Evolution in Iterated Prisoner's Dilemma under Logit Dynamics

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Background

- evolutionary dynamics on Iterated Prisoner's Dilemma
- selection out of a vast set of the repeated game strategies
- Axelrod (1997) round-robin tournaments: Tit-For-Tat winner
- **ecology** of submitted rules critical for success of this direct reciprocity norm
- Brandt and Sigmund (2006): AllD, TFT, AllC with Replicator Dynamics: RSP cycles

Background

- Enriched ecologies of heuristics: GTFT, Pavlov(WSLS) +alternative evolutionary dynamic
- Pavlov= stimulus-response strategy, “Win Stay Lose Shift”
- GTFT= Generous reciprocator: cooperate with certain probability even after opponent defection
- "bifurcations" in the space of 2×2, 3×3 game matrices

Outline

- Prisoner's Dilemma stage game
- Discuss iterated strategies
- Construct a 5x5 IPD game
- Investigate various sub-ecologies under Perturbed BR dynamics
  1. 2x2 (*2-cycle*)
  2. 3x3 (*limit cycle, chaos*)
  3. 4x4 (*co-existence of attractors: RSP and chaotic*)
  4. 5x5 (*"breaking of an invariant circle" route to chaos*)

PD Stage Game

$$\begin{bmatrix} r/c & C & D \\ C & b-c, b-c & -c, b \\ D & b, -c & 0, 0 \end{bmatrix}$$

- b benefits of cooperation
- c costs associated with cooperative behavior
- $b > c > 0$

Iterated PD strategies

- random pairing to play an (infinitely) repeated PD game
- focus on *memory-one* strategies
- for each time t state of play b/w two such randomly drawn opponents $\Omega = \{CC, CD, DC, DD\}$
- Iterated strategy:
  1. start with a first random move C or D
  2. play C with probability (r, s, t, p) conditional on realized state at time $t-1$ being CC, CD, DC, DD, respectively

Deterministic Players

- unconditional cooperators:

$$AllC - (1, 1, 1, 1)$$

- unconditional defectors:

$$AllD - (0, 0, 0, 0)$$

- conditional cooperators "Tit-for-Tatters":

$$TFT - (1, 0, 1, 0)$$

- generous conditional cooperators "Generous-Tit-for-Tat":

$$GTFT - (1, m, 1, n)$$

- stimulus-response (Pavlov) "WinStayLoseShift":

$$WSLS - (1, 0, 0, 1)$$

Stochastic Players

- stochastic players: $\varepsilon-$perturbations of the deterministic ones
- ε-mistakes/errors in implementation of the deterministic strategies
- $S_1 = (r, s, t, p)$ vs. $S_2 = (x, y, z, w)$ Markov chain on state space $\Omega = \{ CC, CD, DC, DD\}$ with transition matrix T.
- $\varepsilon > 0$, ergodic chain, unique invariant distribution Ω_τ
- $\Omega_\tau = \{\tau_{CC}, \tau_{CD}, \tau_{DC}, \tau_{DD}\}$- fraction of time system spends in each of the four states in Ω
- $\pi_i(S_i, S_j), (\forall) i, j \in \{AllD, TFT, GTFT, WSLS, AllC\}$

IPD payoff matrix M

$$
\begin{bmatrix}
r/c & AllD & TFT & GTFT & WSLS & AllC \\
AllD & \varepsilon(b-c) & m_{12} & m_{13} & \frac{1}{2}b - c\varepsilon & b - b\varepsilon - c\varepsilon \\
TFT & m_{21} & \frac{1}{2}b - \frac{1}{2}c & m_{23} & \frac{1}{2}b - \frac{1}{2}c & m_{25} \\
GTFT & m_{31} & m_{32} & \frac{n}{n+\varepsilon}(b-c) & m_{34} & m_{35} \\
WSLS & b\varepsilon - \frac{1}{2}c & \frac{1}{2}b - \frac{1}{2}c & m_{43} & m_{44} & b - \frac{1}{2}c - b\varepsilon \\
AllC & b\varepsilon - c + c\varepsilon & m_{52} & m_{53} & m_{54} & (1-\varepsilon)(b-c)
\end{bmatrix}
$$

Evolutionary dynamics of IPD strategies

- "population games" interpretation
- *IPD* strategy revision opportunities
- updating/switching according to fitness
- *IPD* rules ecology evolution:

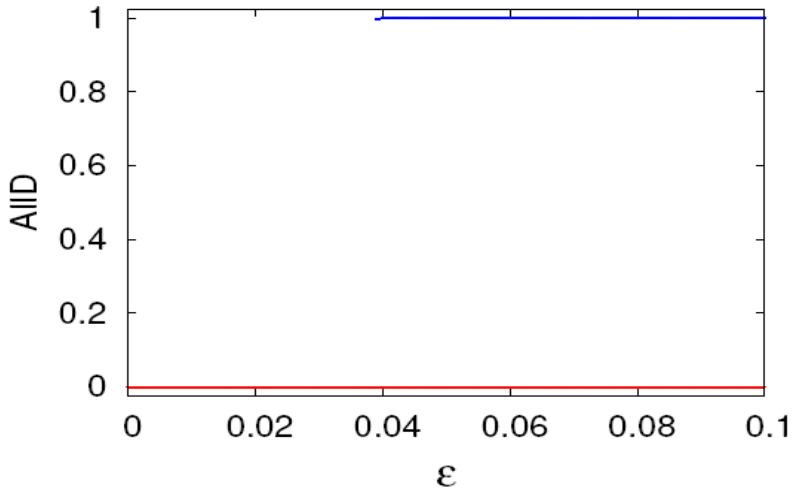
$$x_{i,t+1} = \frac{e^{\beta(M\mathbf{x})_{i,t}}}{\sum_{i=1}^{5} e^{\beta(M\mathbf{x})_{i,t}}}, \sum_{i=1}^{5} x_{i,t} = 1$$

