Universality in the profile of small-dispersion integrable waves: the nonlinear Schrödinger case and other integrable systems

Marco Bertola, Dep. Mathematics and Statistics, Concordia University
Centre de recherches mathématiques (CRM), UdeM.
Alex Tovbis, Dep. of Mathematics University of Central Florida

(IMRN 2009 and arXiv:1004.1828)
Banff, November 1 2010

The notion of "universality"

"Definition"

For a dynamical system/statistical system the notion of "universal behavior" means that a behavior occurs in a certain scaling regime and independently of the solution, or stable under perturbations.

The notion of universality is akin to the *Central limit theorem* in statistics:

$$\frac{\sum_1^N X_j - N\overline{X_j}}{\sigma\sqrt{N}} \to N(0,1) \tag{1}$$

where X_j are IID random variables (with finite second moment $< (X_j - \overline{X_j})^2 >= \sigma^2$) Note the scaling and the scale of the fluctuations (i.e. $\sqrt{N}$). We start with an example of the Korteweg-deVries equation.

The small–dispersion KdV equation (after Dubrovin and Claeys-Grava)

The KdV equation

$$u_t = uu_x + \epsilon^2 u_{xxx} \;, \qquad u(x,0) = u_0(x) \qquad \text{rapidly decaying} \tag{2}$$

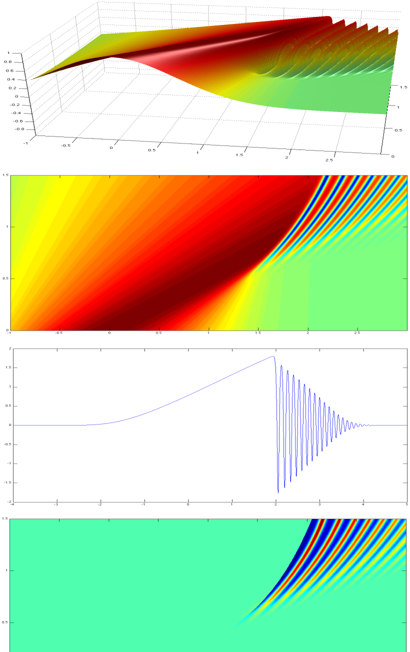
For $\epsilon = 0$ we have Burger's equation $u_t = uu_x$, solved by the hodograph method (characteristics), locally

$$f(u) = x + u\,t \qquad f(u) = u_0^{-1} \tag{3}$$

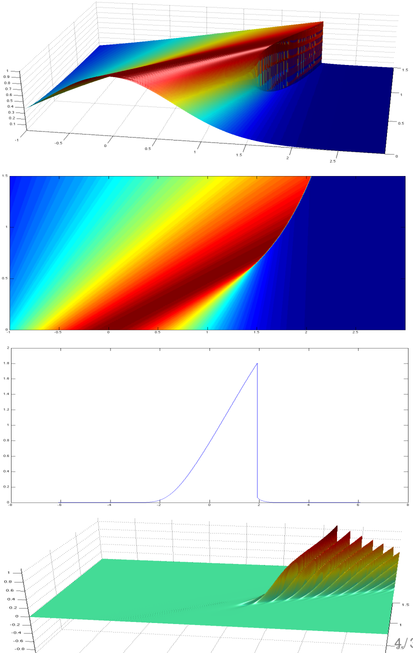
It shocks at $t_0 = \frac{1}{\max u_0'(x)}$.

- Near the point of gradient catastrophe (x_0, t_0) its behavior is described in terms of a generalization of the Painlevé I equation with critical scale $\varepsilon^{\frac{6}{7}}$;
- Near the trailing edge (after the time t_0) it is described by the Hastings-McLeod solution of the Painlevé II equation $y''(s) = sy(s) + 2y^3(s)$ with critical scale $\varepsilon^{\frac{2}{3}}$;
- Near the leading edge the behavior is described in terms of elementary function (superposition of soliton solutions) with scale $\varepsilon \ln \varepsilon$.

KdV-small dispersion



KdV-zero dispersion = Burgers



Focusing Nonlinear Schrödinger (NLS) equation

The focusing Nonlinear Schrödinger (NLS) equation,

$$i\varepsilon\partial_t q = -\varepsilon^2\partial_x^2 q - 2|q|^2 q \tag{4}$$

$$q(x,0,\varepsilon) = A(x)e^{i\Phi(x)/\varepsilon} \tag{5}$$

models self-focusing and self-modulation (*optical fibers*). It is **integrable** by inverse scattering methods (Zakharov–Shabat). We study $\varepsilon \to 0$; in different regions of spacetime, there are different asymptotic behaviors (*phases*) separated by **breaking curves** (or **nonlinear caustics**).

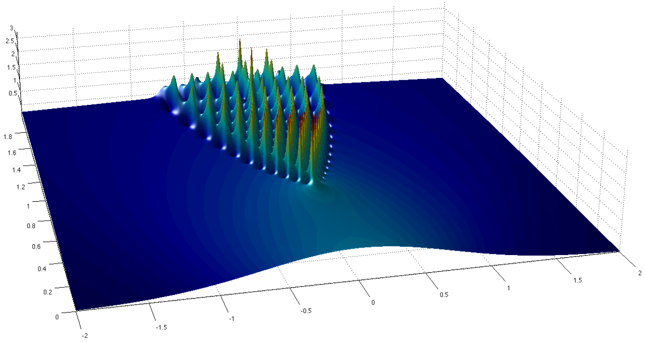


Figure: The case $A(x) = e^{-x^2}$, $\Phi'(x) = \tanh x$ and $\varepsilon = 0.03$

The tip-point of the braking curves is called a point of **gradient catastrophe**, or **elliptic umbilical singularity** [Dubrovin-Grava-Klein].

