Topological methods in toric geometry, symplectic geometry and combinatorics

BIRS, 8–12 November 2010

ORGANIZERS

Tony Bahri, Fred Cohen, Matthias Franz, Samuel Gitler, Megumi Harada

SCHEDULE

Sunday

16:00	Check-in begins (Front Desk - Professional Development Centre - open 24 hours)
	Lecture rooms available after 16:00 (if desired)
17:30–19:30	Dinner
20:00	Informal gathering in 2nd floor lounge, Corbett Hall
	Beverages and a small assortment of snacks are available on a cash honor system.

Monday

7:00– 8:45	Breakfast
8:45– 9:00	Introduction and Welcome by BIRS Station Manager, Max Bell 159
	Session chair: Martin Bendersky
9:00–10:00	Graham Denham: *Topological aspects of partial product spaces: a survey*
10:00–10:30	Coffee break
10:30–11:30	Santiago Lopez de Medrano: *Intersections of quadrics and the polyhedral product functor*
11:30–13:00	Lunch
13:00–14:00	Guided tour of the Banff Centre; meet in the 2nd floor lounge, Corbett Hall
	Session chair: Alexander Suciu
14:00–15:00	Kiumars Kaveh: *Convex bodies associated to reductive group actions*
15:00–15:30	Coffee break
15:30–16:30	Julianna Tymoczko: *Computational tools in torus-equivariant cohomology with applications to Schubert calculus*
17:30–19:30	Dinner

Tuesday

7:00– 9:00	Breakfast
	Session chair: Mario Salvetti
9:00–10:00	Henry Schenck: *Cohomology and Chow rings of toric varieties*
10:00–10:30	Coffee break
10:30–11:30	Mikiya Masuda: *Cohomological rigidity problem, topological toric manifolds and face numbers of simplicial cell manifolds*
11:30–14:00	Lunch
	Session chair: Dietrich Notbohm
14:00–15:00	Michael Davis: *Generalized moment angle complexes, graph products of groups and related constructions*
15:00–15:30	Coffee break
15:30–16:30	Nigel Ray: *Applications of toric methods to cobordism theory*
17:30–19:30	Dinner

Wednesday	
7:00– 9:00	Breakfast
	Session chair: Alberto Verjovsky
9:00–10:00	Victor Batyrev: *On topological invariants of Calabi–Yau* 3*-folds constructed by toric geometry*
10:00–10:30	Coffee break
10:30–11:30	Tom Braden: *Geometry and representation theory of hypertoric varieties*
11:30–11:45	Group photo; meet on the front steps of Corbett Hall
11:45–13:30	Lunch
17:30–19:30	Dinner
Thursday	
7:00– 9:00	Breakfast
	Session chair: Volker Puppe
9:00–10:00	Corrado De Concini: *Infinitesimal index and some cohomology computations*
10:00–10:30	Coffee break
10:30–11:30	Susan Tolman: *The integral cohomology of GKM spaces*
11:30–14:00	Lunch
	Session chair: Dong Youp Suh
14:00–15:00	Carl Lee: *Sweeping the* cd*-index and the toric h-vector*
15:00–15:30	Coffee break
15:30–16:00	Suyoung Choi: *Torus actions on cohomology complex Bott manifolds*
16:00–16:30	Jelena Grbić: *Higher Whitehead products and polyhedral product functors*
17:30–19:30	Dinner
	Session chair: Yasuhiko Kamiyama
20:00–20:30	Dave Anderson: *Transversality in equivariant cohomology*
20:30–21:00	Volker Puppe: *Equivariant cohomology and orbit structure*
Friday	
7:00– 9:00	Breakfast
	Session chair: Jim Carrell
9:00–10:00	Taras Panov: *Complex-analytic structures on moment-angle manifolds*
10:00–10:30	Coffee break
10:30–11:30	Nickolai Erokhovets: *Buchstaber invariant of simple polytopes*
11:30–13:30	Lunch
Checkout by 12 noon.	