- $\beta \in [0, \infty)$: random vs. best response

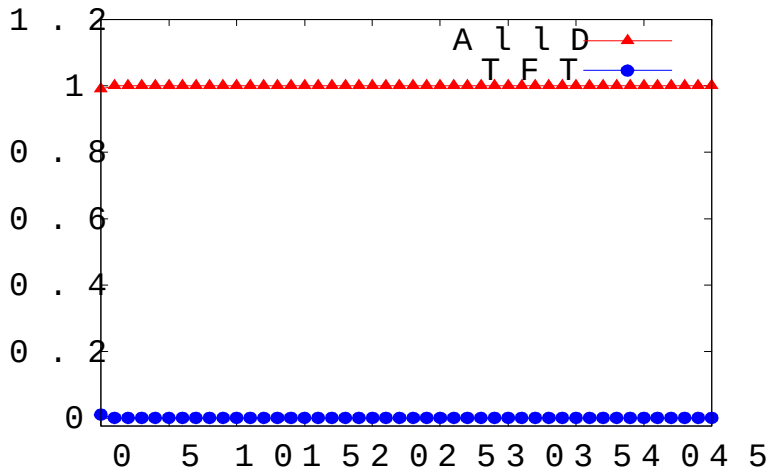
AllD vs. TFT

- WDS(very small ε), Coordination game

$$\begin{bmatrix} & AllD & TFT \\ AllD & \varepsilon(b-c) & -\varepsilon(c-2b+2b\varepsilon) \\ TFT & \varepsilon(b-2c+2c\varepsilon) & \frac{1}{2}b-\frac{1}{2}c \end{bmatrix}$$



(a) $\beta = 200$.Bifurcation diagram

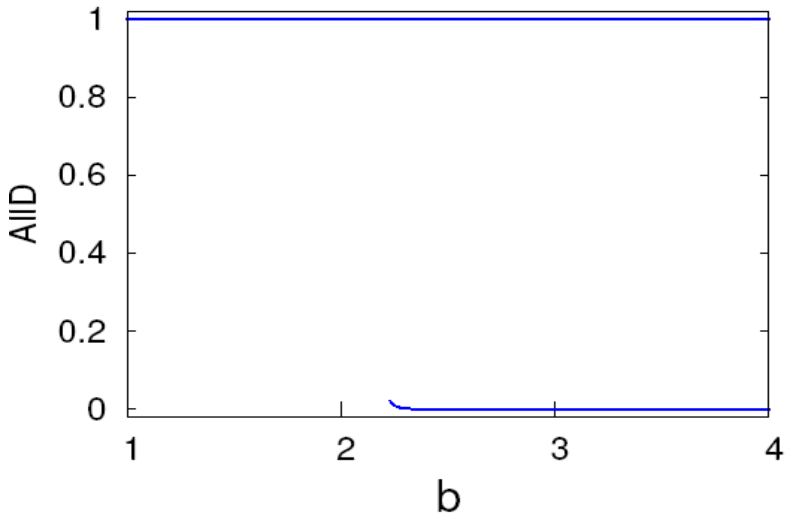


(b) $\varepsilon = 0.05$.

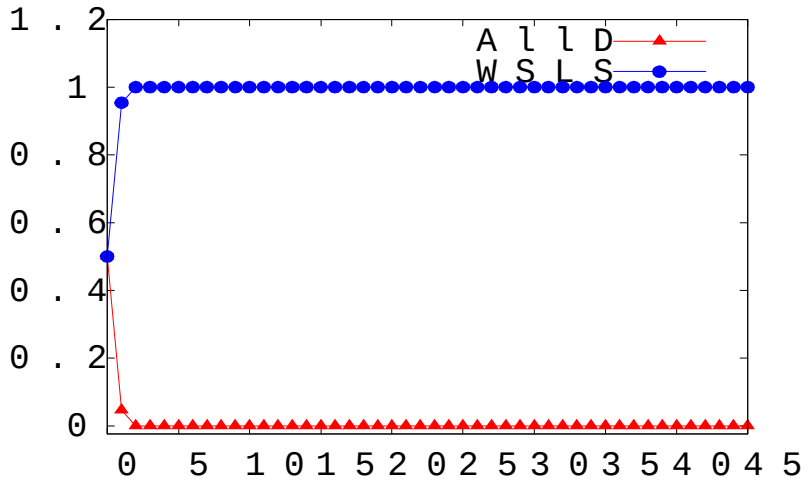
AllD vs. Pavlov

- DS ($b < b^*$), Coordination

$$\begin{bmatrix} AllD & \varepsilon(b-c) & \frac{1}{2}b - c\varepsilon \\ WSLS & b\varepsilon - \frac{1}{2}c & (b-c)\left(1 - 4\varepsilon^3 + 6\varepsilon^2 - 3\varepsilon\right) \end{bmatrix}$$



(a) $\beta = 200, \varepsilon = 0.01$.

