Boundary problems for the second order elliptic equations with rough coefficients

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1 Overview of the Field and the Framework

The main focus of the meeting was on boundary value problems for general differential operators $L = -\mathrm{div}\, A\nabla$. Here A is an elliptic matrix with variable coefficients, given by complex-valued bounded and measurable functions. Such operators arise naturally in many problems of pure mathematics as well as in numerous applications. In particular, they describe a wide array of physical phenomena in rough, anisotropic media. Thus, one of the central questions is: *what kind of medium yields solvable boundary problems*, or, mathematically, *what are the sharp conditions on the matrix A responsible for the solvability of problem $-\mathrm{div}\, A\nabla u = 0$ in a given domain $\Omega \subset \mathbb{R}^n$, $u|_{\partial\Omega} = f$, with boundary data, for instance, in $L^p(\partial\Omega)$*. Despite tremendous advances in the elliptic theory over the past half a century, this question remains largely open.

2 Recent Developments and Open Problems

For purposes of this discussion let us concentrate on the case $\Omega = \mathbb{R}^n_+ = \{(x,t) : x \in \mathbb{R}^{n-1},\, t \in (0,\infty)\}$.

It has been known for a long time that some restrictions on the matrix A are necessary to ensure solvability, and more precisely, certain smoothness in the transversal direction t is needed [4]. This observation naturally leads to two threshold problems.

Problem 1. *Establish solvability of boundary problems for a t-independent matrix A.*

Problem 2. *Investigate the perturbation: which restrictions on $A - A_0$ would allow one to pass from solvability results for A_0 to those for the matrix A.*

The first results towards Problems 1 and 2 date back to the early 80's. However, both of them are still far from being fully understood.

There are three basic types of boundary value problems: the Dirichlet problem with the prescribed trace on the boundary (stated in Section 1), the Neumann problem with the given flow through the boundary, i.e., normal derivative, and the regularity problem when the tangential derivative is known. In each case, most of the results available pertain to *real and symmetric* coefficients. In this context, the solvability for the Dirichlet problem with t-independent matrix A is due to D. Jerison and C. Kenig [10], and for the Neumann and the regularity problems due to C. Kenig and J. Pipher [11], [12]. Aside from these achievements, only a few results exist, addressing real non-symmetric matrices *in dimension two* [7], [13].

In the direction of Problem 2, a few approaches emerged. The situation when A and A_0 are *real and symmetric* and the discrepancy $A - A_0$ has bounded Carleson norm has been treated in [5], [6], [11], [12]. For *complex* matrices analogous perturbation result was established by S. Hofmann and S. Mayboroda [9] and independently in [2] under additional assumptions that A_0 is t-independent and the Carleson norm of $A - A_0$ is small. It is important to observe that while the Carleson condition is, in some sense, sharp [6], it necessarily requires that A and A_0 coincide on the boundary. Thus, one has to investigate independently a complementary problem: *find the optimal conditions on $A - A_0$, with $A \not\equiv A_0$ on the boundary, such that the solvability for A could be deduced from solvability for A_0*. In this context, it has been only known that the smallness of the L^∞ norm of $A - A_0$ is sufficient [1].

It has long been recognized that any further progress would require introduction of some decisively new methods. A big advantage of real symmetric matrices is availability of the Rellich identity. The latter allows one to compare the tangential and normal derivatives of the solution on the boundary, and has proved to be one of the leading tools in the analysis of boundary value problems. In particular, it was extensively used in the aforementioned works. Unfortunately, there is no analogue of the Rellich identity even for real non-symmetric matrices. Moreover, complex coefficients offer a whole new set of challenges: the positivity of the solutions, comparison principle, and thus, harmonic measure techniques essentially fail. At the same time, a few available methods of the analysis of elliptic operators with non-smooth *complex* coefficients, largely emerging from [3], are still extremely limited and await to be fully developed.

3 Scientific Progress Made

The meeting has brought significant advancement in the direction of Problems 1 and 2. Already in the first few days we have established the solvability of the boundary problems under the assumption that the matrix A is t-independent and close to identity in BMO. More precisely, we have showed the following.

Theorem 1. *Let $L = -\mathrm{div}\, A\nabla$ be an elliptic operator in $\mathbb{R}^{n+1}_+$ with complex bounded measurable coefficients independent of the transversal direction t. Then there exists $\varepsilon > 0$ such that the boundary value problems for L are well-posed in L^2 whenever $\|A - I\|_{BMO} < \varepsilon$.*

Note that, *in the case when $A_0 = I$,* Theorem 1 is strictly stronger than all previously available results for complex t-independent matrices. Indeed, the space BMO is strictly larger than L^∞ and, thus, in this context, the perturbation in the L^∞ norm discussed in Section 2 immediately follows from Theorem 1.

At the same time, our result opens several new directions of research. For instance, one would like to know what are the optimal restrictions on A_0. Specifically, it is desirable to have an analogue of Theorem 1 with any “good” A_0 in place of I, i.e., such that $L_0 = -\mathrm{div}\, A_0\nabla$ yields well-posed boundary problems.

Furthermore, during the meeting the participants have actively pursued Problem 1. The main theorems in [7] include sufficient conditions on absolute continuity of the elliptic measure (and thus, the solvability of the Dirichlet problem) for real non-symmetric matrices. The most general results were not, however, stated in terms of the conditions on the matrix A, but rather invoking some a priori estimates on the square function and non-tangential maximal function of solutions. At the time, these estimates could be originally verified only in the two-dimensional case. It seems though that the infusion of new methods from [3], [1], [9] could pave a way to a full higher-dimensional result. At the meeting we have built a possible strategy to attack this problem, and currently we are actively working on its major aspects.

4 Outcome of the Meeting

The meeting has been successful and productive at many levels. Based on the progress outlined above we are now preparing a manuscript of the paper. Moreover, an intensive exchange of the ideas, continuous collaboration and brainstorming allowed us to single out possible strategies for some outstanding problems in the field. In this connection, an opportunity to meet together at BIRS for a full week of an uninterrupted focused research has been invaluable. Now that the foundation for future work is laid, the participants will continue working at their home institutions, and hopefully, many more results are on the way.

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