** 5-day workshops are welcome to use BIRS facilities (2nd Floor Lounge, Max Bell Meeting Rooms, Reading Room) until 3 pm on Friday, although participants are still required to checkout of the guest rooms by 12 noon. **

MEALS

Breakfast (Buffet): 7:00–9:30 am, Sally Borden Building, Monday–Friday
Lunch (Buffet): 11:30 am–1:30 pm, Sally Borden Building, Monday–Friday
Dinner (Buffet): 5:30–7:30 pm, Sally Borden Building, Sunday–Thursday
Coffee breaks: As per daily schedule, 2nd floor lounge, Corbett Hall

Please remember to scan your meal card at the host/hostess station in the dining room for each meal.

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ABSTRACTS
(in alphabetic order by speaker surname)

Speaker: **Victor Batyrev** (University of Tübingen)
Title: *On topological invariants of Calabi–Yau 3-folds constructed by toric geometry*
Abstract: A 3-dimensional complex projective algebraic variety X is called Calabi–Yau 3-fold if X has a nowhere vanishing holomorphic 3-form and the first Betti number of X is zero. Many topologically different Calabi–Yau 3-folds can be constructed using toric geometry. We give a general introduction to this subject and explain how toric geometry allows to compute Hodge (and Betti) numbers of constructed Calabi–Yau 3-folds as well as some applications to mirror symmetry.

Speaker: **Tom Braden** (University of Massachusetts)
Title: *Geometry and representation theory of hypertoric varieties*
Abstract: Hypertoric varieties are holomorphic symplectic analogs of toric varieties. They share many important properties with other important holomorphic symplectic varieties such as cotangent bundles to flag varieties and Hilbert schemes of points on symplectic surfaces, but like toric varieties, they give a test case where many things can be computed "by hand" using elementary combinatorics. After a general introduction to hypertoric geometry I will discuss how modules over a deformation of the ring of functions provide a hypertoric analog of representations of a reductive Lie algebra.

Speaker: **Suyoung Choi** (Osaka City University)
Title: *Torus actions on cohomology complex Bott manifolds*
Abstract: A Bott manifold is a closed smooth manifold obtained as the total space of an iterated $\mathbb{C}P^1$-bundle over a point, where each fibration is a projectivization of the Whitney sum of two complex line bundles. A Bott manifold is known to be a toric manifold, and hence, it admits a natural half dimensional compact torus action. A closed smooth manifold is called a cohomology Bott manifold if its cohomology ring is isomorphic to that of some Bott manifold.

In this talk, we show that any cohomology ring isomorphism between two cohomology Bott manifolds which admit a half dimensional effective smooth compact torus action preserves their Pontrjagin classes. As a corollary, there is only a finite number of manifolds homotopy equivalent to given Bott manifold which admit a smooth action of a torus.

Speaker: **Michael Davis** (Ohio State University)
Title: *Generalized moment angle complexes, graph products of groups and related constructions*
Abstract: I will define generalized moment-angle complexes (also called "polyhedral products") and discuss several basic examples. Then I will focus on the question of computing the fundamental group of a generalized moment-angle complex and determining when it is an aspherical complex. The answer is related to the notion of a graph product of groups or of pairs of groups. Right-angled buildings also play a key role in the discussion. In fact, right-angled buildings can be defined as universal covers of certain generalized moment-angle complexes.

Speaker: **Corrado De Concini** (University of Rome)
Title: *Infinitesimal index and some cohomology computations*
Abstract: (joint with C. Procesi and M. Vergne) Let G be a compact Lie group with Lie algebra $\mathfrak{g}$. Given a G-manifold M with a G-equivariant one form ω we consider the zeroes M^0 of the corresponding moment

map and define a map, called infinitesimal index, of $S[\mathfrak{g}^*]^G$-modules from the equivariant cohomology of M^0 with compact support to the space of invariant distributions on $\mathfrak{g}^*$.

In the case in which G is a torus, N is a linear complex representation of G, $M = T^*N$ with tautological one form we are going to explain how this is used to compute the equivariant cohomology of M^0 with compact support using certain spaces of polynomial which appear in approximation theory.

