Lovely pairs of geometric structures and linearity

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Geometric Theories

We say that a complete theory T is *geometric* if:

1. In every model of T, *acl* has the exchange property.
2. It eliminates the quantifier $\exists^\infty$.

Note: In T there is a notion of independence for *real sets*:
If $M \models T$, $\vec{a} \in M^n$, $B, C \subset M$,
$\vec{a} \mathop{\smile\!\!\!\!\!\mid}_B C$ if $dim(\vec{a}/B) = dim(\vec{a}/BC)$.

Examples:

1. strongly minimal theories.
2. o-minimal theories extending DLO
3. *SU*-rank one simple theories.

Lovely pairs

Let T be a geometric theory. Let P be a new unary predicate and let $L_P = L \cup \{P\}$.

Definition

We say that a structure $(M, P(M))$ is a lovely pair of models of T if

1. $P(M) \preceq M \models T$
2. *(Coheir property) If $A \subset M$ is algebraically closed and finite dimensional and $q \in S_1(A)$ is non-algebraic, there is $a \in P(M)$ such that $a \models q$.*
3. *(Extension property) If $A \subset M$ is algebraically closed and finite dimensional and $q \in S_1(A)$ is non-algebraic, there is $a \in M$, $a \models q$ and $a \notin \mathrm{acl}(A \cup P(M))$.*

Previous work

Extends Poizat's notion of *beautiful* pairs.
Simple theories: Ben Yaacov, Pillay, Vassiliev,
O-minimal theories: van den Dries.
Geometric theories: Hils: rich fusions.

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Basic Facts I

Let $(M, P(M))$, $(N, P(N))$ be lovely pairs of models of T, then

$$(M, P(M)) \equiv (N, P(N))$$

We write T_P for the common theory.

The class of lovely pairs is not elementary, but if $(M, P(M)) \models T_P$ is $|L|^+$-saturated, then $(M, P(M))$ is a lovely pair.

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Basic Facts II

Proposition

The theory T_P is near model complete: every L_P-formula is equivalent to a boolean combination of formulas of the form:

$$\exists y_1 \in P \ldots \exists y_k \in P \varphi(\bar{y}, \bar{x})$$

where $\varphi(\bar{y}, \bar{x})$ is an L-formula.

If $\psi(\bar{x})$ defines a subset of P, then there is $\varphi(\bar{x})$ L-definable such that $\psi(\bar{x}) = P(x) \wedge \varphi(\bar{x})$.

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Examples of lovely pairs

$$T = ACF_0 \quad (\mathbb{C}, +, \times, 0, 1, \overline{\mathbb{Q}(e_0, e_1, \dots)})$$

$$T = DLO \quad (\mathbb{R}, \leq, \mathbb{Q})$$

$$T = RCF \quad (\mathbb{R}, +, \times, 0, 1, \leq, \overline{\mathbb{Q}(e_0, e_1, \dots)}^r) \models RCF_P$$

The o-minimal case

Let T be o-minimal expansion of $Th(\mathbb{R}, +, <, 0, 1)$, where 1 stands for a positive constant.

Definition

A dense pair of models of T is a pair $(M, P(M))$ of models of T such that $P(M) \preceq M$,$P(M) \neq M$ and $P(M)$ is dense in M.

It follows from work of van den Dries: *dense pairs of models of T are the models of T_P.*

It has nice topological features: o-minimal open core (Dolich, Miller, Steinhorn).

Properties induced by T on T_P:

If T is strongly minimal, then T_P is ω-stable and $MR(T_P) \leq \omega$ (Poizat, Buechler).

If T is simple of SU-rank one, then T_P is supersimple and $SU(T_P) \leq \omega$ (Vassiliev).

If T is a rosy theory of thorn-rank one, then T_P is super-rosy and thorn-rank$(T_p) \leq \omega$ (Boxall).

If T is (strongly) dependent, T_P is also (strongly) dependent (B., Dolich, Onshuus).

Question: What if T has the NTP_2 property?