Main goal

Leading order asymptotic $q(x,t,\varepsilon)$ on and around the gradient catastrophe point (x_0,t_0).

The behavior in the bulk is described in terms of slow modulation of exact quasi-periodic solutions (**genus 2**), while outside by slow modulation equations for the amplitude. There are (generically) two types of **transitional regions**

- A strip region of scale $\mathcal{O}(\varepsilon \ln \varepsilon)$ around the *breaking curves* (nonlinear caustics);
- a circular region of scale $\mathcal{O}(\varepsilon^{\frac{4}{5}})$ around the gradient catastrophe point.

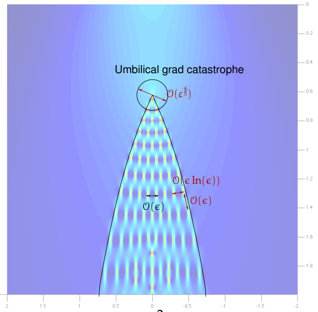


Figure: $A(x) = e^{-x^2}$, $\Phi'(x) = \tanh x$ and $\varepsilon = 0.03$

Common features of transitional regions

The scale of the quasi-periodic structure in the oscillatory region is $\mathcal{O}(\varepsilon)$ while in the transitional regions a different (longer) scale is typically involved; the **critical exponent** of this scale depends on the region.

Around the breaking curve

"Universal" expression for the behavior of the first oscillations as we egress from the genus zero region into the genus two one; does not depend upon the details of the initial data, or rather, it depends on it only through a few parameters that are explicitly computable.

- The first oscillations have nonzero amplitude ($\varepsilon \to 0$);
- they are periodic of period $\mathcal{O}(\varepsilon)$ in the **tangential direction** to the breaking curve;
- the relative correction to the amplitude is $\mathcal{O}(1)$ only at (discrete) distances in the **transversal** direction with separation of order $\mathcal{O}(\varepsilon|\ln\varepsilon|)$

The gradient catastrophe point

Separating amplitude and phase

$$q(x,t) = b(x,t)e^{\frac{i}{\varepsilon}\Phi(x,t)}\ , \quad U := |q|^2, V = \Phi_x \tag{6}$$

the equation is recast

$$U_t + (UV)_x = 0\ , \qquad V_t + VV_x - U_x + \frac{\varepsilon^2}{2}\left(\frac{1}{2}\frac{U_x^2}{U^2} - \frac{U_{xx}}{U}\right)_x = 0 \tag{7}$$

Neglecting the green term yields an elliptic system, with a finite lifespan; they develop singularities in the derivatives at (x_0, t_0).

What is the behavior in the vicinity of (x_0, t_0)?

The gradient catastrophe point

Separating amplitude and phase

$$q(x,t) = b(x,t)\mathrm{e}^{\frac{i}{\varepsilon}\Phi(x,t)}\ , \quad U := |q|^2, V = \Phi_x \tag{6}$$

the equation is recast

$$U_t + (UV)_x = 0\ , \qquad V_t + VV_x - U_x + \frac{\varepsilon^2}{2}\left(\frac{1}{2}\frac{U_x^2}{U^2} - \frac{U_{xx}}{U}\right)_x = 0 \tag{7}$$

Neglecting the green term yields an elliptic system, with a finite lifespan; they develop singularities in the derivatives at (x_0, t_0).

Conjecture (Dubrovin-Grava-Klein (2007); Theorem in B.-Tovbis (2010))

Let $x = x_0 + \varepsilon^{\frac{4}{5}}X\ ,\ \ t = t_0 + \varepsilon^{\frac{4}{5}}T$*; then (* $\alpha := -2V + i\sqrt{U}$*)*

$$U + i\sqrt{U_0}V = U_0 + i\sqrt{U_0}V_0 + \varepsilon^{\frac{2}{5}}\frac{4ib_0}{C}y(v) + \mathcal{O}(\varepsilon^{\frac{3}{5}}) \tag{8}$$

where

$$v = -i\sqrt{\frac{2i\sqrt{U_0}}{C}}\left(X + 2(i\sqrt{U_0} - 4V_0)T\right)(1 + \mathcal{O}(\varepsilon^{\frac{1}{5}})) \tag{9}$$

and $y(v)$ *is the* **tritronquée solution** *of the Painlevé II equation*

$$y'' = 6y^2 - v \tag{10}$$

The Tritronquée solution

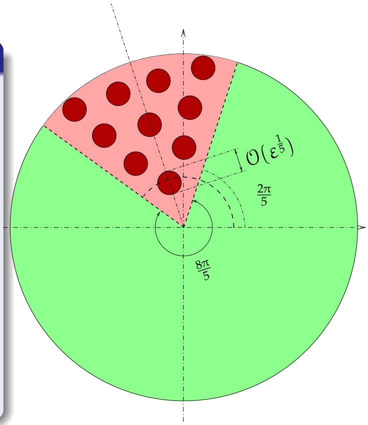
$$y'' = 6y^2 - v \tag{11}$$

Theorem (Kapaev (2004))

There exists a unique solution $y(v)$ with the asymptotics

$$y = \sqrt{\frac{e^{-i\pi}}{6} v} + \mathcal{O}\left(v^{-2}\right), \quad v \to \infty,$$
$$\arg(v) \in \left[-\frac{6\pi}{5} + 0, \frac{2\pi}{5} - 0\right]. \tag{12}$$

Such a solution has no poles for $|v|$ large enough in the above sector (or –equivalently– has at most a finite number of poles within said sector).