Speaker: **Graham Denham** (University of Western Ontario)
Title: *Topological aspects of partial product spaces: a survey*
Abstract: The notion of a partial product space is a relatively recent unification of various combinatorial constructions in topology. This construction is variously known as the generalized moment-angle complex, or (more euphoniously) as the polyhedral product functor. Some instances of it are closely related to Davis and Januszkiewicz's quasitoric manifolds: these include the moment-angle complexes (Buchstaber and Panov) and homotopy orbit spaces for quasitoric manifolds. By making suitable choices, one also obtains classifying spaces for right-angled Artin groups and Coxeter groups, as well as certain real and complex subspace arrangements.

One advantage to this generality is that some topological information about such spaces can sometimes be expressed directly in combinatorial terms: presentations of cohomology rings; a homotopy-theoretic decomposition of the suspension of a partial product space; descriptions of rational homotopy Lie algebras and the Pontryagin algebra. I will give an introductory overview of some remarkable results along these lines.

Speaker: **Nickolai Erokhovets** (Moscow State University)
Title: *Buchstaber invariant of simple polytopes*
Abstract: Surfaces and convex polytopes lie in the focus of a scientific study since antiquity. Toric topology gives a new fruitful connection between polytopes and surfaces. Namely, for a given convex simple n-dimensional polytope P with m facets there is a canonical way to build an $(m+n)$-dimensional *moment-angle manifold* $\mathcal{Z}_P$ with a canonical action of the torus T^m such that the topological type of $\mathcal{Z}_P$ depends only on the combinatorial type of P, and $\mathcal{Z}_P/T^m = P$. Then combinatorial properties of simple polytope P can be investigated from the point of view of the topology of $\mathcal{Z}_P$ and vice versa. For example, *bigraded Betti numbers* $\beta^{-i,2j}$ defined by the canonical moment-angle cell structure on $\mathcal{Z}_P$ are combinatorial invariants of P.

Definition. A *Buchstaber number* $s(P)$ is the maximal dimension of a torus subgroup $H \cong T^s \subset T^m$, which acts freely on $\mathcal{Z}_P$.

The problem stated by Victor M. Buchstaber in 2002 is to find a simple combinatorial description of the s-number.

From the definition it is not difficult to see that $1 \leqslant s(P) \leqslant m - n$.

A very important case is $s(P) = m - n$. Then the quotient space $M^{2n} = \mathcal{Z}_P/T^m$ is a so-called *quasitoric manifold*. It is known that $b_{2i}(M^{2n}) = h_i(P)$, where $\{b_{2i}\}$ are classical Betti numbers, and $(h_0, h_1, \ldots, h_n)$ is an h-vector of a polytope. However $s(P) = m - n$ is not the general case. For example, this equality is not valid for the polytope, dual to the cyclic polytope $C^n(m)$ for $m \geqslant 2^n$.

The s-number can be defined for any simplicial complex K in such a way that for a simple polytope P we have $s(P) = s(K)$, where $K = \partial P^*$ – the boundary complex of the dual simplicial polytope.

At present moment the following problems in this field are actual: to find a simple combinatorial description that gives an *effective* method to calculate the s-number in important *special* cases, to find a connection between values of $s(K)$ of different simple polytopes and complexes, to find a connection with other combinatorial invariants, to study the behavior of the s-number under factor mappings of simplicial complexes (see the definition below).

We will discuss the results about $s(P)$ like the following:

1. $s(P) = 1$ if and only if P is a simplex.

