Theories without tree property of the second kind (NTP_2)

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Dividing patterns

We say that $(\phi_i(x, y_i), I_i \,:\, i < \kappa)$ with $I_i = (a^i_j \,:\, j < \omega)$ is a **dividing pattern of depth** κ if:

- for each $i < \kappa$: $\bigwedge_{j<\omega} \phi_i(x, a^i_j)$ is k_i-inconsistent for some $k_i < \omega$

- for each $f \in \omega^\kappa$: $\bigwedge_{i<\kappa} \phi_i(x, a^i_{f(i)})$ is consistent.

$\phi_0(x, a^0_0)$	$\phi_0(x, a^0_1)$	$\phi_0(x, a^0_2)$	$\cdots$	$-k_0-$ *inconsistent*
$\phi_1(x, a^1_0)$	$\phi_1(x, a^1_1)$	$\phi_1(x, a^1_2)$	$\cdots$	$-k_1-$ *inconsistent*
$\phi_2(x, a^2_0)$	$\phi_2(x, a^2_1)$	$\phi_2(x, a^2_2)$	$\cdots$	$-k_2-$ *inconsistent*
$\cdots$	*consistent*	$\cdots$	$\cdots$	

$\kappa_{inp}(T)$ and NTP_2

$\kappa_{inp}(T) :=$ supremum of all possible depths of dividing patterns or ∞ if it does not exist.

T is **strong** if there is no dividing pattern of infinite depth

T is NTP_2 if $\kappa_{inp}(T) < \infty$ (equivalently there is no dividing pattern of infinite depth with $\phi = \phi_i$ and $k = k_i$ for all i.

Indiscernible arrays

We say that an array $I_{\in O}$ is indiscernible if its rows are mutually indiscernible, that is I_i is indiscernible over $I_{\neq i}$.

Multi-dimensional "Erdös-Rado":
For every $c \in \mathbb{M}$ and cardinal κ there is some λ such that for any array $I_{<n}$, $I_i = (a^i_j : j \in O)$ with $|O| \geq \kappa$ (and $|a^i_j| \leq \kappa$) there is some c-indiscernible array $J_{<n}$, $J_i = (b^i_j : j < \omega)$ and such that

for each $m < \omega$: $b^0_{<m} b^1_{<m} ... b^n_{<m} \equiv_c a^1_{\in k_1} a^2_{\in k_2} ... a^n_{\in k_n}$ for some $k_1, k_2, ..., k_n \subseteq O$

Indisernible dividing patterns

So when computing $\kappa_{inp}(T)$ it is enough to look only at indiscernible dividing patterns. Besides we can assume that rows are 2-inconsistent (by changing ϕ_i's at worst)

Place in the classification hierarchy

1. *NIP* $\implies$ NTP_2 (and actually *NIP* = NTP_2 + ”bounded non-forking”)
2. simple $\implies$ NTP_2 (and actually simple = NTP_2 + NTP_1)
3. strong $\implies$ NTP_2 (and actually is a uniform version of it, ”superNTP_2”)
4. strong + *NIP* = strongly dependent
5. strong + simple = every type has finite weight

Some examples: NTP_2

- Of course, reducts and interpretations preserve NTP_2
- T_1, T_2 are $NTP_2 \implies T_1 \times T_2$ is NTP_2 (so e.g. product of simple and dependent groups)
- Chatzidakis-Pillay expansions by random predicate preserve NTP_2
- The main candidate (unfortunately no proof yet) – VFA_0

Some examples: not everything is NTP_2

- triangle free random graph, atomless boolean algebra, etc
- ω-free *PAC* fields
- any non-simple NTP_1 theory

Enough to check formulas in a single variable

Folklore (?): If T is unstable then there is a formula in one variable with *the order property*.

Folklore (?): T is not simple if and only if there is a formula in one variable with the *tree property*.

Theorem of Shelah: If T has *IP* then already some formula in a single variable does.

Theorem: If T has TP_2 then there is some formula in a single variable with TP_2.

Why?: rotation of indiscernibles and arrays

- We say that two indiscernible sequences I and J are **rotation-equivalent** if $I \equiv_a J$ where a is the first element of the sequences.