Discussion

The conjecture was formulated in the genus-zero region;

Question

How far into the oscillatory region can the conjecture be pushed?

Discussion

The conjecture was formulated in the genus-zero region;
The function $y(v)$ has **double poles** in a region of the v–plane (conjecturally) contained within a sector of with $\frac{2\pi}{5}$; this region corresponds to the oscillatory region near the grad. cat.

$$y(v) = -\frac{1}{(v - v_0)^2} + \mathcal{O}(v - v_0)^2 \tag{13}$$

Since the correction is $\mathcal{O}(\varepsilon^{\frac{2}{5}} y(v))$ we see that

The "correction term" becomes a **leading order term** as $v - v_0 = \mathcal{O}(\varepsilon^{\frac{1}{5}})$

Therefore

The asymptotics is different in a region $\mathcal{O}(\varepsilon^{\frac{1}{5}})$ around the pole in the v–plane = $\mathcal{O}(\varepsilon)$ in the physical plane around a *spike*

Zooming in on a peak

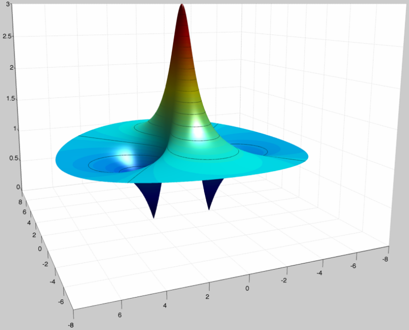
If we scale by ε around each peak we find the **Peregrine breather**

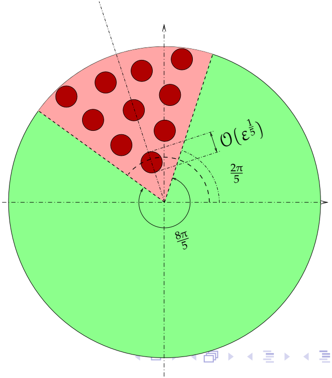
$$q(x,t,\mathrm{e}) = \mathrm{e}^{\frac{i}{\varepsilon}\Phi(x_p,t_p)} Q_{br}\left(\frac{x-x_{p,j}}{\varepsilon}, \frac{t-t_{p,j}}{\varepsilon}\right)(1+\mathcal{O}(\varepsilon^{\frac{1}{5}})), \tag{14}$$

where the rational breather

$$Q_{br}(\xi,\eta) = \mathrm{e}^{-2i\left(a\xi+(2a^2-b^2)\eta\right)} b\left(1-4\frac{1+4ib^2\eta}{1+4b^2(\xi+4a\eta)^2+16b^4\eta^2}\right) \tag{15}$$

$$i\partial_\eta Q_{br} + \partial_\xi^2 Q_{br} + 2|Q_{br}|^2 Q_{br} = 0 \tag{16}$$



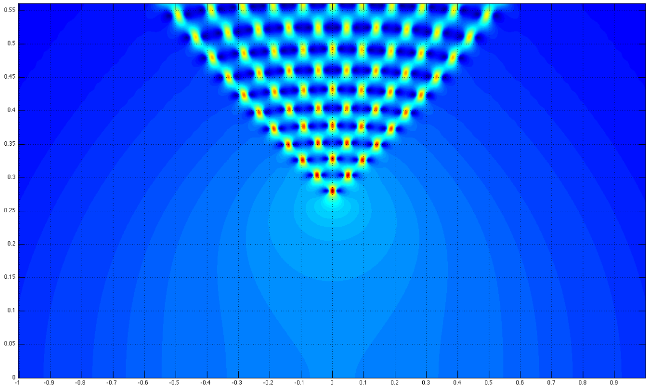


In our case it is obtained from the "stationary" breather

$$Q^0_{br}(\xi,\eta) = e^{2i\eta}\left(1 - 4\frac{1+4i\eta}{1+4\xi^2+16\eta^2}\right) \tag{17}$$

by applying the transformations (mapping solutions into solutions)

$$\widetilde{Q}(\xi,\eta) = \lambda Q(\lambda\xi,\lambda^2\eta), \qquad \widehat{Q}(\xi,\eta) = e^{i(kx-k^2\eta)}Q(\xi-2k\eta,\eta). \tag{18}$$



Ideally these peaks will get very “ sparse” near the gradient catastrophe:

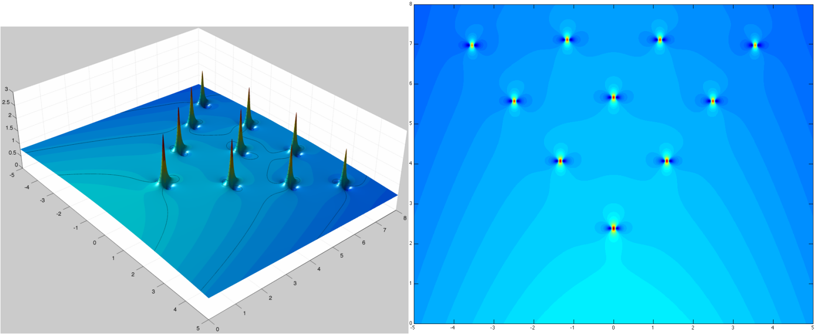


Figure: A mock-up of what would happen for very small ε (location of peaks modeled after numerics for the poles of the tritronquée)