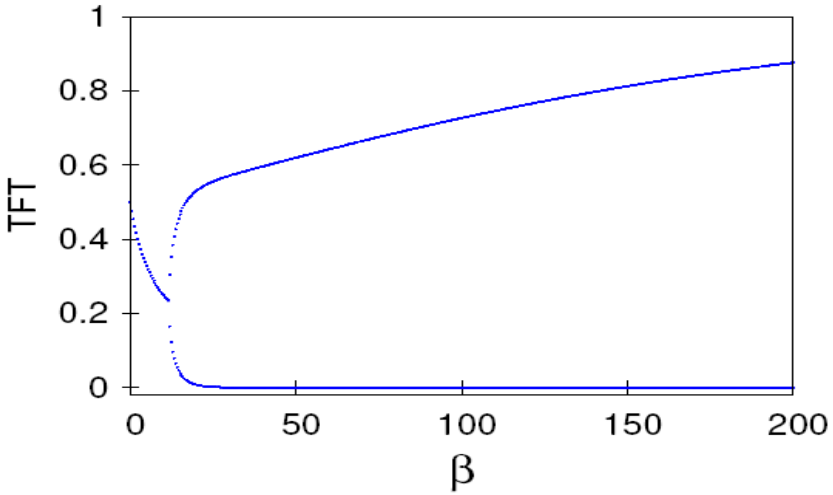


(b) $b = 4.(x_0, y_0) = (50\%, 50\%)$

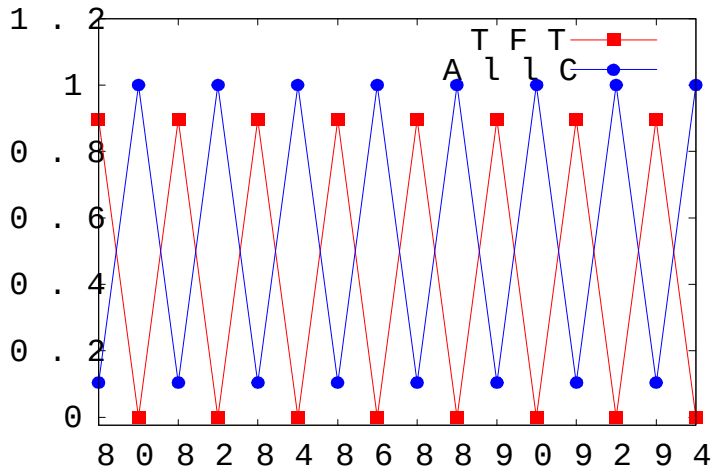
TFT vs. AllC

- HD

$$\begin{bmatrix} r/c & TFT & AllC \\ TFT & \frac{1}{2}(b-c) & b-c-b\varepsilon+2c\varepsilon-2c\varepsilon^2 \\ AllC & b-c-2b\varepsilon+c\varepsilon+2b\varepsilon^2 & (1-\varepsilon)(b-c) \end{bmatrix}$$



(a) $\varepsilon = 0.01$. Bifurcation diagram



(b) large $\beta = 220$.

AllD vs. AllC

- IPD is a PD game itself

$$\begin{bmatrix} r/c & AllD & AllC \\ AllD & \varepsilon(b-c) & b-b\varepsilon-c\varepsilon \\ AllC & b\varepsilon-c+c\varepsilon & (1-\varepsilon)(b-c) \end{bmatrix}$$

Pavlov vs. TFT

- WSLS weakly dominates TFT

$$\begin{bmatrix} r/c & TFT & WSLS \\ TFT & \frac{1}{2}b - \frac{1}{2}c & \frac{1}{2}b - \frac{1}{2}c \\ WSLS & \frac{1}{2}b - \frac{1}{2}c & (b-c)\left(1 - 4\varepsilon^3 + 6\varepsilon^2 - 3\varepsilon\right) \end{bmatrix}$$

GTFT vs. Pavlov

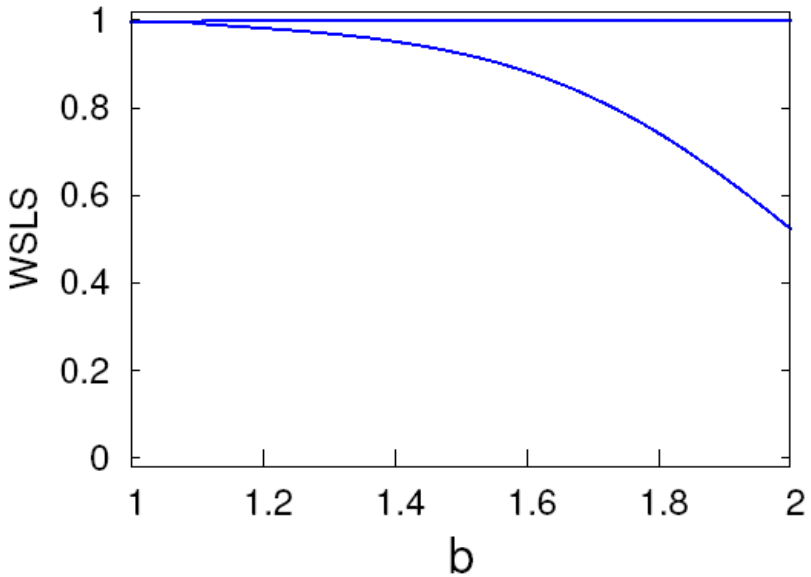
- Coordination

$$\begin{bmatrix} r/c & GTFT & WSLS \\ GTFT & \frac{n}{n+\varepsilon}(b-c) & m_{34} \\ WSLS & m_{43} & (b-c)\left(1-4\varepsilon^3+6\varepsilon^2-3\varepsilon\right) \end{bmatrix}$$

