Stability Theoretic Methods in Unstable Theories

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1 Overview of the Field, Recent Developments and Open Problems.

The idea of applying methods and results from stability theory to unstable theories has been an important theme over the past 25 years, with o-minimality, smoothly approximable structures, and simple theories being key examples.

But there have been some key recent developments which bring new ideas and techniques to the table. One of these is the investigation of abstract notions of independence, leading for example to the notions of thorn forking and rosiness. Another is the discovery that forking, weight, and related notions from stability are meaningful in dependent theories. Another is the formulation of notions of stable, compact, or more general domination, coming from the analysis of theories such as algebraically closed valued fields and o-minimal theories.

The level of different approaches and techniques which end up overlapping was the reason we decided it would be a perfect time for a research meeting where the most prominent researchers would come together and discuss the ideas, results and goals that were showing up in different contexts. The dominant subjects of the meeting were the following.

1.1 Dominating local structure.

In the talks given by Hrushovski and Macpherson about stable domination and by Hasson about stable types in theories interpretable in o-minimal structures, there were clear indications towards the possible applications of understanding and using the stable-like "pieces" (types or sorts) within a particular theory. This is a completely new approach towards traditional stability theory (and general classification theory) where the main idea is not anymore to find dividing lines between different theories but instead one tries to find well behaved parts within a particular model. It is clear by results from Haskell, Hrushovski, Pillay and Macpherson that whenever the stable sorts "dominate" (the precise definition of this can be found in [HHM08]) all the types in the structure, many of the stability theory results can be applied to understand the global structure. In [HP09] these ideas of finding particularly well behaved sorts that could give global information about the whole structure proved to be particularly promising when one expanded the well behaved part from stable to compact in order to apply the results to structures which were groups interpretable in theories with an underlying order.

There are two possible lines of research that stem out from the results described here. On the one hand, there should be an ongoing research in stably dominated and compactly dominated structures. There are quite a few interesting examples in this areas and the results so far have been very impressive and promising. The other venue that should be explored is to find other possible definitions that can work as well (as part of the structures in a "local" manner) as local stability and local compactness have worked so far. Some possible ideas would be to find definitions for local dependence that are good enough to be significant (in the sense of finding examples) and strong enough to prove for example some of Shelah's theorems concerning types in dependent theories.

1.2 Measures.

In 1987 Keisler ([Kei87]) wrote a paper where he showed that many aspects of forking could be understood and generalized when one studied measures on the definable sets as an extension of types (types would just be a 0-1 measure on the space, where the measure of a formula is 1 if and only if the formula is in the type). Work by Hrushovski, Peterzil and Pillay and lately by Simon shows that this analysis can provide a deeper understanding of many theories which had an underlying order (bounded o-minimal theories) and actually conclude many interesting and far reaching results in the definable groups of such theories.

It is quite possible that a deeper understanding of invariant Keisler measures will shed more light upon how forking can affect the type definable subsets in particular theories.

1.3 Pregeometries.

There were two aspects in which pregeometries (matroids) have been studied lately. The first is the possible equivalences to linearity. Linearity is a concept which is model theoretically hard to define, to prove and to use. In (forking-)minimal theories there are concept like one-basedness and local modularity which are equivalent to linearity and much easier to use. Using pairs of structures one can define notions which are equivalent to linearity and the study of applications of these new equivalences seems very promising.

Matroids have played a role in model theory since Zilber started studying the different possible matroids that were defined by algebraic closure in strongly minimal sets. However, in many ways when one tried to use this sort of arguments in higher dimensions one had to use quite sophisticated model theoretic tools such as weight and regular types in order to apply them. In combinatorics, a generalization of matroids (greedoids) has been developed during the last couple of decades. Even though there is no good definition for infinite dimensional greedoids, a definition which would be essential if one wanted to apply greedoids to model theory, the development of these definitions and the possible applications are quite promising ideas for the near future.

1.4 Dependent Theories.

Dependent (also called "NIP") theories have been studied intensively during the last decade. Original work of Shelah has had many applications and has been very influential in oncoming work. Some of the ongoing questions include.

- Number of non forking extensions. It is known that one can prove that if for any set A of cardinality κ the number of non forking extensions of any type over A is at most 2^κ then the theory is dependent. It is also known that in a theory T the number of non forking extensions of a type p as described above is 2^{2^κ}. But there is a big gap between the two and the study of this gap should reveal the properties of non forking in dependent theories.