1. **poles of *tritronquée*** $\Leftrightarrow$ spikes of amplitude of q; can be used to find location in spacetime of the peaks after the grad. cat.;

$$v(x,t,\mathrm{e}) = \frac{e^{-i\pi/4}}{\varepsilon^{\frac{4}{5}}}\sqrt{\frac{2b}{C}}\,[\delta x + 2(2a+ib)\delta t]\left(1+\mathcal{O}(\varepsilon^{\frac{2}{5}})\right) \tag{19}$$

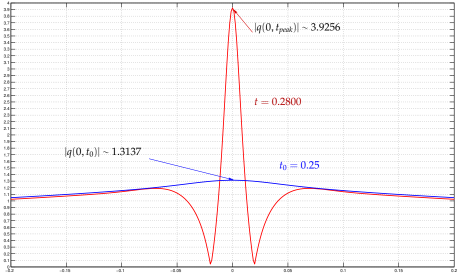


Figure: $q(x,0) = \frac{1}{\cosh(x)}$ and $\varepsilon = \frac{1}{33}$; note that $3|q_0| = 3.9411$. In this case $\mu = 0$ and $t_0 = \frac{1}{4}$. The time of the first peak (numerically 0.2800) matches the prediction from the Tritronquée (0.2791260482)

Summarizing: B.-Tovbis (2010)

1. **poles of *tritronquée*** $\Leftrightarrow$ spikes of amplitude of q; can be used to find location in spacetime of the peaks after the grad. cat.;

$$v(x,t,\mathrm{e}) = \frac{e^{-i\pi/4}}{\varepsilon^{\frac{4}{5}}}\sqrt{\frac{2b}{C}}\,[\delta x + 2(2a+ib)\delta t]\left(1+\mathcal{O}(\varepsilon^{\frac{2}{5}})\right) \tag{19}$$

2. Height of each spike $= 3|q_0(x_0,t_0)| + \mathcal{O}(\varepsilon^{1/5})$;
3. **Universal shape**

$$q(x,t,\mathrm{e}) = \mathrm{e}^{\frac{i}{\varepsilon}\Phi(x_p,t_p)}Q_{br}\left(\frac{x-x_{p,j}}{\varepsilon},\frac{t-t_{p,j}}{\varepsilon}\right)(1+\mathcal{O}(\varepsilon^{\frac{1}{5}})), \tag{20}$$

The two "roots" and the maximum are synchronous.

4. Away from the spikes

$$q(x,t,\mathrm{e}) = \left(b - 2\varepsilon^{\frac{2}{5}}\Im\left(\frac{y(v)}{C}\right) + \mathcal{O}(\varepsilon^{\frac{3}{5}})\right)\times$$
$$\exp\frac{2i}{\varepsilon}\left[\frac{1}{2}\Phi(x_0,t_0) - (a\,\delta x - (2a^2-b^2)\delta t) + \varepsilon^{\frac{6}{5}}\Re\left(\sqrt{\frac{2i}{Cb}}H_I(v)\right)\right] \tag{21}$$

$H_I = \frac{1}{2}(y'(v))^2 + vy(v) - 2y^3(v)$. Equation (21) is consistent with the conjecture.

Summarizing: B.-Tovbis (2010)

1. **poles of *tritronquée*** $\Leftrightarrow$ spikes of amplitude of q; can be used to find location in spacetime of the peaks after the grad. cat.;

$$v(x,t,\mathrm{e}) = \frac{e^{-i\pi/4}}{\varepsilon^{\frac{4}{5}}}\sqrt{\frac{2b}{C}}\,[\delta x + 2(2a+ib)\delta t]\left(1+\mathcal{O}(\varepsilon^{\frac{2}{5}})\right) \qquad (19)$$

2. Height of each spike $= 3|q_0(x_0,t_0)| + \mathcal{O}(\varepsilon^{1/5})$;

3. **Universal shape**

$$q(x,t,\mathrm{e}) = \mathrm{e}^{\frac{i}{\varepsilon}\Phi(x_p,t_p)} Q_{br}\left(\frac{x-x_{p,j}}{\varepsilon}, \frac{t-t_{p,j}}{\varepsilon}\right)(1+\mathcal{O}(\varepsilon^{\frac{1}{5}})), \qquad (20)$$

The two "roots" and the maximum are synchronous.

4. Away from the spikes

$$q(x,t,\mathrm{e}) = \left(b - 2\varepsilon^{\frac{2}{5}}\Im\left(\frac{y(v)}{C}\right) + \mathcal{O}(\varepsilon^{\frac{3}{5}})\right)\times$$
$$\exp\frac{2i}{\varepsilon}\left[\frac{1}{2}\Phi(x_0,t_0) - (a\,\delta x - (2a^2-b^2)\delta t) + \varepsilon^{\frac{6}{5}}\Re\left(\sqrt{\frac{2i}{Cb}}H_I(v)\right)\right] \qquad (21)$$

$H_I = \frac{1}{2}(y'(v))^2 + vy(v) - 2y^3(v)$. Equation (21) is consistent with the conjecture.