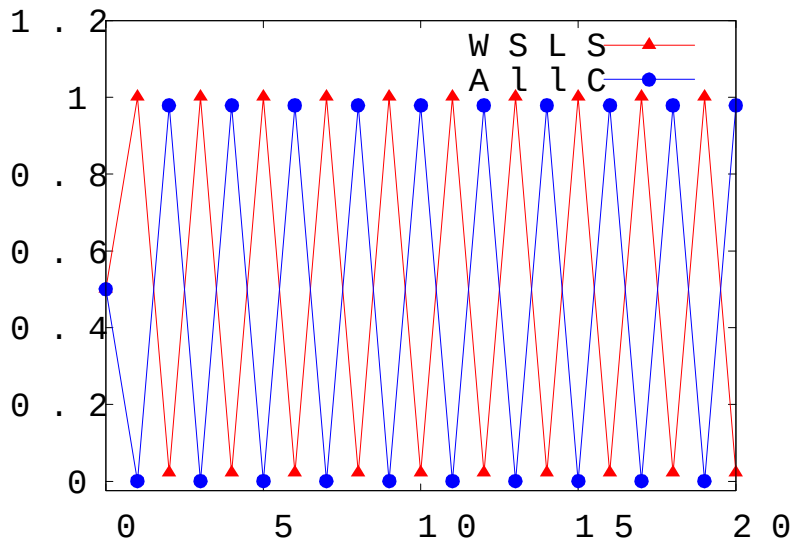
Pavlov vs. AllC

- DS ($b < b^* \approx 2c$), HD

$$\begin{bmatrix} & \textit{WSLS} & \textit{AllC} \\ \textit{WSLS} & (1-\varepsilon)(1-2\varepsilon+4\varepsilon^2)(b-c) & b-\frac{1}{2}c-b\varepsilon \\ \textit{AllC} & b-c-2b\varepsilon+c\varepsilon+2b\varepsilon^2 & (1-\varepsilon)(b-c) \end{bmatrix}$$



(a) $\beta = 500$. Bifurcation diagram



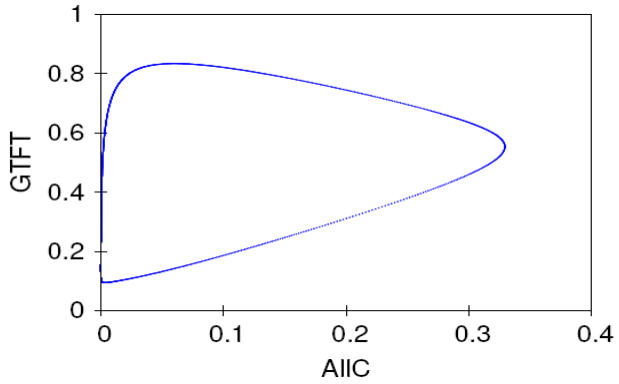
(b) $\beta = 200$.

2x2 Ecologies

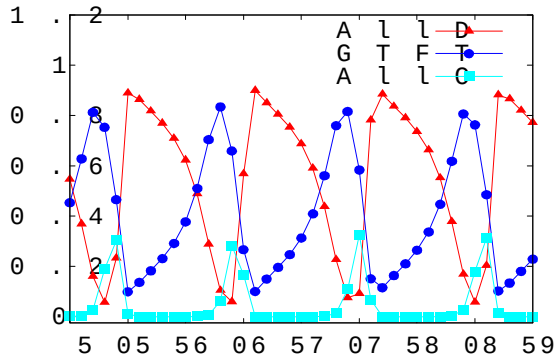
No.	2x2	Game	Bif.	Attractors	Path-
				β large	Dependence
1	AllD-TFT	WDS,C	SN	multiple SS	yes, $\varepsilon = 0.05$
2	TFT-AllC	HD	PD	2-cycle	no
3	AllD-AllC	DS	none	unique SS	no
4	AllD-GTFT	C	SN	multiple SS	yes, $\varepsilon = 0.01$
5	AllD-WSLS	DS,C	SN	multiple SS	yes, $b = 4$
6	GTFT-AllC	HD	PD	2-cycle	yes
7	GTFT-WSLS	C	SN	multiple SS	yes
8	TFT-GTFT	DS, C	PD	2-cycle	no
9	TFT-WSLS	WDS	none	unique SS	no
10	WSLS-AllC	DS,HD	PD	2-cycle	no

AIID-GTFT-AIIC

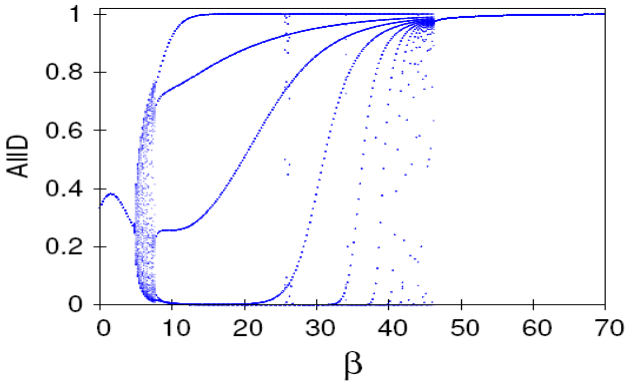
- RSP (moderate β), DS



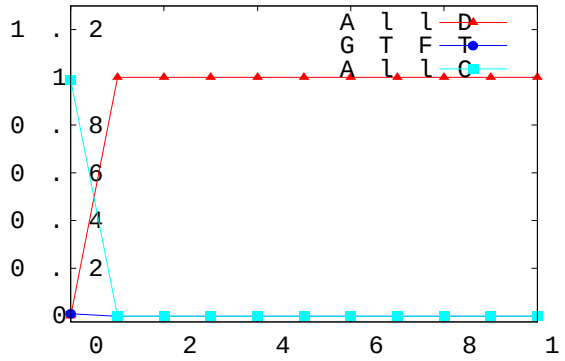
(a) $b = 1.72$.



