LOCC in Operator Algebra Language and the NPT conjecture

Mary Beth Ruskai
marybeth.ruskai@tufts.edu

Tufts University

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Basic set-up

$$\mathcal{H} = \mathbf{C}_d \qquad \mathfrak{A} = \mathcal{B}(\mathcal{H}) = M_d$$

will use tensor products $\mathfrak{A}_1 \otimes \mathfrak{A}_2 \otimes \ldots \mathfrak{A}_n$ or $\mathfrak{A} \otimes \mathfrak{B}$

local means acts only on components of a tensor product

$\mathfrak{C} \subset M_d$ denotes classical algebra of diagonal matrices in some M_d

algebra associated with "Alice" $\mathfrak{A}_A$ or $\mathfrak{A}$

algebra associated with "Bob" $\mathfrak{A}_B$ or $\mathfrak{B}$

identify pure state with vector $|\psi\rangle \in \mathcal{H}$ or better

rank one projection $\rho = |\psi\rangle\langle\psi|$

Mixed states and Entanglement

Mixed state is convex comb of pure $\rho = \sum_k p_k |\chi_k\rangle\langle\chi_k|$

$|\chi_k\rangle$ need not be O.N. – not nec spectral decomp.

Identify state with density matrix ρ, i.e., $\rho > 0, \quad \text{Tr}\,\rho = 1$

defines pos lin fctnl $\mathfrak{A} \mapsto \mathbf{C}$ given by $A \mapsto \text{Tr}\,\rho A$

pure $|\psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$ entangled if it is not product $|\phi_A \otimes \phi_B\rangle$

Def: ρ is separable if convex comb of prods $\rho = \sum_k p_k |\phi_k^A \otimes \phi_k^B\rangle$

$|\phi_k\rangle$ need not be O.N. - spectral decomp not prod in general

Pure $|\psi\rangle \in \mathcal{H} \otimes \mathcal{H}$ is maximally entangled if $\text{Tr}_B\,|\psi\rangle\langle\psi| = \frac{1}{d} I_A$

Examples: $|\psi\rangle = \sum_k \frac{1}{\sqrt{d}} e^{i\theta_k} |\phi_k^A \otimes \chi_k^B\rangle \qquad d_A = d_b = d.$

Can find O.N. basis for $\mathcal{H} \otimes \mathcal{H}$ consisting of max entang states.

Example: Teleportation

$d = 2$ Max entangled Bell state: e-bit or EPR pair

Def: $|\beta_k\rangle = (I \otimes \sigma_k)|\beta_0\rangle$ $\quad |\beta_0\rangle = \frac{1}{\sqrt{2}}\big(|00\rangle + |11\rangle\big)$

For prod of qubits $\mathcal{H}_{A'} \otimes \mathcal{H}_A \otimes \mathcal{H}_B$, i.e., all $\mathcal{H} = \mathbf{C}_2$

$$|\phi \otimes \beta_0\rangle = \sum_{k=0}^{3} \tfrac{1}{2} |\beta_k\rangle \otimes \sigma_k |\phi\rangle$$

- Alice and Bob share max entang $|\beta_0\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$
- Alice also has unknown state $|\phi\rangle \in \mathcal{H}_{A'}$ get state $|\phi\rangle \otimes |\beta_0\rangle$
- A makes "Bell" meas.; gets one of $|\beta_k\rangle \otimes \sigma_k|\phi\rangle$ each with prob $\frac{1}{4}$
- Alice learns k from meas – calls and tells Bob to apply σ_k
- Bob ends up with $\sigma_k^2|\phi\rangle = |\phi\rangle$ exactly what Alice had in $\mathcal{H}_{A'}$.

LOCC

Teleportation transfers $|\phi\rangle$ from A' to B using only LOCC

e-bit or EPR pair is important resource in quant info proc.

LOCC = Local Operations and Classical Communication

$$\mathfrak{A}_A \otimes \mathfrak{A}_B \qquad \text{or} \qquad \mathfrak{A} \otimes \mathfrak{B}$$

LO just means a trace-decreasing CP map of form $\Phi_A \otimes \Phi_B$

$$\Phi_A : \mathfrak{A} \mapsto \mathfrak{A}' \qquad \Phi_B : \mathfrak{B} \mapsto \mathfrak{B}'$$

not all authors agree – can be more restrictive wlog

but lose flavor of process

to explain CC review measurement

Quantum basics and von Neumann measurement

Fund Postulate of Q.M.: Observable represented by self-adj op A

spectral decomp $A = \sum_k a_k E_k = \sum_k a_k |\alpha_k\rangle\langle\alpha_k|$

Measurement of A with system in some state ψ.