Summarizing: B.-Tovbis (2010)

1. **poles of *tritronquée*** $\Leftrightarrow$ spikes of amplitude of q; can be used to find location in spacetime of the peaks after the grad. cat.;

$$v(x,t,\mathrm{e}) = \frac{e^{-i\pi/4}}{\varepsilon^{\frac{4}{5}}}\sqrt{\frac{2b}{C}}\,[\delta x + 2(2a+ib)\delta t]\left(1+\mathcal{O}(\varepsilon^{\frac{2}{5}})\right) \tag{19}$$

2. Height of each spike $= 3|q_0(x_0,t_0)| + \mathcal{O}(\varepsilon^{1/5})$;
3. **Universal shape**

$$q(x,t,\mathrm{e}) = \mathrm{e}^{\frac{i}{\varepsilon}\Phi(x_p,t_p)} Q_{br}\left(\frac{x-x_{p,j}}{\varepsilon}, \frac{t-t_{p,j}}{\varepsilon}\right)(1+\mathcal{O}(\varepsilon^{\frac{1}{5}})), \tag{20}$$

The two "roots" and the maximum are synchronous.

4. Away from the spikes

$$q(x,t,\mathrm{e}) = \left(b - 2\varepsilon^{\frac{2}{5}}\Im\left(\frac{y(v)}{C}\right) + \mathcal{O}(\varepsilon^{\frac{3}{5}})\right)\times$$
$$\exp\frac{2i}{\varepsilon}\left[\frac{1}{2}\Phi(x_0,t_0) - (a\,\delta x - (2a^2-b^2)\delta t) + \varepsilon^{\frac{6}{5}}\Re\left(\sqrt{\frac{2i}{Cb}}H_I(v)\right)\right] \tag{21}$$

$H_I = \frac{1}{2}(y'(v))^2 + vy(v) - 2y^3(v)$. Equation (21) is consistent with the conjecture.

Some details on the proof

- Uses inverse scattering plus nonlinear steepest descent;
- Involves some new analysis for Painlevé I near a pole, following Masoero (2009);
- Shows clearly that higher breaks involve the PI hierarchy.

The g–function and the geometry of the breaking curve

The exact evolution

Initial data $q(x, 0, \varepsilon)$	$\xrightarrow{\text{Spectral transform}}$	Spectral data $r(z, \epsilon) = e^{-\frac{i}{2\varepsilon} f_0(z)}$
fNLS $\downarrow$		$\downarrow$ spacetime evolution
$q(x, t, \varepsilon)$	$\xleftarrow{\text{RHP}}$	$r(z, x, t, \epsilon) = e^{-\frac{i}{2\varepsilon} f(z,x,t)}$ $f(z, x, t) := f_0(z) - xz - 2tz^2$

The g–function and the geometry of the breaking curve

The approximate evolution

Initial data $q(x,0,\varepsilon)$	$\overset{\text{Spectral transform}}{\longrightarrow}$	Spectral data $r(z,\epsilon) = e^{-\frac{i}{2\varepsilon} f_0(z)}$
fNLS $\downarrow$		$\downarrow$ spacetime evolution
$q(x,t,\varepsilon)$	$\overset{\text{RHP}}{\longleftarrow}$	$r(z,x,t,\epsilon) = e^{-\frac{i}{2\varepsilon} f(z;x,t)}$
$\downarrow$		$\downarrow$ g–function
Approx $\widetilde{q}(x,t,\varepsilon)$	$\overset{\text{Deift–Zhou nonlin. steep.st dsnt}}{\longleftarrow}$	$e^{\frac{i}{2\varepsilon}(2g-f)}$

$$h(z;x,t) := 2g(z;x,t) - f_0(z) + xz + 2tz^2 \tag{22}$$

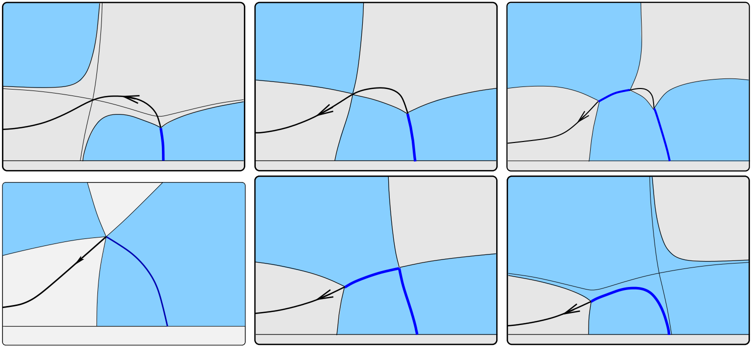
The g–function then is obtained as solution of a scalar RHP on a free boundary with the jump $f(z)$

(Nonlinear) Steepest descent contour

[In the lower half plane symmetric statements]

The contour(s) of the RHP must be homologic to a contour $\gamma = \gamma_m \cup \gamma_c$ where

- $g_+(z) + g_-(z) = f(z; x, t)$ for $z \in \gamma_m$ (**blue** contour);
- $g(z)$ is analytic off γ_m;
- $h(z; x, t) := 2g(z; x, t) - f(z; x, t)$ is such that $\Im h < 0$ on **both sides of** γ_m;
- $\Im h(z) \geqslant 0$ on γ_c (**black** contour).



Generically in (x,t) we have

$$h(z;x,t) = C_0(x,t)(z-\alpha)^{\frac{3}{2}} + C_1(x,t)(z-\alpha)^{\frac{5}{2}} + \dots \quad (23)$$

$$C_0 = \frac{\sqrt{\alpha-\overline{\alpha}}}{3\pi}\oint \frac{f'(\zeta)\,\mathrm{d}\zeta}{(\zeta-\alpha)(R(\zeta)+} \quad (24)$$

At the g.c. point (x_0,t_0) we have $C_0 = 0$;

$$h(z;x,t) = (z-\alpha)^{\frac{5}{2}}\,(C_1(x,t) + \dots) \quad (25)$$

$$C_1 = \frac{2\sqrt{\alpha-\overline{\alpha}}}{15\pi}\oint \frac{f''(\zeta)\,\mathrm{d}\zeta}{(\zeta-\alpha)(R(\zeta)+} \quad (26)$$

In a neighborhood of (x_0,t_0) C_0 is a **deformation (unfolding)** of the critical point.