2. For any $q \geqslant 2$ there exists a polytope P such that $m - n = q$ and $s(P) = 2$. In fact, $s(C^n(m)^*) = 2$ for $n \geqslant 13$, and

$$n + 2 \leqslant m \leqslant n + 2 + \frac{n - 13}{48}$$

3. A surjective mapping $\pi\colon \operatorname{Vert}(K_1) \to \operatorname{Vert}(K_2)$ of the sets of vertices of two simplicial complexes K_1 and K_2 is said to be a *factor mapping* if $\sigma \in K_2$ if and only if $\pi^{-1}(\sigma) \in K_1$. For any factor mapping $\pi\colon K_1 \to K_2$ one can define a set of all sections – mappings $\zeta\colon \operatorname{Vert}(K_2) \to \operatorname{Vert}(K_1)$ such that $\pi \circ \zeta = \mathrm{id}$. Any section induces a simplicial mapping $K_2 \to K_1$, while the map π itself does not necessary induce a simplicial mapping $K_1 \to K_2$. Using this observation one can prove that

$$s(K_2) \leqslant s(K_1) \leqslant s(K_2) + m_1 - m_2, \quad m_i = |\operatorname{Vert}(K_i)|$$

4. There are two polytopes P and Q such that their face vectors $f(P)$ and $f(Q)$, and the chromatic numbers $\gamma(P)$ and $\gamma(Q)$ are equal, but $s(P) = 2$, and $s(Q) = 3$. Therefore $s(P)$ can not be calculated using only $f(P)$ and $\gamma(P)$.

We also investigate the properties of simple polytopes with $m = n+3$ vertices. In this case we calculate the bigraded cohomology ring of the moment-angle manifold $\mathcal{Z}_P$, find the value of $s(P)$, and prove that it can be expressed in terms of the bigraded Betti numbers.

Speaker: **Jelena Grbić** (University of Manchester)
Title: *Higher Whitehead products and polyhedral product functors*
Abstract: In this talk I'll look at the unstable homotopy type of moment-angle complexes related to shifted complexes. In addition, I'll describe how higher Whitehead products arise in the homotopy theory of moment-angle complexes and their relations with the loop homology of Davis–Januszkiewicz spaces.

Speaker: **Kiumars Kaveh** (University of Pittsburgh)
Title: *Convex bodies associated to reductive group actions*
Abstract: I will discuss how to associate three different convex bodies (usually polytopes) to a wide class of graded G-algebras where G is a connected reductive group. Most important examples are the algebra of sections of a G-linearized line bundle on a projective G-variety. They include and generalize several important convex polytopes appearing in algebraic geometry and symplectic geometry, e. g. the moment polytope of a Hamiltonian action, the Newton polytope of a toric variety and the Gelfand–Cetlin polytopes for the flag variety. These convex bodies give information about the Hilbert function as well as the multiplicities of irreducible representations appearing in the graded algebra. Using these we can extend the notion of Duistermaat–Heckman measure to graded G-algebras and prove a Fujita type approximation theorem as well as a Brunn–Minkowski inequality for this measure. This in particular gives an equivariant version of the theory of volumes of line bundles. Our approach follows some of the previous works of A. Okounkov. We use the asymptotic theory of semigroups of integral points and Newton–Okounkov bodies. The talk is based on a joint work with Askold Khovanskii.

Speaker: **Carl Lee** (University of Kentucky)
Title: *Sweeping the* cd*-index and the toric h-vector*
Abstract: By sweeping a hyperplane across a simple convex d-polytope P, the h-vector, $h(P^*) = (h_0, \ldots, h_d)$, of its dual can be computed—the edges in P are oriented in the direction of the sweep and h_i equals the number of vertices of outdegree i. Moreover, the nonempty faces of P can be partitioned to explicitly reflect the formula for the h-vector. For a general convex polytope, in place of the h-vector, one often considers the flag f-vector and flag h-vector as well their encoding into the cd-index, and also the toric h-vector (which does not contain the full information of the flag h-vector, but provides the middle perversity intersection homology Betti numbers of the associated toric variety when P is rational). Given a convex polytope P, we describe formulas for the cd-index of P and for the toric h-vector of P^* from a

sweeping of P. These arise from analyzing Stanley's S-shelling of P^*. We describe a partition of the faces of the complete truncation of P to provide an interpretation of what the components of the cd-index are counting. One corollary is a quick way to compute the toric h-vector directly from the cd-index.