(b) $b = 1.72$.



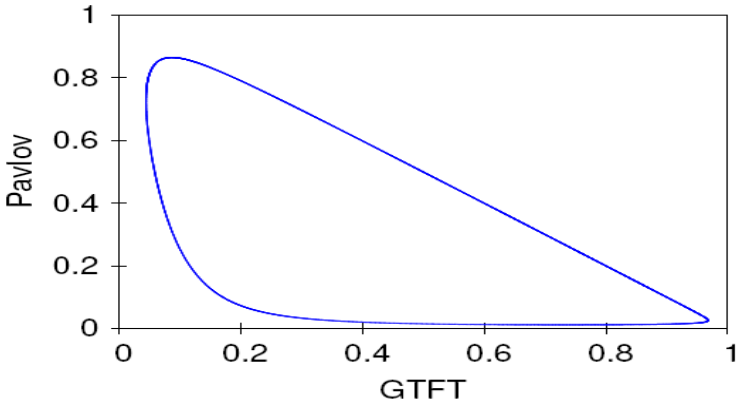
(c) $\varepsilon = 0.01$.



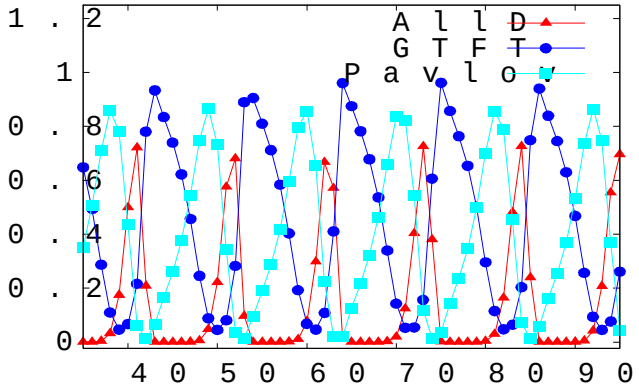
(d) $\beta = 100$.

AllD-GTFT-Pavlov

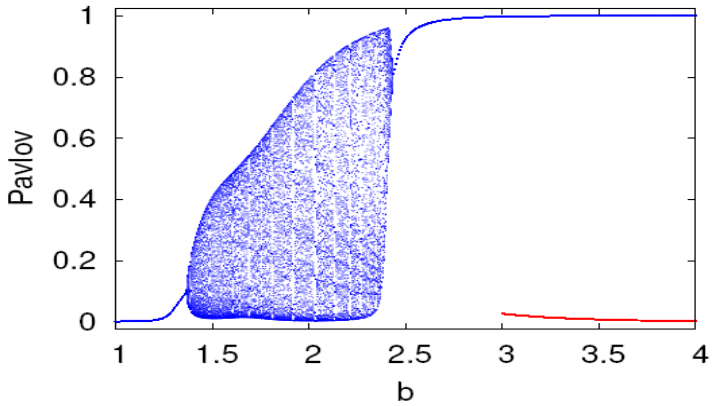
- RSP(low b and moderate β), Coordination (high b,β)



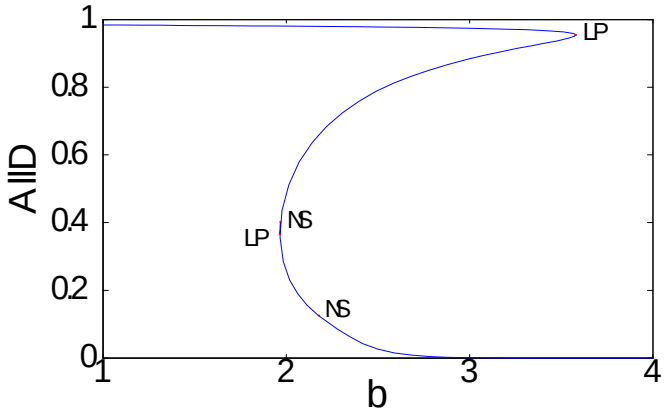
(a) $b = 2.16$.



(b) $b = 2.16$.



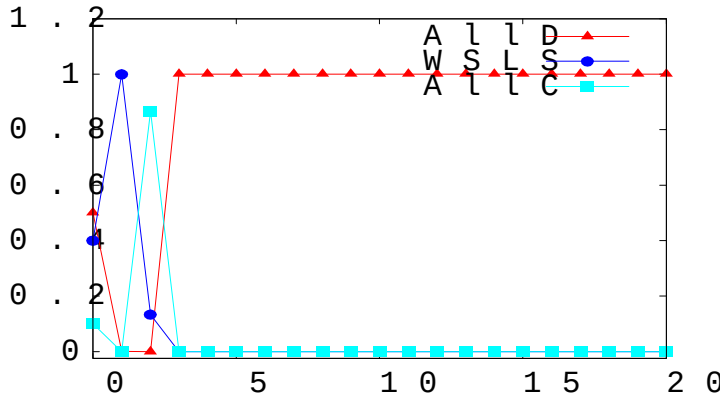
(c) $\beta = 15$.