Theorem

We can find a conformal change of coordinate and an analytic function τ of C_0 s.t.

$$\frac{i}{\varepsilon}h(z;x,t) = \frac{4}{5}\zeta^{\frac{5}{2}} + \tau(x,t)\zeta^{\frac{3}{2}} \quad (27)$$

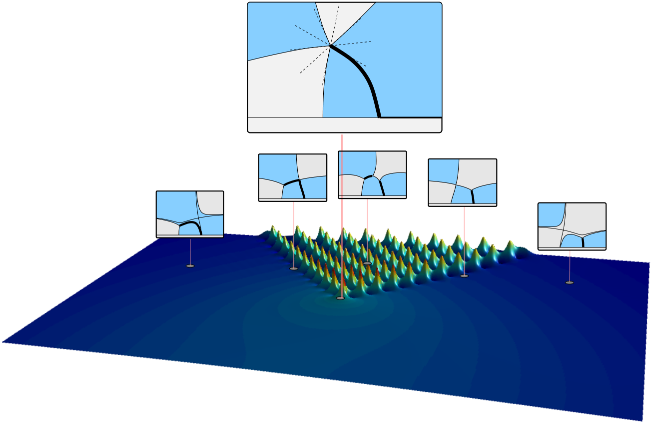
$$v := \frac{3}{8}\tau^2(x,t;\varepsilon) = -i\sqrt{\alpha_0 - \overline{\alpha}_0}\sqrt[5]{\frac{4}{5C_1}}\left(\frac{\delta x + 2(\alpha_0 + a_0)\delta t}{\varepsilon^{\frac{4}{5}}}\right)(1 + \mathcal{O}\varepsilon^{\frac{2}{5}}) \quad (28)$$

$$C_1 = \frac{2\sqrt{\alpha - \overline{\alpha}}}{15\pi}\oint\frac{f''(\zeta)\,d\zeta}{(\zeta - \alpha)(R(\zeta) +} \quad (29)$$

This expression travel its way to the phase of the PI ψ function

Parametrix with poles

Using the nonlinear steepest descent:



Parametrix with poles

Using the nonlinear steepest descent:

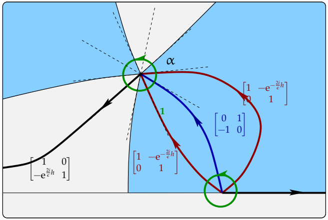


Figure: The jumps for the RHP for Y. The shaded region is where $\Im h < 0$ (the "sea"). The blue contour is the main arc, the black contour is the complementary arc.

Near the point α we need to solve the RHP in exact form ($\longrightarrow$ Painlevé I)

Problem (Painlevé 1 RHP (Kapaev))

The matrix $\mathbf{P}(\xi;v)$ is locally bounded, admits boundary values on the rays shown in Fig. 5 and satisfies

$$\mathbf{P}_+ = \mathbf{P}_- M,$$
$$\mathbf{P}(\xi) = \frac{\xi^{\sigma_3/4}}{\sqrt{2}}\begin{bmatrix}1 & -i\\ 1 & i\end{bmatrix}\left(I + \frac{H_I\sigma_3}{\sqrt{\xi}} + \dots\right),$$
$$H_I = \frac{1}{2}(y')^2 + vy - 2y^3 = \int y(v)\,\mathrm{d}v$$

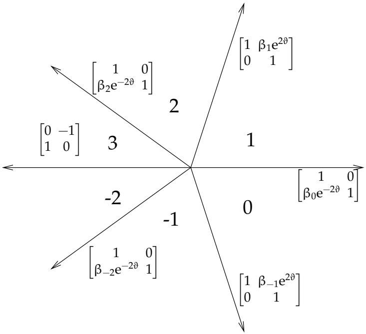


Figure: $\vartheta := \vartheta(\xi;v) := \frac{4}{5}\xi^{\frac{5}{2}} - v\xi^{\frac{1}{2}}.$

$$1 + \beta_0\beta_1 = -\beta_{-2}, \quad 1 + \beta_0\beta_{-1} = -\beta_2, \quad 1 + \beta_{-2}\beta_{-1} = \beta_1, \tag{30}$$

For exceptional values of v the RHP has **no solution** ; these values correspond to (double) poles of $y(v)$

Tritronqées solutions

They correspond to (cyclic permutations of)

$$\beta_{-1} = 0 = \beta_0 \qquad \beta_1 = 1 = -\beta_2 = -\beta_{-2} \tag{31}$$

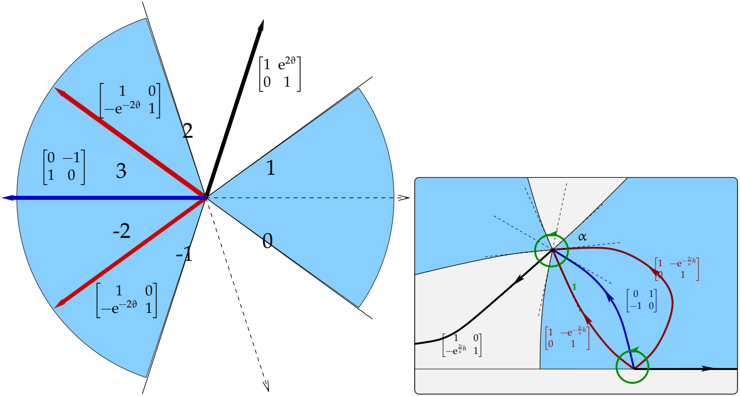


Figure: $\vartheta := \vartheta(\xi; v) := \frac{4}{5}\xi^{\frac{5}{2}} - v\xi^{\frac{1}{2}}$.