- Counting types. Shelah conjectured that any type in a dependent theory had boundedly many coheir extensions (where the bound depended on the model, not on the size of the parameter set of the different types). Using this he was able to prove existence of indiscernible subsequences of large enough sequences, but Kaplan and Shelah disproved this conjecture. However, it is possible that one can have a bound in the number of definable heirs or other significant extensions which are very important to the study of the best known unstable dependent theories (o-minimal theories).

- Generic pair conjecture. The generic pair conjecture in dependent theories was proved in [Sh08-2], but it has been studied in the more general context of abstract elementary classes (which is a categorical approach to model theory) and this may provide a way to define dependent theories in this more abstract setting.
- Weight, strong dependence and dp-minimality. Strong dependence and dp-minimal theories have been some of the most successful strengthenings of dependent theories. Both of this notions can be defined using the appropiate definition of weight.
- Other notions of independence. Studying NTP_2 structures (see the item below) Chernikov, Kaplan and Usvyatsov came up with the notion of strong non forking. How this notion relates to non forking, to non th-forking and to the notion of non splintering defined by Grossberg, Van Dieren and Villaveces for abstract elementary classes seems to be a meaningful area of research that can help us understand the extensions of types. Studying this within dependent theories and even restricting oneself to generically stable types seems to be particularly promising.

1.5 Other divding lines in classification theory; ideas from Shelah's "Classification Theory".

Much of stability theory has been developed based on ideas from Shelah's book [She78] and even notions like dependent theories described above have the origins in this extremely influential book. However, there are still many unexplored ideas from this book, some of which have resurfaced lately and seem to be quite promising.

$\mathbf{NTP_2}$.

Kaplan and Chernikov have been able to extend many of the results which are true for both simple and dependent theories. The study of coheirs, heirs, and non forking seems to be quite meaningful in this context. Also, viewing things in this level of generality may help understanding and defining interesting concepts in dependent thories. The new definition of strong forking by Kaplan and Usvyatsov seems to be part of this. One should also try to understand the relation between the notion of non splintering (defined by Grossberg, Van Dieren and Villaveces) and strong non forking, even in the context of dependent theories.

$\mathbf{NTP_1}$

Kim gave a very nice talk where he tried to analyse NTP_1 theories. The best example of NTP_1 theories are the ω-free PAC fields studied by Chatzidakis in [Cha02]. Once again, studying in this level of generality is also helpful to find significant independence notions which may coincide with forking in stable theories and therefore be hidden by it. In his talk, Kim suggested studying the independence notion that comes from defining that a formula $\phi(x, a)$ "strong divides" over A (not to be confused with strong dividing defined for th-forking, although it would seem that in many cases they would coincide) if given any Morley sequence $\langle a_i \rangle$ of a over A the conjunction

$$\bigwedge_i \phi(x, a_i)$$

is k-inconsistent.

The main idea was that if this notion has local character in NTP_1 theories (which it appears to have) one can develop in NTP_1 theories an internal analysis of this independence notion.

Amalgamation. There have been many approaches, particularly in Abstract Elementary Classes, about amalgamation problems (finding a common realization to a family of types). The principal impetus in this area was Hrushovski's paper [Hr06] where he studied, in stable theories, the relation between the existence problem for the amalgamation, the uniqueness, and the existence of certain definable groupoids. He also proved that for any structure M of a stable theory T one could find an expansion M^* of M by new sorts such that in M^* one had existence and uniqueness for all the amalgamation problems, further suggesting that this notion is related to something close to higher dimensional groupoids.

Perhaps the best way to describe the possible impact of this is to say that in stable theories, the concepts of 1-uniqueness and 2-existence are related to the very important notion of imaginaries. In this same direction, Hrushovski proved that 2-uniqueness and 3-existence was related to the definition of groupoids which can be seen as a generalization of imaginaries. It is therefore quite likely that studying the n-existence and the n-uniqueness problems can point out to objects which are inherent to the structure but which we have yet to analyse as independent sorts.

1.6 Other topics discussed in the meeting.

Other subjects discussed during the meeting which are quite interesting and where one can find model theoretic results but which are not quite as well structured yet include

(i) Other dividing lines in the classification of first order theories include rosiness, strict order property, SOP_n.