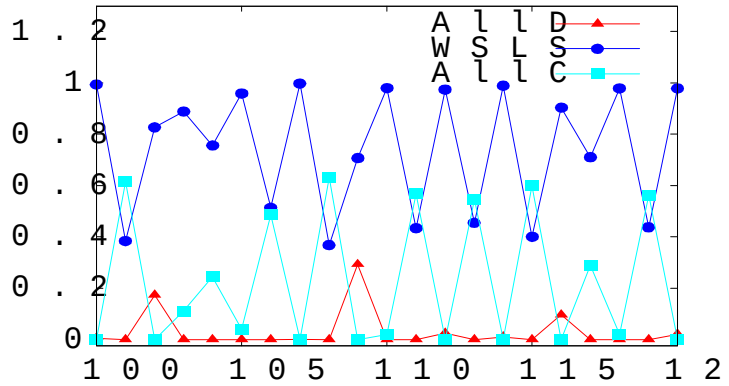
(d) Equilibria curve

AllD-Pavlov-AllC

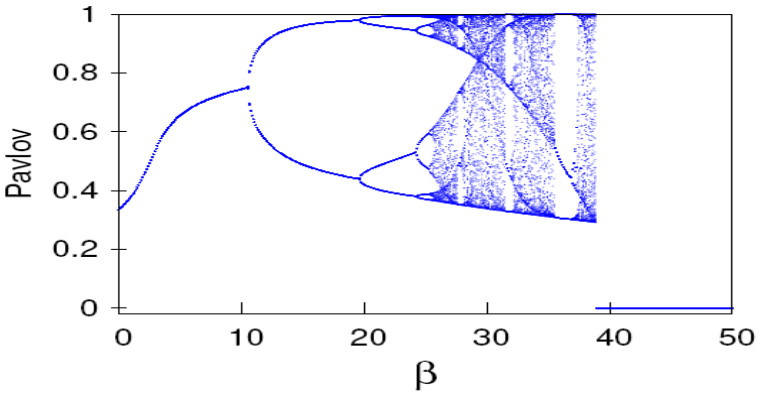
- chaotic co-existence(moderate β), DS



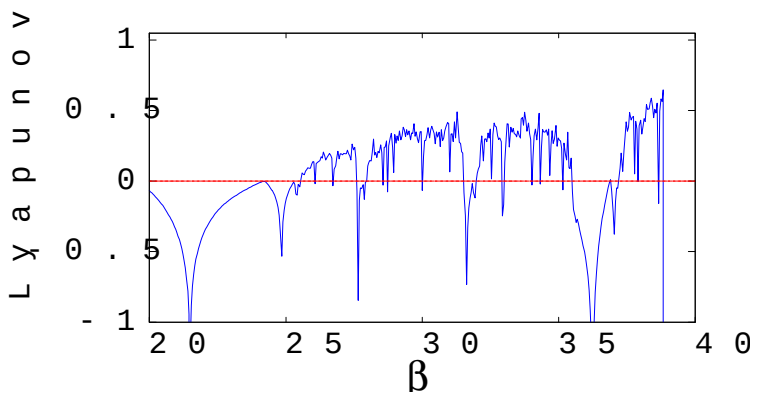
(a) $b > 2c, \beta = 300$.



(b) $b > 2c, \beta = 30$.



(e) $b = 4.4$.



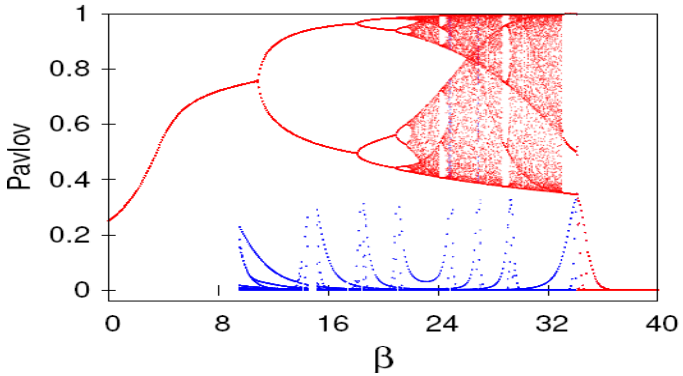
(f) $b = 4.4$.

3x3 Ecologies

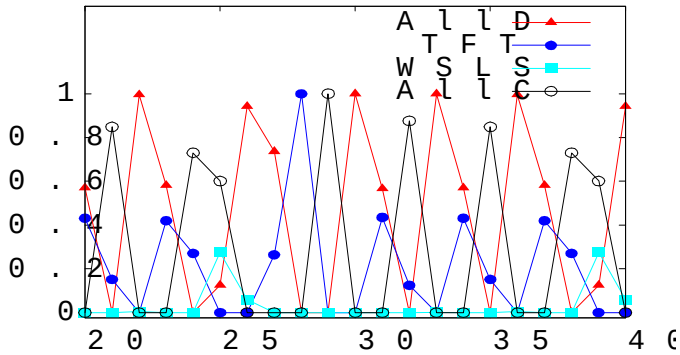
3×3	**Bif**	**Attractors**	
		moderate β	$\beta \to \infty$
AllD-TFT-AllC	NS	limit cycle	3-cycle
AllD-GTFT-WSLS	NS,LP	limit cycle	3 steady states
AllD-GTFT-AllC	NS	limit cycle	1 steady state (AllD)
AllD-TFT-WSLS	NS	limit cycle	1 steady state (Pavlov)
AllD-TFT-GTFT	NS	limit cycle	1 steady state (GTFT)
AllD-WSLS-AllC	PD	2-cycle, chaos	1 steady state (AllD)
TFT-WSLS-AllC	PD	2-cycle	stable steady state
TFT-GTFT-WSLS	PD	2-cycle	2 steady states
GTFT-WSLS-AllC	PD	2-cycle	2-cycle
TFT-GTFT-AllC	PD	2-cycle	2-cycle