Painlevé I near a pole

For exceptional values of v the RHP has **no solution** , i.e. $\mathbf{P}(\xi;v)$ has a pole; these values correspond to (double) poles of $y(v)$.
In a neighborhood of $v = v_0$ a pole of $y(v)$ (Masoero (2009))

$$\widehat{\mathbf{P}}(\xi;v) := G(\xi;v)\mathbf{P}(\xi;v), \tag{32}$$

$$G(\xi;v) := \begin{bmatrix} 0 & 1 \\ 1 & -\frac{1}{2}\left(y' + \frac{1}{2(\xi-y)}\right) \end{bmatrix} (\xi - y)^{\sigma_3/2}.$$

has **no pole!**

We need some information of how the solution $\mathbf{P}$ becomes $\widehat{\mathbf{P}}$, i.e. some asymptotics valid for **both** ξ and y large;

Theorem (B.-Tovbis 2010)

$$\hat{\mathbf{P}}(\xi, v) = \xi^{-\frac{3}{4}\sigma_3} \frac{1}{\sqrt{2}} \left(\begin{bmatrix} 1 & i \\ 1 & -i \end{bmatrix} + \mathcal{O}\left(\xi^{-\frac{1}{2}}, y^{-4}, e^{-p_2 \frac{|y|^{5/2}}{|\xi_0|^{5/2}}} \right) \right) \left(\frac{\sqrt{\xi} + \sqrt{y}}{\sqrt{\xi - y}} \right)^{\sigma_3}$$

The blue term is crucial: if ξ, y are of the same order then it is not the identity matrix; it forces modifications of the model-parametrix.
The exponent $-\frac{3}{4}\sigma_3$ is responsible for the amplitude at the top of the peak; the three comes from the shearing of the ODE (**Cubic Schrödinger**)

$$f''(\xi) - \left(2\xi^3 - v_0\xi - 14\beta\right) f(\xi) = 0 \tag{33}$$

To match the behavior of the parametrix

$$\hat{\mathbf{P}}(\xi, v) = \xi^{-\frac{3}{4}\sigma_3} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & i \\ 1 & -i \end{bmatrix} (\mathbf{1} + \dots) \tag{34}$$

we need a different Model problem because of the exponent $\frac{3}{4}$

Schlesinger chain

$$\Psi_K(z) := \frac{1}{2} \begin{bmatrix} -i & -1 \\ 1 & i \end{bmatrix} \left(\frac{z - \alpha}{z - \overline{\alpha}} \right)^{\left(\frac{1}{4} - K\right)\sigma_3} \begin{bmatrix} i & 1 \\ -1 & -i \end{bmatrix}, \qquad K \in \mathbb{Z}, \tag{35}$$

are related by a left-multiplication by a rational matrix

$$\Psi_K(z) = R_K(z) \Psi_0(z), \tag{36}$$

Modified Model parametrix

Schlesinger chain

$$\Psi_K(z) := \frac{1}{2}\begin{bmatrix} -i & -1 \\ 1 & i \end{bmatrix}\left(\frac{z-\alpha}{z-\overline{\alpha}}\right)^{\left(\frac{1}{4}-K\right)\sigma_3}\begin{bmatrix} i & 1 \\ -1 & -i \end{bmatrix}, \qquad K \in \mathbb{Z}, \qquad (34)$$

1. $K = 0 \quad \leftrightarrow \quad$ Airy Parametrix$/PI_{(1)}$ away from pole;
2. $K = 1 \quad \leftrightarrow \quad PI_{(1)}$ at the pole $(-\frac{3}{4}\sigma_3)$;
3. $K = -1 \quad \leftrightarrow \quad PI_{(2)}$ at the pole $(\frac{5}{4}\sigma_3)$;
4. $K = 2 \quad \leftrightarrow \quad PI_{(3)}$ at the pole $(-\frac{7}{4}\sigma_3)$;
5. $K = -2 \quad \leftrightarrow \quad PI_{(4)}$ at the pole $(\frac{9}{4}\sigma_3)$;
6. etc.

Approximation and Error

$$\mathcal{P}_{1;\alpha}(z) = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ i & i \end{bmatrix} \zeta^{\frac{3}{4}\sigma_3} \widehat{\mathbf{P}}\left(\zeta + \frac{\tau}{2}; \frac{3}{8}\tau^2\right) \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} e^{\left(\frac{i}{\varepsilon}h - \vartheta\right)\sigma_3}, \tag{35}$$

$$\Upsilon(z) = \begin{cases} \mathcal{E}(z)\Psi_1(z) & \text{for } z \textbf{ outside} \text{ of the disks } \mathbb{D}_\alpha, \mathbb{D}_{\overline{\alpha}}, \\ \mathcal{E}(z)\Psi_1(z)\mathcal{P}_{1;\alpha}(z) & \text{for } z \textbf{ inside} \text{ of the disk } \mathbb{D}_\alpha, \\ \mathcal{E}(z)\Psi_1(z)\mathcal{P}_{1;\overline{\alpha}}(z) & \text{for } z \textbf{ inside} \text{ of the disk } \mathbb{D}_{\overline{\alpha}}. \end{cases} \tag{36}$$