Speaker: **Santiago Lopez de Medrano** (UNAM Mexico)
Title: *Intersections of quadrics and the polyhedral product functor*
Abstract: For many years certain intersections of quadrics were studied without knowing that essentially the same objects were being considered as part of a very active area known as Toric Topology. These intersections of quadrics are very concrete real algebraic varieties $Z = Z(\Lambda) \subset \mathbf{R}^n$ or $Z^{\mathbf{C}}(\Lambda) \subset \mathbf{C}^n$ given by $k+1$ equations

$$\sum_{i=1}^{n} \Lambda_i x_i^2 = 0, \qquad \sum_{i=1}^{n} x_i^2 = 1$$

or

$$\sum_{i=1}^{n} \Lambda_i \left|z_i\right|^2 = 0, \qquad \sum_{i=1}^{n} \left|z_i\right|^2 = 1,$$

where, in both cases,

$$\Lambda_i \in \mathbf{R}^k,\ i = 1, \ldots, m,\ \Lambda = (\Lambda_1, \ldots, \Lambda_m).$$

Starting in 1984, I studied these manifolds in the case $k = 2$ using their polytope quotients under some obvious group actions, computing their homology groups and applying techniques of differential topology. After sometime I was able to give a rather complete answer: a complete classification and an (almost) complete description of the smooth topological type of the generic cases ([LdM]).

Most of the topological ideas and constructions worked also for arbitrary k, but extending the above complete results for $k > 2$ ran into extremely difficult combinatorial problems. As a result, the study continued in other directions, especially to their projective versions which had important implications in the theory of complex manifolds, starting from [LdM-V]. An important article in this line ([B-M]) also opened new possibilities of research on the differential topology of some large families of intersections of quadrics.

Meanwhile, the subject of Toric Topology had started with the article [D-J] and continued for two decades with the work of many authors (see [B-P] for a crucial survey). It included, among many other things, an abstract construction of the same manifolds and generalizations of them. Further successive generalizations led to the recent *Polyhedral Product Functor* in [B-B-C-G], a very abstract homotopy theoretical construction.

When the connection between the two points of view was established it turned out that the Polyhedral Product Functor was the precise generalization where all the constructions with intersections of quadrics fitted in a single global picture. From this perspective, new topological results were obtained ([G-LdM]), answering important questions raised in [B-M] and with surprising consequences in the general abstract framework.

In this talk I will present mainly my side of this story, confident that the Toric Topology point of view will be present in many forms in this Workshop. I will start by recalling the origin of these questions in dynamical systems and in the rest of the talk I will point out some connections with other parts of Mathematics.

[B-M], F. Bosio and L. Meersseman, *Real quadrics in* $\mathbf{C}^n$*, complex manifolds and convex polytopes.* Acta Math. 197 (2006), no. 1, 53–127.
[B-P], V.M. Buchstaber and T.E. Panov, *Torus actions and their applications in Topology and Combinatorics*, University Lecture Series, AMS (2002).
[D-J], M. Davis and T. Januszkiewicz, *Convex polytopes, Coxeter orbifolds and torus actions*, Duke Math. Journal 62 (1991), 417–451.
[LdM], S. López de Medrano, *Topology of the intersection of quadrics in* $\mathbf{R}^n$, in *Algebraic Topology* (Arcata Ca, 1986), Springer Verlag LNM **1370** (1989), pp. 280–292, Springer Verlag.

[LdM-V], S. López de Medrano and A. Verjovsky, *A new family of complex, compact, nonsymplectic manifolds.* Bol. Soc. Brasil. Mat., 28 (1997), 253–269.

Speaker: **Ernesto Lupercio** (Centro de Investigacion del IPN)
Title: *The moduli space of toric manifolds*
Abstract: In this talk I will explain how the Gale transform as it appears in results of Lopez de Medrano, Verjovsky and Meersseman can be used to define the moduli stack of toric manifolds and how to endow it with a complex structure. This is work in progress (joint with A. Verjovsky and L. Meersseman).