AllD-GTFT-Pavlov-AllC

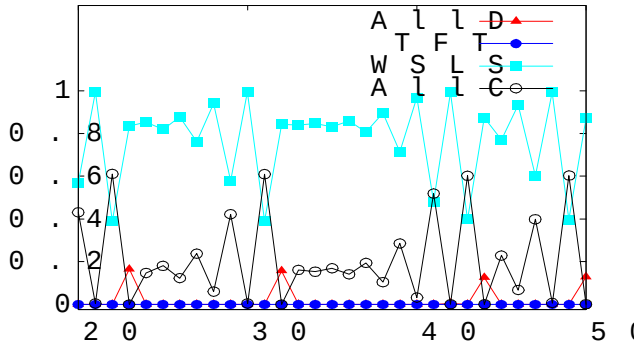
- Co-existence of attractors (RSP and chaos), moderate β



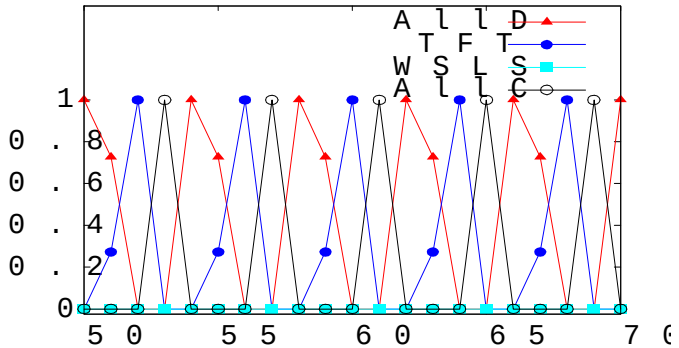
(a) $b = 4$.



(b) small $\beta = 27$.



(c) small $\beta = 27$.



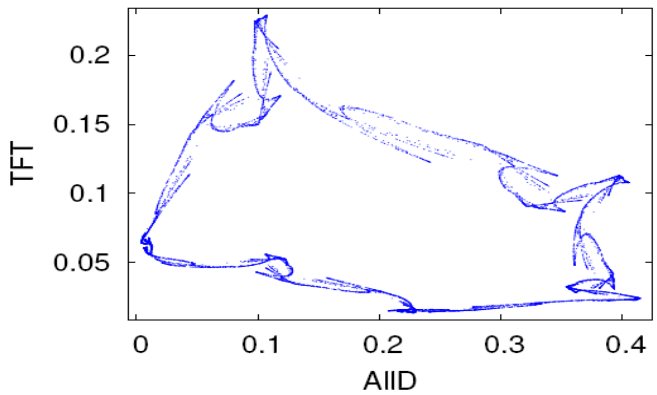
(d) large $\beta = 100$.

Summary

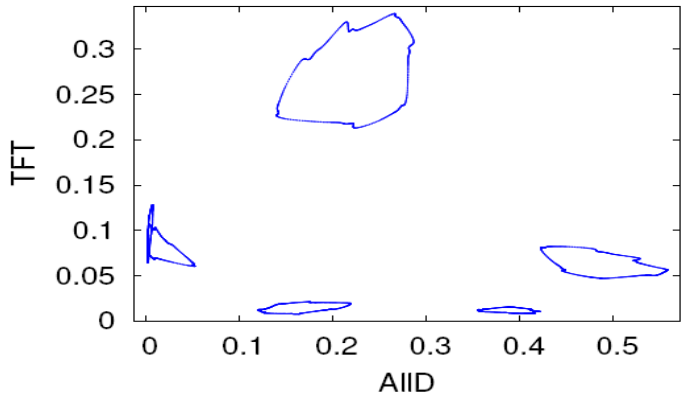
4x4	Bif	Attractors	
		small β	**large β**
No AllD	PD	2-cycle	2-cycle/unique SS
No TFT	PD, NS	co-existence SS and chaos	unique SS (AllD)
No GTFT	PD, NS	co-existence RSP and chaos	unique SS (AllD)
No WSLS	NS	limit cycles, chaos	4-cycle
No AllC	NS, PD	limit cycles, chaos	multiple SS

AllD-TFT-GTFT-Pavlov-AllC

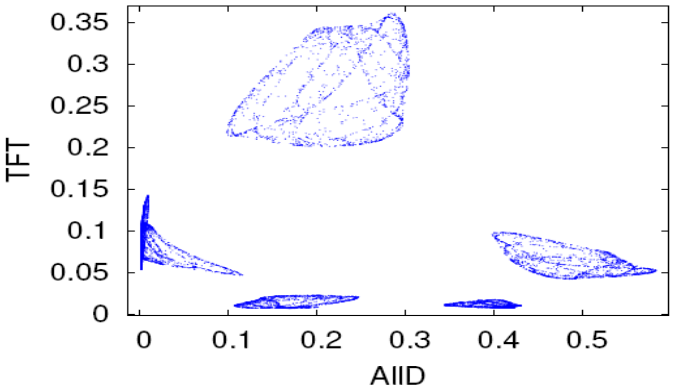
- Full 5x5: "breaking of an invariant circle" route to chaos



