BIRS Workshop
Banff, 030809

From codes to matroids and back

Thomas Britz*

*University of New South Wales
Funded by an ARC Discovery Grant

Definition (Whitney 1935).

$M = (E, \mathcal{I})$ is a *matroid* iff

I1: $\emptyset \in \mathcal{I}$

I2: $I \subseteq J \in \mathcal{I} \;\Rightarrow\; I \in \mathcal{I}$

I3: $I, J \in \mathcal{I}$ and $|I| < |J| \;\Rightarrow\; I \cup e \in \mathcal{I}$ for some $e \in J - I$.

$\mathcal{I} =$ the *independent sets* of M

$$E = \{1 \quad 2 \quad 3 \quad 4 \quad 5\}$$

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

$$\mathcal{I} = \big\{\emptyset, \{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{1,2\}, \{1,3\}, \{1,4\}, \{1,5\}, \{2,3\}, \{2,4\}, \{2,5\}, \{3,4\}, \{3,5\}, \{1,2,4\}, \{1,2,5\}, \{1,3,4\}, \{1,3,5\}, \{2,3,4\}, \{2,3,5\}\big\}$$

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$\mathcal{I} =$ the *independent sets* of M
$\mathcal{B} =$ *bases* $=$ the *maximally independent sets* of M
$\mathcal{C} =$ *circuits* $=$ the *minimally dependent sets* of M
$\rho =$ *rank function*: $\rho(T) = \max\{|I| : I \in \mathcal{I},\, I \subseteq T\}$

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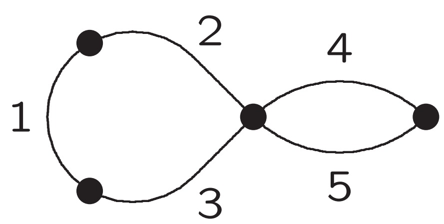

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M = a matroid on the set E

The *dual* of M: $M^* = (E, \mathcal{B}^*)$ where $\mathcal{B}^* = \{E - B \,:\, B \in \mathcal{B}(M)\}$

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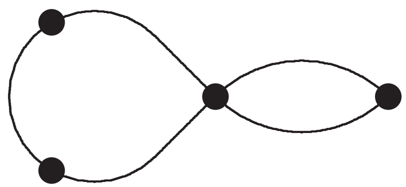

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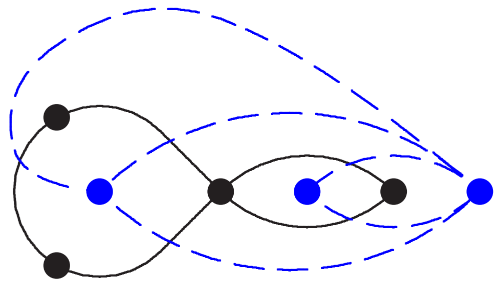

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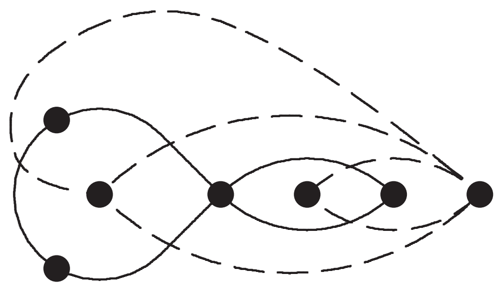
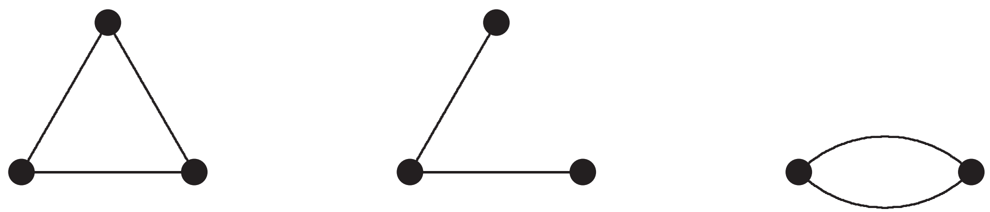


Deletion of M to a set T: $M|T = (T, \mathcal{I}')$ where $\mathcal{I}' = \{I \ : \ I \in \mathcal{I}(M), I \subseteq T\}$

Contraction of M to a set T: $M.T = (M^*|T)^*$



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$E = \{e_1, \ldots, e_n\}$

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Hamming weight: $w(v) = d(v, 0)$