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Strongly minimal theories

Assume T is strongly minimal, $M \models T$. Then TFAE:

1. T is 1-based: whenever $A, B \subset M$,
$$A \underset{acl^{eq}(A)\cap acl^{eq}(B)}{\perp} B$$
2. T is linear: for $a, b \in M$, $C \subset M$ such that $b \in acl(aC)$, then $MR(Cb(\text{tp}(a, b/C))) \leq 1$.
3. T is locally modular: for $a \in M$ not algebraic, and $A, B \subset M$,
$$aA \underset{acl(aA)\cap acl(aB)}{\perp} aB$$

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Linearity

We have the following characterization of linearity in the SU-rank 1 case (note that in the SU-rank 1 case linearity is strictly weaker than local modularity):

Theorem (Vassiliev)
For an SU-rank 1 theory T the following are equivalent:

1. T is linear (Cb of any plane curve has rank ≤ 1)
2. T is 1-based (A is independent from B over $acl^{eq}(A) \cap acl^{eq}(B)$)
3. T_P has SU-rank ≤ 2 (=2 if non-trivial)
4. for any lovely pair (M, P) the quotient pregeometry $(M, acl(\_ \cup P(M)))$ is modular
5. $acl = acl_P$ in T_P

Main example, Loveys-Peterzil

Let $N = (\mathbb{R}, +, 0, f, \leq)$, where

$$f(x) = \begin{cases} \pi x & \text{for } x \in (-1, 1) \\ 0 & \text{otherwise.} \end{cases}$$

Let $T = Th(N)$, $(M, P(M)) \models T_P$ saturated. Then:

1. It has the CF property: every interpretable NORMAL family of plane curves has dimension ≤ 1.
2. It is NOT 1-based, there are sets $A, B \subset M$ such that $A \not\perp_{acl^{eq}(A) \cap acl^{eq}(B)} B$.
3. thorn-rk$(T_p) = 2$.
4. It is not modular after adding any set as parameters.
5. The quotient geometry $acl(\_ \cup P(M))$ is modular.

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Main theorem,

Theorem (B.,Vassiliev)
Let $(M, P(M)) \models T_P$ saturated. Then TFAE:

1. $acl(\_ \cup P)$ is modular.
2. For A, B sets there is $C \downarrow_{\emptyset} AB$ such that $A \downarrow_{acl(AC) \cap acl(BC)} B$.
3. $acl = acl_P$ in the home sort.

We call such a T *linear*.

Examples: SU-rank one linear structures, "linear" o-minimal structures: global addition and CF-property.

Questions I

Let $(M, P) \models T_P$ be saturated. If $acl(\_ \cup P)$ is modular non-trivial, when is there a group interpretable (or something weaker) in M?

Positive answers:

1. T strongly minimal (group conf.)
2. T simple of SU-rank 1 (group conf.) but not *interpretable*.
3. T o-minimal (trichotomy-group interval).
4. T geometric C-minimal under modularity (Fares Malouf).

Questions II

1. A geometric *rosy* theory has CF property if any interpretable family of plane curves has dimension at most one. Our notion of linearity implies CF. Is linearity equivalent to CF? True if the theory has almost Cb.

2. We can define a weak version of 1-basedness for geometric theories by requiring that for any $\bar{a}$ and B, there exists $\bar{a}' \models tp(\bar{a}/B)$ such that $\bar{a}' \downarrow_B \bar{a}$ and $\bar{a} \downarrow_{\bar{a}'} B$. Then it implies linearity. Is the converse true?

Ranks and Geometry

Theorem (Buechler, Vassiliev)
Let T be a strongly minimal theory ($SU(T) = 1$), $(M, P) \models T_P$ be saturated. Then

1. If T is trivial, $MR(T_P) = 1$.
   $(SU(T_P) = 1)$
2. If T is linear non-trivial, $MR(T_P) = 2$.
   $(SU(T_P) = 2)$
3. If T is not linear, $MR(T_P) = \omega$.
   $(SU(T_P) = \omega)$

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o-minimal case

Theorem (B., Vassiliev)

Let T be an o-minimal theory extending DLO and let $(M, P) \models T_P$ be saturated. Then

1. If $a \in M$ is trivial, SU-thorn $(\mathrm{tp}_P(a)) \leq 1$ ($= 1$ iff $a \notin \mathrm{dcl}(\emptyset)$).
2. If M has global addition (i.e. expands the theory of ordered abelian groups) and does not interpret an infinite field, $a \notin P(M)$, then SU-thorn $(\mathrm{tp}_P(a)) = 2$.
3. If M induces the structure of an o-minimal expansion of a real closed field in a neighborhood of $a \notin P(M)$, then SU-thorn $(\mathrm{tp}_P(a)) = \omega$.

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Questions III

1. For T rosy, is there a correspondence with the pregeometry type and the thorn-rk of T_P?

Understand thorn forking in T_P.
Let $B \subset C \subset M$ be sets, $a \in acl(CP(M)) \setminus acl(BP(M))$.
Does $\text{tp}(a/C)$ thorn forks over B?

2. Can we characterize thorn rank one theories? Are they characterized by property (E)?

3. If T is ω-categorical and linear, is T_P ω-categorical?

4. If $Th(M)$ is dependent, $A \subset M$ is small, and the induced structure on A is dependent, is $Th(M, A)$ dependent?

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