The jump of $\mathcal{E}(z)$ on the boundary is (leading term)

$$\mathcal{E}_+ = \mathcal{E}_-\Psi_1 \left(\frac{\sqrt{1-\zeta/y}}{1+\sqrt{\zeta/y}}\right)^{\sigma_3} \Psi_1^{-1} \tag{37}$$

The jump of $\mathcal{E}(z)$ on the boundary is (leading term)

$$\mathcal{E}_+ = \mathcal{E}_- \Psi_1 \left(\frac{\sqrt{1-\zeta/y}}{1+\sqrt{\zeta/y}} \right)^{\sigma_3} \Psi_1^{-1} \tag{38}$$

On the boundary $|\zeta| = \mathcal{O}(\varepsilon^{\frac{2}{5}})$ (from Thm. 4)

- If $\zeta/y > 1$ then the local parametrix has a singularity within the local disk $\Rightarrow$ "standard" PI needed;
- if $|\zeta/y| << 1$ (e.g. $y = \infty$) then we are at the pole: the jump is identity and Ψ_1 is a good approx (see later)
- if $1 > \zeta/y = \mathcal{O}(1)$ then the jump is not small! Luckily this RHP is **exactly solvable** and the solution affects the model parametrix (and yields the shape in the end).

The jump of $\mathcal{E}(z)$ on the boundary is (leading term)

$$\mathcal{E}_+ = \mathcal{E}_- \Psi_1 \left(\frac{\sqrt{1-\zeta/y}}{1+\sqrt{\zeta/y}} \right)^{\sigma_3} \Psi_1^{-1} \tag{38}$$

On the boundary $|\zeta| = \mathcal{O}(\varepsilon^{\frac{2}{5}})$ (from Thm. 4)

- If $\zeta/y > 1$ then the local parametrix has a singularity within the local disk $\Rightarrow$ "standard" PI needed;
- if $|\zeta/y| << 1$ (e.g. $y = \infty$) then we are at the pole: the jump is identity and Ψ_1 is a good approx (see later)
- if $1 > \zeta/y = \mathcal{O}(1)$ then the jump is not small! Luckily this RHP is **exactly solvable** and the solution affects the model parametrix (and yields the shape in the end).

The jump of $\mathcal{E}(z)$ on the boundary is (leading term)

$$\mathcal{E}_+ = \mathcal{E}_- \Psi_1 \left(\frac{\sqrt{1 - \zeta/y}}{1 + \sqrt{\zeta/y}} \right)^{\sigma_3} \Psi_1^{-1} \tag{38}$$

On the boundary $|\zeta| = \mathcal{O}(\varepsilon^{\frac{2}{5}})$ (from Thm. 4)

- If $\zeta/y > 1$ then the local parametrix has a singularity within the local disk $\Rightarrow$ "standard" PI needed;
- if $|\zeta/y| << 1$ (e.g. $y = \infty$) then we are at the pole: the jump is identity and Ψ_1 is a good approx (see later)
- if $1 > \zeta/y = \mathcal{O}(1)$ then the jump is not small! Luckily this RHP is **exactly solvable** and the solution affects the model parametrix (and yields the shape in the end).

The amplitude of the peak: $y = \infty$

For $y = \infty$ (i.e. exactly on a pole of the tritronqée) Ψ_1 is a good approx for the solution ($\alpha = a + ib$)

$$\Psi_1(z) = \frac{1}{2}\begin{bmatrix} -i & -1 \\ 1 & i \end{bmatrix}\left(\frac{z-\alpha}{z-\overline{\alpha}}\right)^{\frac{3}{4}\sigma_3}\begin{bmatrix} i & 1 \\ -1 & -i \end{bmatrix}, \qquad K \in \mathbb{Z}, \tag{39}$$

$$\Psi_1(z) = \left(\mathbf{1} + \frac{3}{2}\frac{b}{z}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} + \mathcal{O}(z^{-2})\right). \tag{40}$$

$$q(x,t,\mathrm{e}) = -2\mathrm{e}^{\frac{i}{\varepsilon}\Phi(x,t)} \lim_{z\to\infty} z(\Psi_1)_{12} = -3\mathrm{e}^{\frac{i}{\varepsilon}\Phi(x,t)} b(x,t) \tag{41}$$

Conjectures and outlook

1. The method applies to higher breaks; necessity of analysis near a pole of the PI hierarchy;

Conjectures and outlook

1. The method applies to higher breaks; necessity of analysis near a pole of the PI hierarchy;
2. For poles of $PI_{(k)}$ we have Schrödinger equations with polynomials of degree $2k+1$; the shearing yields amplitudes (and alternating phases).

Conjectures and outlook

1. The method applies to higher breaks; necessity of analysis near a pole of the PI hierarchy;
2. For poles of $PI_{(k)}$ we have Schrödinger equations with polynomials of degree $2k+1$; the shearing yields amplitudes (and alternating phases).
3. The phenomenon of "poles in the local parametrix that disappear in the solution" should be general to problems with conjugate Riemann invariants;

Conjectures and outlook

1. The method applies to higher breaks; necessity of analysis near a pole of the PI hierarchy;
2. For poles of $PI_{(k)}$ we have Schrödinger equations with polynomials of degree $2k+1$; the shearing yields amplitudes (and alternating phases).
3. The phenomenon of "poles in the local parametrix that disappear in the solution" should be general to problems with conjugate Riemann invariants;
4. Two-humps: what happens at the crossroad of two breaking curves?