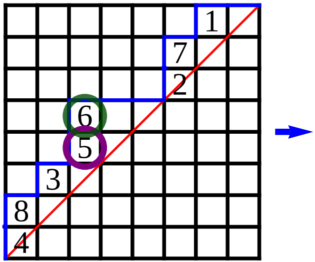
$$f_3 : \mathcal{A}_{\{n-1,1\}} \to \mathcal{A}_{\{1,n-1\}}$$



4	8	3	7	5	6	1	2
0	1	1	1	1	2	1	0

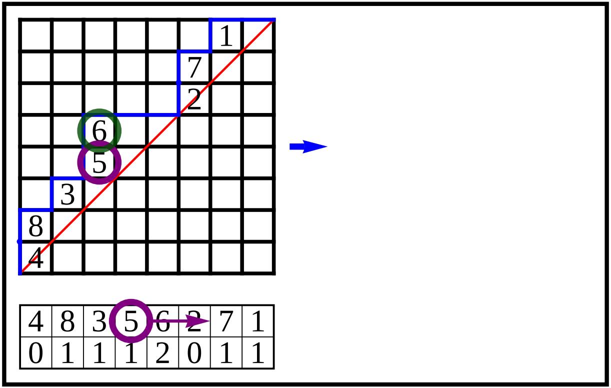
4	8	3	5	6	2	7	1
0	1	1	1	2	0	1	1

$$f_3 : \mathcal{A}_{\{n-1,1\}} \to \mathcal{A}_{\{1,n-1\}}$$

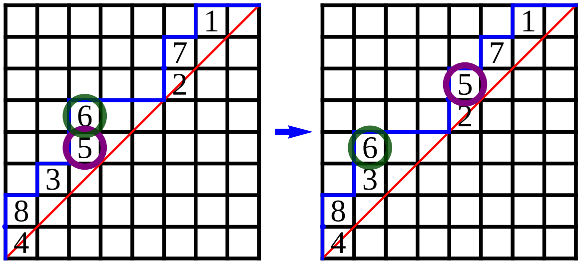


4	8	3	5	6	2	7	1
0	1	1	1	2	0	1	1

$$f_3 : \mathcal{A}_{\{n-1,1\}} \to \mathcal{A}_{\{1,n-1\}}$$



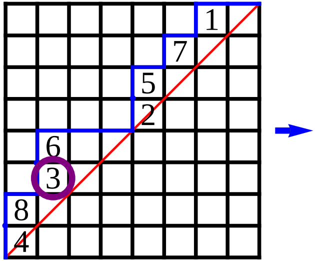
$$f_3 : \mathcal{A}_{\{n-1,1\}} \to \mathcal{A}_{\{1,n-1\}}$$



4	8	3	5	6	2	7	1
0	1	1	1	2	0	1	1

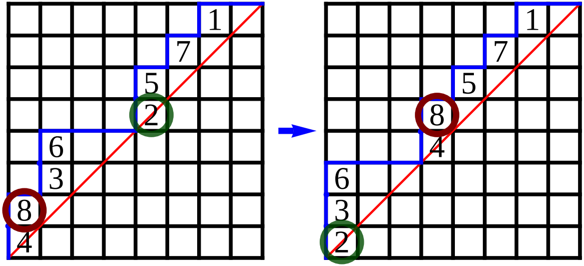
4	8	3	6	2	5	7	1
0	1	1	2	0	1	1	1

$$f_3 : \mathcal{A}_{\{n-1,1\}} \to \mathcal{A}_{\{1,n-1\}}$$



4	8	3	6	2	5	7	1
0	1	1	2	0	1	1	1

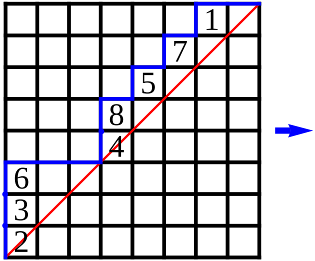
$$f_3 : \mathcal{A}_{\{n-1,1\}} \to \mathcal{A}_{\{1,n-1\}}$$