- Two indiscernible arrays $I_{\in O}$ and $J_{\in O}$ are **rotation-equivalent** if $I_{\in O} \equiv_{a_{i \in O}} J_{\in O}$ where a_i is the first element of I_i

- Two indiscernible arrays $I_{\in O}$ and $J_{\in O}$ (with O endless infinite) are **almost rotation-equivalent** if there is some $h \in O$ such that $I_{>h}$ and $J_{>h}$ are rotation-equivalent.

Why?: lifting indiscernibility by rotation

Define $\kappa^n_{inp}(T)$ to be the maximal depth of dividing patterns $(\phi(x, y_i), I_i)$ with $|x| \leq n$.

Lemma: TFAE

- $\kappa^n_{inp}(T) \leq \kappa$
- $(*)^\kappa_n$: If $I_{<\kappa^+}$ is an indiscernible array and $c \in \mathbb{M}$, $|c| \leq n$ then we can make it indiscernible over c by almost-rotation.

Question: Do we really need rotation? Maybe its possible to find an actual subarray indiscernible over c?

One variable is enough

Its easy to see that $(*)^\kappa_1 \implies (*)^\kappa_2 \implies \ldots \implies (*)^\kappa_\kappa$ and so we can answer a question of Shelah from the book:

Theorem: $\kappa_{inp}(T) = \kappa^1_{inp}(T)$

and so in paticular
- TP_2 is always witnessed by some formula in a single variable
- *strong* = *strong*^{1}
- new proof for *strongly dependent* = *strongly dependent*^{1}

Forking in NTP_2

We say that $a \mathop{\smile\!\!\!\!\!\!|}_{c}^{ist} b$ if there is a global type $p \supseteq tp(a/bc)$ invariant over c and for each $B \supset bc$ if $a' \models p|_B$ then $B \mathop{\smile\!\!\!\!\!\!|}_{c}^{d} a'$ (so invariant non-co-dividing)

Some facts from [CheKap]:

Let T be NTP_2. Then

- $\mathop{\smile\!\!\!\!\!\!|}^{ist}$ exists over models, that is $a \mathop{\smile\!\!\!\!\!\!|}_{M}^{ist} \emptyset$ for each a and M.
- Any $\mathop{\smile\!\!\!\!\!\!|}^{ist}$-free sequence witnesses dividing.
- $\mathop{\smile\!\!\!\!\!\!|}^{f} = \mathop{\smile\!\!\!\!\!\!|}^{d}$ over any extension base.
- T is NIP iff it is NTP_2 and non-forking is bounded.

Pseudo-local character

We say that dividing in T has **pseudo-local character** w.r.t. $\mathop{\smile\hskip-0.9em\mid}$ if

Let $p \in S(A), A_0 \subseteq A$. Then there is some $A' \subseteq A$, $|A'| \leq |T|$ such that

for each $B \subseteq A$: $B \mathop{\smile\hskip-0.9em\mid}_{A_0} A' \implies p|_B$ does not divide over $A_0 A'$.

Of course local character (and so simplicity) implies pseudo-local character w.r.t. any $\mathop{\smile\hskip-0.9em\mid}$.

Strong pseudo-local character is when we can find finite A'.

Pseudo-local character characterizes NTP_2

Theorem: The following are equivalent
- T is NTP_2
- dividing has pseudo-local character w.r.t. $\mathop{\smile\hskip-0.9em\raise0.6ex\hbox{$|$}}^{ist}$
- If $(a_i : i < |T|^+)$ is an $\mathop{\smile\hskip-0.9em\raise0.6ex\hbox{$|$}}^{ist}$-free sequence over A and b some tuple then $b \mathop{\smile\hskip-0.9em\raise0.6ex\hbox{$|$}}^{d}_{A} a_i$ for some (equivalently almost all) $i < |T|^+$

Analogously strongness is equivalent to strong pseudo-local character.

Question: Can we replace $\mathop{\smile\hskip-0.9em\raise0.6ex\hbox{$|$}}^{ist}$ by something weaker?

Philosofical question: Need to work with two different relations – problem or feature?

Pseudo-local character: example

Consider $M \models DLO$ and $p \in S(M)$. So p corresponds to some cut. If, say, cofinality is high on both sides then local character fails for p. But let A_0 be some small subset of M and let (a, b) be some interval containing this cut and not containing anything from A_0. Set $A' = \{a, b\}$.

So essentially pseudo-local character means that local character holds in ”large/generic pieces”.