(ii) Use of stability-style techniques in some of the "good" classes of theories from (i) thorn rank, meta-analysability, definable types,

Model theory and group dynamics. This talk, given by Prof. Newelski was one of the most interesting talks of the meeting, but since the approach is quite new (even for this very new branch of model theory) it was hard to place among the previous subject headings. The main idea is to interpret the basic notions of group dynamics in the model theoretic setting and use these to understand extensions of types. In his talk, Newelski used some of the notions of group dynamics to define weak generics and understand how co-heir extensions can also be studied under this approach; but it is quite likely that this approach can be used in order to define new types of interesting extensions of types or give a deeper understand of the existing ones.

2 Presentation Highlights

Before commenting on the highlights of the meeting, we should describe the structure of it. The most innovative aspect of the meeting was the scheduling of some "structured working time" where the classrooms would be preassigned in order to have people work on and discuss different questions or interesting subjects and each participant would choose which of the "working groups" he would like to join. During most days we would have talks in the morning, and one talk after both lunch and dinner after each of which we would get together and then break up into working groups. Both the attendance of the "gathering" talks and of the working groups was quite impressive and all around the meeting was a great success.

The greatest achievement of the meeting (more of which we will talk about in the following section) was the understanding of the structure of a new branch of model theory which is starting to take shape after many researchers from diverse overlapping subjects got together for this meeting. It became clear that this new approach to model theory has two main subbranches. The first consists to generalize the results from stability and simplicity theory to even broader contexts like dependent, NTP_1 and NTP_2 theories, bringing the traditional "dividing lines" from classification theory even further and being able to conclude quite interesting results in even broader contexts. The second one is to use this dividing lines within a single theory. By this we mean that understanding the "stable parts" of a particular structure can have amazing consequences that in many cases can even say quite important things about the general structure.

We asked the participants what they thought were the highlights of the meeting. Hrushovski's and Simon's talks about invariant measures and stably dominated types and Chernikov's talk about NTP_2 structures were the three talks that most people considered highlights of the meeting, whereas canonical bases in NTP_2 structures and trying to define a linear order in an unstable dependent theory (or find a counterexample) were the working sessions people most frequently mentioned.

3 Scientific Progress Made and Outcome of the Meeting

As mentioned before, the biggest scientific progress made and outcome of the meeting was the awareness that was built around this fast growing subject, and the beginnings of the understanding of the structure of the subject and how the different approaches interact with each other. This feeling was reinforced when we asked the participants what they thought was the outcome of the meeting and we got replies such as "...awareness that a lot is happening very fast around generalisations of stability.", "...this meting managed to describe the state of the art in NIP and related areas, a currently rapidly evolving subject." and "It was a great success, many interesting projects were begun and it will be very interesting to reconvene in Banff in the future to hear the outcome of these investigations!" among many other similar replies. We all have a strong feeling that this meeting will provide a very important first step in a very promising area of model theory, and many of the participants mentioned that organizing a follow-up meeting in few years would be worthwhile.

As for scientific progress per se, many of the participants (Dzamonja, Gismatullin, Hart, Hrushovski, Kim, Macpherson, Malliaris, Onshuus, Usvyatsov) explicitly claimed to have made progress in research they had either started before the meeting or started working during the meeting, and it is quite likely that there will be many upcoming publications and results that were made (or significantly advanced) during the meeting. We should include the following partial reports.

- Hrushovski mentioned achieving a better understanding of the metastability in dependent theories and how much more went through that he thought at first.
- Gismatullin solved one of the questions he asked about groups without proper subgroups of finite index.
- Hart included the following report:

  "Suppose that p is a depth zero, regular type over a model M and N is dominated by a realizations of p over M. Moreover, assume that $M \subseteq_{na} N$. This situation arises on the leaves of a decomposition tree for any model of a countable, classifiable theory. In the calculation of the uncountable spectrum for countable theories, for cardinal arithmetic reasons, it was unnecessary to understand the exact structure of N over M. Hrushovski has shown that in the case where p is not locally modular, N is prime over M and any realization of p in N. There are examples to show that this is not true when p is locally modular. During the Banff meeting, Bouscaren, Hart and Laskowski worked to finalize the details of suggestions of Hrushovski's that in the case where p is non-trivial and locally modular then N is "controlled" over M by the generic of a definable group non-orthogonal to p. Progress was made and a paper should be forthcoming."

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