(i) get some e-value (only possibility)

(ii) leave system in e-state α_k

(iii) probability is $|\langle\alpha_k, \psi\rangle|^2 = \mathrm{Tr}\, E_k |\psi\rangle\langle\psi|$

Write $|\psi\rangle = \sum_k c_k |\alpha_k\rangle$ as a superposition of e-states, $c_k = \langle\alpha_k, \psi\rangle$

Coefficients c_k in superpos. give probs $|c_k|^2$ not classical

Average result of meas in state $|\psi\rangle$ is $\langle\psi, A\psi\rangle = \mathrm{Tr}\, A|\psi\rangle\langle\psi|$

Av result of meas in mixed state $\rho = \sum_k p_k |\psi_k\rangle\langle\psi_k|$ is $\mathrm{Tr}\, A\rho$

set $\{E_k\}$ orthog projections $E_j E_k = E_k \delta_{jk}$ with $\sum_k E_k = I$ called

von Neumann measurement or projection valued measure (PVM)

corresponds to “yes-no” experiment (e.g., polarization filter)

CPT map $\Omega_{\mathcal{M}} : \mathfrak{A} \mapsto \mathfrak{C}$ givees result of PVM or vN measurement

$$\Omega_{\mathcal{M}} : \rho \mapsto \sum_j E_j \rho E_j = \sum_j |\alpha_j\rangle\langle\alpha_j, \rho\, \alpha_j\rangle\langle\alpha_j| = \sum_j E_j \,\mathrm{Tr}\, \rho\, E_j$$

QC quantum-classical $\{E_j\}$ O.N. $\Rightarrow$ output in subalg iso to $\mathfrak{C}$

Now consider two non-commuting observables

$$A = \sum_j a_j |\alpha_j\rangle\langle\alpha_j| = \sum_j a_j E_j, \qquad B = \sum_k b_k |\beta_k\rangle\langle\beta_k| = \sum_k b_k F_k$$

- measure A, then B ends in e-state $|\beta_k\rangle$ or F_k of B
- measure B, then A ends in e-state $|\alpha_j\rangle$ or E_j of A

Measure B, then A ends with $F_k \mapsto \Omega_{\mathcal{M}}(F_k) = \sum_j E_j F_k E_j$

Quantum measurement: POVM

$$\sum_{jk} E_j F_k E_j = \sum_j E_j I E_j = I$$

$\{E_j F_k E_j\}$ example of POVM positive operator valued measurement

Def: (Davies-Lewis) POVM $\mathcal{M} = \{G_m\}$ $\quad G_m > 0,\ \sum_m G_m = I$

Result of POVM depends on order in which G_m performed

QC map using instrument with class "pointer" $|f_m\rangle$ O.N.

$$\begin{aligned} \Omega_{\mathcal{M}} : \rho \ &\mapsto \ \sum_m (\operatorname{Tr} \rho\, G_m) |\phi_m\rangle\langle\phi_m| \otimes |f_m\rangle\langle f_m| \\ &= \ \bigoplus_m (\operatorname{Tr} \rho\, G_m)\, |\phi_m\rangle\langle\phi_m| \end{aligned}$$

$$\Omega_{\mathcal{M}} : \mathfrak{A} \mapsto \mathfrak{A} \otimes \mathfrak{C} \simeq \bigoplus \mathfrak{A}$$

Classical Communication

Recall POVM meas. $\Omega_{\mathcal{M}} : \mathfrak{A} \mapsto \mathfrak{A} \otimes \mathfrak{C} \simeq \bigoplus \mathfrak{A}$

have state $\rho_{AB} \in \mathfrak{A} \otimes \mathfrak{B}$

$$(\Omega_{\mathcal{M}} \otimes \mathcal{I})(\rho_{AB}) = \bigoplus_{M} |\phi_m\rangle\langle\phi_m| \otimes \operatorname{Tr}_A \rho_{AB} G_m$$

local meas $(\Omega_{\mathcal{M}} \otimes \mathcal{I}) : \mathfrak{A} \otimes \mathfrak{B} \mapsto (\mathfrak{A} \otimes \mathfrak{C}) \otimes \mathfrak{B} \simeq \mathfrak{A} \otimes \mathfrak{C} \otimes \mathfrak{B}$

math trivial equiv $(\mathfrak{A} \otimes \mathfrak{C}) \otimes \mathfrak{B} \simeq \mathfrak{A} \otimes \mathfrak{C} \otimes \mathfrak{B} \simeq \mathfrak{A} \otimes (\mathfrak{C} \otimes \mathfrak{B})$