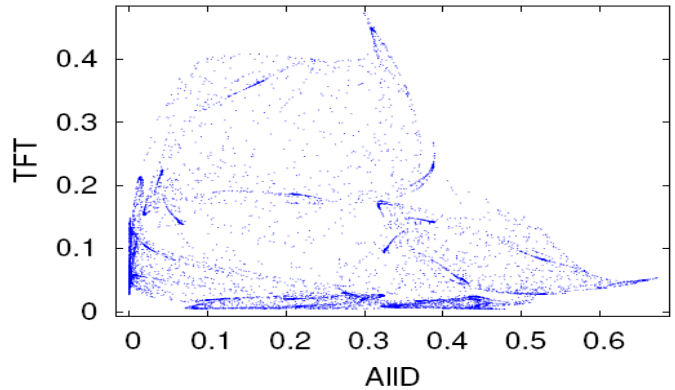
(a) $\beta = 9.05$. 1-piece



(b) $\beta = 10.05$. 6-piece

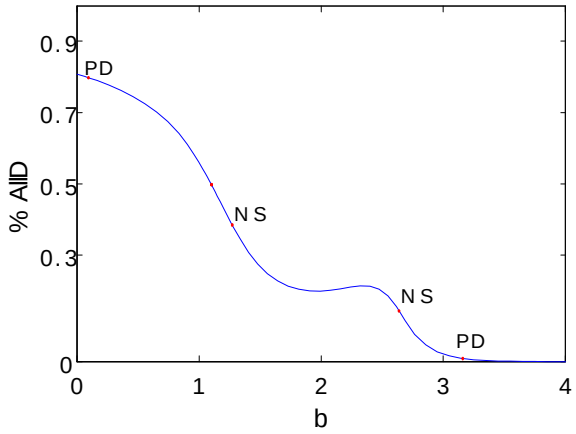


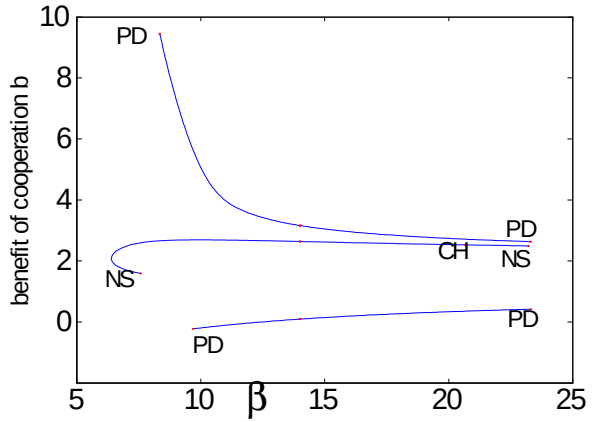
(c) $\beta = 10.2$. 6-piece

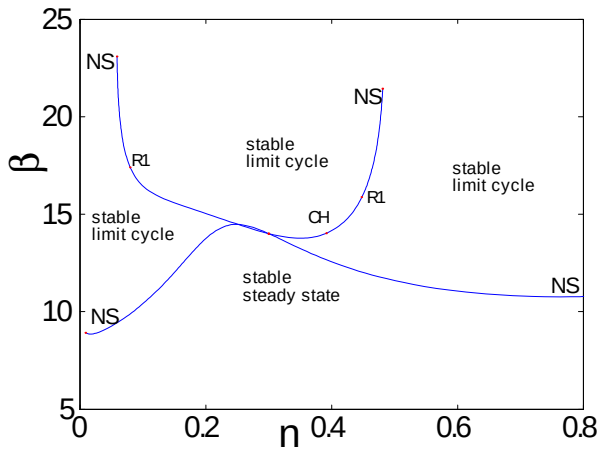


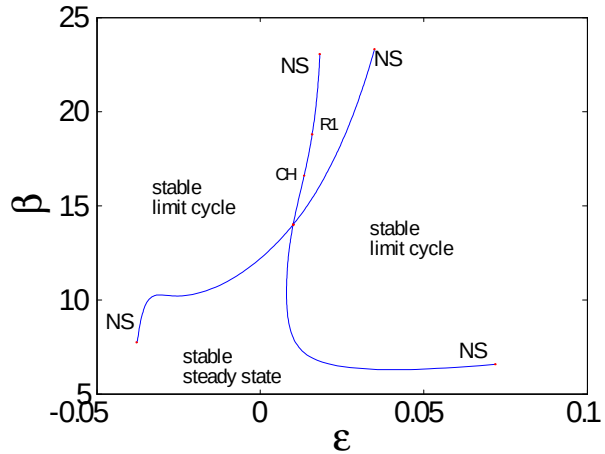
(d) $\beta = 10.8$. 1-piece

Numerical Bifurcation Curves









Concluding remarks

- Abundance of RSP patterns in the 3x3 ecologies of heuristics
- finite β leads to path-dependence and co-existence of cyclical/monomorphic and chaotic attractors
- AllC is detrimental to the discriminating types (TFT, GTFT and WSLS) in the 4x4 ecologies leading to an AllD monomorphism
- mixed evidence for Pavlov: wins the evolutionary competition in those 4x4 environments with hard defectors (AllD) but no AllC players...
- ...but, in the full 5x5 ecology almost goes extinct for $\beta \longrightarrow \infty$
- however, with boundedly rational players, fractions of Pavlov stay high even within the complete ecology of rules.

Future directions

- beyond direct-reciprocity norms
- beyond memory-one strategies
- towards a "bifurcation theory" of games