4	8	3	6	2	5	7	1
0	1	1	2	0	1	1	1

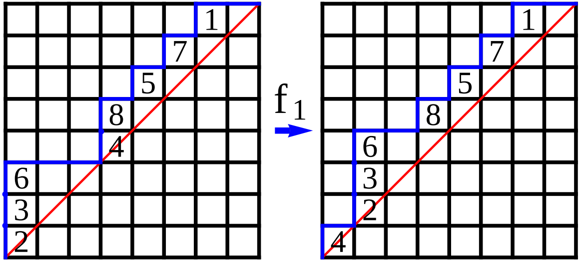
2	3	6	4	8	5	7	1
0	1	2	0	1	1	1	1

$$f_3 : \mathcal{A}_{\{n-1,1\}} \to \mathcal{A}_{\{1,n-1\}}$$



2	3	6	4	8	5	7	1
0	1	2	0	1	1	1	1

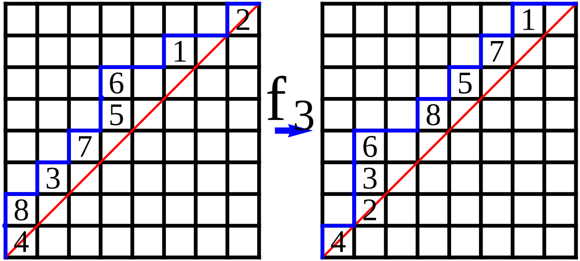
$$f_3 : \mathcal{A}_{\{n-1,1\}} \to \mathcal{A}_{\{1,n-1\}}$$



2	3	6	4	8	5	7	1
0	1	2	0	1	1	1	1

4	2	3	6	8	5	7	1
0	0	1	2	1	1	1	1

$$f_3 : \mathcal{A}_{\{n-1,1\}} \to \mathcal{A}_{\{1,n-1\}}$$



4	8	3	7	5	6	1	2
0	1	1	1	1	2	1	0

4	2	3	6	8	5	7	1
0	0	1	2	1	1	1	1

Why f_3 works:

1. The procedure terminates, since $|T|$ decreases by one with each iteration.
2. One of the four cases will always produce a valid parking function.
   - (Recursive Condition) The last car before C_L is either small or not in the first diagonal.
3. The $\text{dinv}(PF^i) = \text{dinv}(PF^{i+1})$ and f_1 decreases the dinv by exactly one, so $\text{dinv}(f_3(PF)) = \text{dinv}(PF) + 1$
4. The $\text{comp}(f_3(PF)) = (1, n-1)$
5. The $\text{ides}(PF^i) = \text{ides}(PF^{i+1})$.
6. f_3 is invertible.

Final Conclusions

Theorem (H.)

Then there exists a bijective map

$$f : \mathcal{F}_{\{1,n-1\}} \cup \mathcal{F}_{\{n-2,2\}} \Leftrightarrow \mathcal{F}_{\{n-2,2\}} \cup \mathcal{F}_{\{n-1,1\}}$$

such that $q\,\mathrm{wt}(f(PF)) = \mathrm{wt}(PF)$ *and* $\mathrm{diag}(f(PF)) = \mathrm{diag}(PF)$.

Corollary

$$q(\mathcal{A}_{\{1,n-1\}} + \mathcal{A}_{\{n-2,2\}}) = \mathcal{A}_{\{n-1,1\}} + \mathcal{A}_{\{2,n-2\}}$$

Theorem (H.)

Then there exists a bijective map

$$f : \mathcal{F}_{\{1,n-1\}} \cup \mathcal{F}_{\{n-2,2\}} \Leftrightarrow \mathcal{F}_{\{n-2,2\}} \cup \mathcal{F}_{\{n-1,1\}}$$

such that $q\,\mathrm{wt}(f(PF)) = \mathrm{wt}(PF)$ *and* $\mathrm{diag}(f(PF)) = \mathrm{diag}(PF)$.

Corollary

For p, p' *compositions,*

$$q(\mathcal{A}_{\{p,1,n-1,p'\}} + \mathcal{A}_{\{p,n-2,2,p'\}}) = \mathcal{A}_{\{p,n-1,1,p'\}} + \mathcal{A}_{\{p,2,n-2,p'\}}$$

For Further Reading

J. Haglund and N. Loehr.
A conjectured combinatorial formula for the Hilbert series for diagonal harmonics.
Discrete Math., 298(1-3):189–204, 2005.

James Haglund, Jennifer Morse, and Mike Zabrocki.
Dyck paths with forced and forbidden touch points and q,t-catalan building blocks, 2010.