class algebra is shared – gives (one-way) classical communication

- one-way: A or B does all measurements, e.g., $\Omega_{\mathcal{M}_A} \otimes \mathcal{I}_B$
- two-way: either A or B can measure, $\Omega_{\mathcal{M}_A} \otimes \mathcal{I}_B$ or $\mathcal{I}_A \otimes \Omega_{\mathcal{M}_B}$

next LO can be conditioned on classical algebra

Teleportation revisited

$$|\phi \otimes \beta_0\rangle\langle\phi \otimes \beta_0| \xrightarrow{\Omega_{\mathcal{M}_{A'A}-\text{Bell}}} \frac{1}{2}\bigoplus_k |\beta_k\rangle\langle\beta_k| \otimes \sigma_k|\phi\rangle\langle\phi|\sigma_k$$

$$= \frac{1}{2}\bigoplus_k |\beta_k\rangle\langle\beta_k| \otimes \Gamma_k(|\phi\rangle\langle\phi|)$$

$$\xrightarrow{\mathcal{I}_{A'A}\otimes(\oplus\Gamma_k)} \left(\frac{1}{2}\oplus_k |\beta_k\rangle\langle\beta_k|\right) \otimes |\phi\rangle\langle\phi|$$

$\Gamma_k(\rho) \equiv \sigma_k \rho \sigma_k^*$ unitary conj

More gen CC step $(\Omega_{\mathcal{M}} \otimes \mathcal{I})$: yields $\oplus_k \rho_k \in \oplus \mathfrak{A} \otimes \mathfrak{B}$

Apply cond LO of form $\mathcal{I} \otimes \Phi_B = \mathcal{I} \otimes \oplus_k \Phi_k$ with $\Phi_k : \mathfrak{B} \mapsto \mathfrak{B}'$

Distillation

Typical situation:

$$\mathfrak{A} = \mathfrak{A}_1 \otimes \mathfrak{A}_2 \otimes \ldots \mathfrak{A}_n = \mathfrak{A}_1^{\otimes n} \qquad \mathfrak{B} = \mathfrak{B}_1^{\otimes n}$$

$\mathfrak{A} \otimes \mathfrak{B}$ iso to $(\mathfrak{A}_1 \otimes \mathfrak{B}_1)^{\otimes n}$.

start with n copies of $\rho \equiv \rho_{AB} \in \mathfrak{A}_1 \otimes \mathfrak{B}_1$ i.e.,

$$\rho^{\otimes n} = \rho_{AB}^{\otimes n} \in \mathfrak{A} \otimes \mathfrak{B}$$

goal: create e-bits by applying sequence of

- local measurements with CC (classical communication)
- LO (local operations) conditioned on shared class alg

Entanglement measures

Entanglement of distillation: asymptotic rate $\dfrac{\#\text{ e-bits}}{\#\text{ copies of }\rho}$

$$\rho^{\otimes n} \mapsto (|\beta\rangle\langle\beta|)^{\otimes m}$$

Entanglement cost: asymptotic rate $\dfrac{\#\text{ e-bits}}{\#\text{ copies of }\rho}$

$$(|\beta\rangle\langle\beta|)^{\otimes m} \mapsto \rho^{\otimes n} \mapsto$$

LOCC not nec reversible: In general entang cost $>$ entang of dist

Entang cost $= \lim_{n\to\infty} \mathrm{EoF}\big(\rho^{\otimes n}\big)$ Entanglement of Formation

$$\mathrm{EoF}(\rho) \equiv \sup\Big\{ \textstyle\sum_j p_j S\big(\mathrm{Tr}_B|\psi_j\rangle\langle\psi_j|\big) : \rho = \sum_j p_j|\psi_j\rangle\langle\psi_j| \Big\}$$

Open question: NPT bound entanglement

Recall partial transpose $\mathcal{I} \otimes T$

PPT: state ρ_{AB} satisfies $(\mathcal{I} \otimes T)(\rho_{AB)} \geq 0$

for $d > 2$ can be separable or entangled

NPT: state ρ_{AB} for which $(\mathcal{I} \otimes T)(\rho_{AB} < 0$ always entangled

Thm: (Horodecki) If ρ_{AB} is PPT but not separable, then

no useful entanglement can be distilled

not even one e-bit or EPR pair – called bound entanglement

Question: Can at least one e-bit be distilled from every NPT state?

Or Are there NPT states which are "bound entangled" ?