Speaker: **Mikiya Masuda** (Osaka City University)
Title: *Cohomological rigidity problem, topological toric manifolds and face numbers of simplicial cell manifolds*
Abstract: I will talk about three topics in the title. The first one is the cohomological rigidity problem for (real) toric manifolds, which asks whether cohomology rings distinguish the diffeomorphism types of (real) toric manifolds. The second one is the introduction of what we call a *topological toric manifold* which seems a right topological analogue of a toric manifold. The third one is about counting face numbers of simplicial cell manifolds. Simplicial cell manifolds naturally arise in toric topology like simplicial polytopes arise in toric geometry.

Speaker: **Taras Panov** (Moscow State University)
Title: *Complex-analytic structures on moment-angle manifolds*
Abstract: Moment-angle complexes are spaces acted on by a torus and parametrised by finite simplicial complexes. They are central objects in toric topology, and currently are gaining much interest in the homotopy theory. Due the their combinatorial origins, moment-angle complexes also find applications in combinatorial geometry and commutative algebra. After an introductory part describing the general properties of moment-angle complexes we shall concentrate on the complex-analytic aspects of the theory. We show that the moment-angle manifolds corresponding to complete simplicial fans admit non-Kähler complex-analytic structures. This generalises the known construction of complex-analytic structures on polytopal moment-angle manifolds, coming from identifying them as LVM-manifolds. We proceed by describing the Dolbeault cohomology and certain Hodge numbers of moment-angle manifolds by applying the Borel spectral sequence to holomorphic principal bundles over toric varieties. The complex-analytic part of the talk is based on the joint work with Yuri Ustinovsky, arXiv:1008.4764.

Speaker: **Volker Puppe** (Universität Konstanz)
Title: *Equivariant cohomology and orbit structure*

Speaker: **Nigel Ray** (University of Manchester)
Title: *Applications of toric methods to cobordism theory*
Abstract: In this talk I shall outline some basic aspects of real and complex cobordism, and explain how toric and quasitoric manifolds have enhanced our understanding of certain cobordism phenomena. The story begins in 1986, and extends to current investigations into equivariant cobordism via universal toric genera. I shall attempt to make the subject matter reasonably accessible to all participants.

Speaker: **Henry Schenck** (University of Illinois)
Title: *Cohomology and Chow rings of toric varieties*
Abstract: I will begin with an overview of recent developments on the cohomology and Chow rings of toric varieties. Then I will focus on the integral, equivariant Chow ring: several years ago, Payne showed this ring consists of piecewise polynomial functions on the fan. In the simplicial case, combinatorics tells the whole story, but in the nonsimplicial case the geometry of the fan enters the picture. This involves an analysis of the structure of the Chow ring as a reflexive sheaf on projective space. I will also discuss an interesting connection to a famous open problem in approximation theory: the dimension of the space

of C^1 splines on a planar simplicial complex. From an algebro-geometric perspective, this translates into bounding the Castelnuovo–Mumford regularity of a certain vector bundle on the projective plane.

Speaker: **Susan Tolman** (University of Illinois at Urbana-Champaign)
Title: *The integral cohomology of GKM spaces*
Abstract: Chang–Skjelbred gives a straight-forward way to compute the rational cohomology of GKM spaces in terms of the GKM graph. Unfortunately, this formula does not work for integral cohomology. Nevertheless, the integral cohomology of a GKM space is determined by the GKM graph. This talk is based on a work in progress.

Speaker: **Julianna Tymoczko** (University of Iowa)
Title: *Computational tools in torus-equivariant cohomology with applications to Schubert calculus*
Abstract: In its classical form, Schubert calculus asks for the structure constants of the cohomology ring of the Grassmannian of k-dimensional subspaces of an n-dimensional complex vector space, with respect to a special geometric basis of Schubert classes. In its modern form, Schubert asks for the structure constants of generalized cohomology rings of more general varieties (like G/P), with respect to nice combinatorial bases. I will describe how localization techniques in torus-equivariant cohomology have be used to attack problems in modern Schubert calculus, as well as directions for future work. My approach will be concrete, explicit, and combinatorial, and my talk will involve many examples.