Amalgamation of types

Fact (Kim): If T has TP_1 then the independence theorem for Lascar strong types fails. (and modulo set-theoretic assumption fails over models).

So is there anything to say about NTP_2?

Preindependence relations with amalgamation: setting

Let $(p_i(x, a_i) : i \in O)$ be a familiy of $\mathop{\smile\!\!\!\!\!\!|}\,$-free types over M extending some $p \in S(M)$.

We say that it is **amalgamable** if $\bigwedge_{i \in O} p(x, a_i)$ is consistent and $\mathop{\smile\!\!\!\!\!\!|}\,$-free over M.

In this terms independence over models $\iff$ we can amalgamate when a_i is an M-independent set.

Generic amalgamation / Chain condition

We say that $\downfall$ has **generic amalgamation** over models if for any family of $\downfall$-free types $p_i(x, a_i)$ over M large enough ($\geq 2^{2^{|M|}}$) at least two of them amalgamate.

Observation (Adler, Casanovas): TFAE

1. $\downfall$ has generic amalgamation
2. $\downfall$ has amalgamation if a_i's form an indiscernible sequence over M
3. Let $a \downfall_b I$ with I a b-indiscernible sequence. Then there is $a' \equiv_b a$ and such that $a' \downfall_b I$, I is $a'b$-indiscernible.

Generic amalgamation in NTP_2?

Remark: 1) $\mathop{\smile\!\!\!\!\!\!|}$ satisfies independence theorem $\implies$ it has generic amalgamation.
2) $\mathop{\smile\!\!\!\!\!\!|}$ is bounded $\implies$ it has generic amalgamation,
so both in simple and dependent theories non-forking has generic amalgamation, but for orthogonal reasons.

Conjecture: $\mathop{\smile\!\!\!\!\!\!|}^f$ satisfies generic amalgamation over models in NTP_2?

Also, what is known in NTP_1?

Kim-Pillay for simple theories

T is simple if and only if there is a pre-independence relation $\mathop{\smile\!\!\!\!\!\!\,\mid}$ satisfying

- **left transitivity**, **base monotonicity**, **extension**, **finite character**
- $\mathop{\smile\!\!\!\!\!\!\,\mid}$ has **local character**
- **Independence theorem over models**

and in this case $\mathop{\smile\!\!\!\!\!\!\,\mid}$ is **exactly non-dividing / non-forking.**

Let us define a shortcut: $a \mathop{\smile\!\!\!\!\!\!|}\,'_c\, b$ if exists a global type $p \supseteq tp(a/bc)$ invariant over c and for each $B \supset bc$ if $a' \models p|_B$ then $B \mathop{\smile\!\!\!\!\!\!|}_c a'$.

Theorem: T is $NTP_2 \iff$ exists an **invariant** relation $\mathop{\smile\!\!\!\!\!\!|}$ satisfying

- **left transitivity**, **base monotonicity**, **finite character**
- $\mathop{\smile\!\!\!\!\!\!|}'$ satisfies **existence over models**
- $\mathop{\smile\!\!\!\!\!\!|}$ has **pseudo-local character** with respect to $\mathop{\smile\!\!\!\!\!\!|}'$
- **weak generic amalgamation**: if $(a_i)_{i<\omega}$ is $\mathop{\smile\!\!\!\!\!\!|}'$-free over M and $p(x, a_0)$ is $\mathop{\smile\!\!\!\!\!\!|}$-free over M, then $\bigwedge_{i<\omega} p(x, a_i)$ is consistent.

and in this case $\mathop{\smile\!\!\!\!\!\!|}$ is **exactly non-dividing / non-forking** when restricted to models

NIP and strongness

T is strong $\iff$ instead of pseudo-local character we have **strong pseudo-local character**

T is *NIP* $\iff$ in addition we have

- **boundedness**: for each M there are boundedly many global types $\mathop{\smile}\limits^{|}$-free over M.

And, of course, T is strongly dependent if we have both.

Questions / research directions

- Groups with NTP_2 (strong) - is there anything to say?
- Low NTP_2(strong) theories (include simple low theories and NIP)
- Are there TP_2 theories with bounded non-forking?
- Study (generically-) dependent and (generically-) simple types in NTP_2 (strong) theories. Could there be any decomposition? Simply-dominated types?