Can reduce question to consideration of special states

$$a \sum_j |f_j \otimes f_j\rangle + b \sum_{j<k} |\phi_{jk}^+\rangle\langle\phi_{jk}^+| + c \sum_{j<k} |\phi_{jk}^-\rangle\langle\phi_{jk}^-|$$

$$|\phi_{jk}^\pm\rangle = \tfrac{1}{\sqrt{2}}\big(|f_j f_k\rangle \pm |f_k f_j\rangle\big)$$

Watrous showed that here are states $\rho = \rho_{AB}$ such that

- no entanglement can be distilled from $\rho^{\otimes n}$, but
- one e-bit can be distilled from $\rho^{\otimes(n+1)}$

Operator algebra reformulation

recall iso ρ_{AB} and CP map Φ give by Choi matrix

For ρ_{AB} NPT define $\Lambda = \Phi \circ T$

Claim: $\rho_{AB}^{\otimes n}$ is not distillable $\forall\ n \quad \Leftrightarrow \quad \Lambda^{\otimes m}$ is 2-positive $\forall\ m$

Challenge for Op Alg: Find a CP map Φ for which $(\Phi \circ T)^{\otimes m}$

$= (\Phi \circ T) \otimes (\Phi \circ T) \otimes \ldots \otimes (\Phi \circ T)$ is 2-positive for all m

OR show that no such map exists.

BANFF: NPT in purely Op Alg form

Is there a CP map Φ such that $T \circ \Phi$ is not CP,

but $(T \circ \Phi)^{\otimes n}$ is 2-positive for all n? $T =$ transpose

If yes, NPT conjecture is true because

Φ defines an NPT state from which no entang can be distilled

If no, entang can be distilled from any state which is not PPT

Watrous showed there are maps for which $(T \circ \Phi)^{\otimes m}$

is not 2-pos for some $m = n$ but is 2-pos for all $m < n$.

Reformulate as operator Ineq

Choi showed linear map Ω is 2-pos if and only if

$$\Omega(X^*)\,[\Omega(A)]^{-1}\,\Omega(X) \le \Omega(X^* A^{-1} X) \qquad \forall\, X, \quad \forall\, A > 0$$

Apply to $(T \circ \Phi)^{\otimes n} = T^{\otimes n} \circ \Phi^{\otimes n}$ to get 2-pos $\Leftrightarrow\ \forall\, X,\ \forall\, A > 0$

$$T^{\otimes n}\circ\Phi^{\otimes n}(X^*)\,[T^{\otimes n}\circ\Phi^{\otimes n}(A)]^{-1}\,T^{\otimes n}\circ\Phi^{\otimes n}(X) \;\le\; T^{\otimes n}\circ\Phi^{\otimes n}(X^* A^{-1} X)$$

$$\Longleftrightarrow \qquad \Phi^{\otimes n}(X)\,[\Phi^{\otimes n}(A)]^{-1}\Phi^{\otimes n}(X^*) \;\le\; \Phi^{\otimes n}(X^*\, A^{-1} X)$$

But always $\quad \Phi^{\otimes n}(X^*)\,[\Phi^{\otimes n}(A)]^{-1}\Phi^{\otimes n}(X) \;\le\; \Phi^{\otimes n}(X^* A^{-1} X)$

very, very weak non-commutativity

Can show suffices to consider special class of X.

NPT and Operator Schwarz Ineq.

Djokovic arXiv:1005.4247 has a Schwarz Ineq. approach to NPT.

Don't know if equivalent to above or not.

Lieb-Rusk (1974) Ω CP $\Rightarrow\ \Omega(X^*)\,[\Omega(A)]^{-1}\,\Omega(X) \leq \Omega(X^*A^{-1}X)$

Choi (197?) Ω 2-pos $\Leftrightarrow\ \Omega(X^*)\,[\Omega(A)]^{-1}\,\Omega(X) \leq \Omega(X^*A^{-1}X)$

Lieb-Ruskai (1974) showed special case for matrices

$$\sum_k M_k^* \Big[\sum_k A_k\Big]^{-1} \sum_k M_k \leq \sum_k M_k^* A_k^{-1} M_k$$

or, equi., $(M, A) \mapsto M^* A^{-1} M$ is jointly convex

proved earlier by Kiefer (1959) MBR learned June, 2010

References

D. P. DiVincenzo, P. W. Shor, J. A. Smolin, B. M. Terhal, A. V. Thapliyal "Evidence for Bound Entangled States with Negative Partial Transpose" *Phys. Rev. A* **61**, 062312 (2000). arXiv:quant-ph/9910026

John Watrous, arXiv:quant-ph/0312123 "Many copies may be required for entanglement distillation"

R., P., M., and K. Horodecki arXiv:quant-ph/0702225 "Quantum entanglement" *Rev. Math. Phys.* **81**, (2009)

Dragomir Z. Djokovic, arXiv:1005.4247 "Generalized distillability conjecture and generalizations of Cauchy-Bunyakovsky-Schwarz inequality and Lagrange identity"

J. Kiefer, "Optimum experimental designs", *J. Roy. Statist. Soc. Ser. B* **21** 272–310 (1959),