Support: $S(v) = \{e \in E : v_e \neq 0\}$

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Hamming distance: $d(u, v) = \#\{e \in E : u_e \neq v_e\}$

Hamming weight: $w(v) = d(v, 0)$ $\quad w(v) = 3$

Support: $S(v) = \{e \in E : v_e \neq 0\}$ $\quad S(v) = \{e_2, e_3, e_5\}$

$$v = \begin{pmatrix} 0 & \overset{e_2}{\mathbf{1}} & \overset{e_3}{\mathbf{2}} & 0 & \overset{e_5}{\mathbf{-1}} \end{pmatrix}$$

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Definition: $M_C = M[G]$ is the *vector matroid* of C.

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$A_i^{(r)}$ = # r-dimensional subcodes $D \subseteq C$ with $|\bigcup_{v \in D} S(v)| = i$

The *rth higher weight enumerator* of C: $W_C^{(r)}(z) = \sum_{i=0}^{n} A_i^{(r)} z^i$

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$$\begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \qquad W_C^{(2)}(z) = 3z^5 + 3z^4 + z^3$$

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3	5	5	5	4	4	4

Problem: Calculate these enumerators for interesting linear codes

The extremal doubly-even, self-dual binary codes:

$$1 \times [24, 12, 8] \text{ code (the extended binary Golay code)}$$

$$5 \times [32, 16, 8] \text{ codes}$$

$$1 \times [48, 24, 12] \text{ code}$$

$$\text{At most } 1 \times [72, 36, 16] \text{ code}$$

$$\text{At least } 12579 \times [40, 20, 8] \text{ codes}$$

etc.

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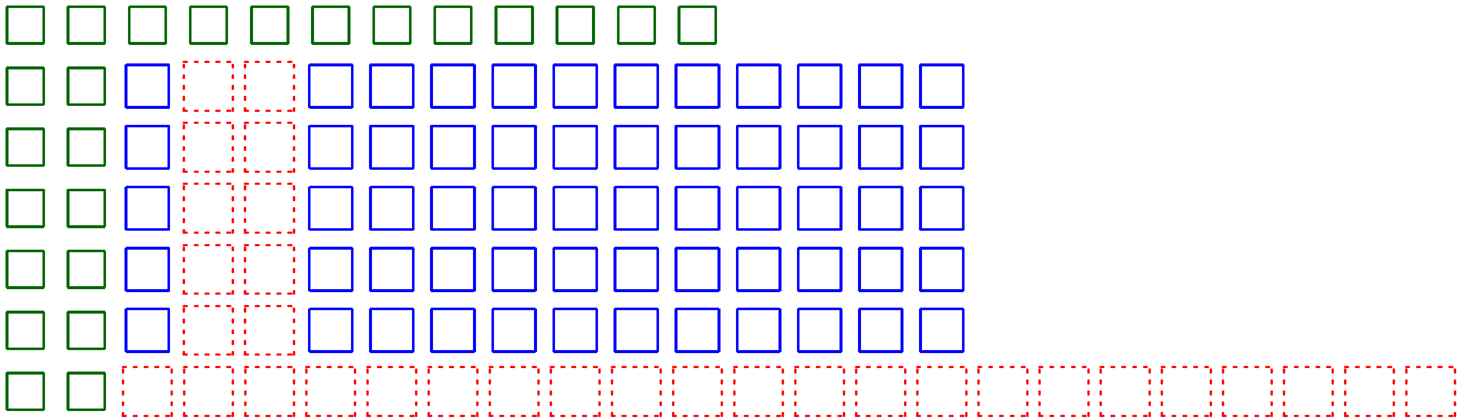
$W_C^{(r)}(z)$:



□ : [Dougherty and Gulliver 2001]

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⬚ : missing

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Theorem (Britz 2005) For a linear code $C \subseteq \mathbb{F}_q^E$ of dimension k,

$$W_C^{(r)}(z) = z^{n-k}(1-z)^k \sum_{i=0}^{r} \frac{(-1)^{r-i}}{[r]_r} q^{\binom{r-i}{2}} \begin{bmatrix} r \\ i \end{bmatrix} T_{M_C}\left(\frac{1+(q^i-1)z}{1-z}, \frac{1}{z}\right)$$

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$$T_M(x,y) = \begin{cases} y\,T_{M/e}(x,y), & \rho_M(e) = 0; \\ x\,T_{M/e}(x,y), & \rho_{M^*}(e) = 0; \\ T_{M\backslash e}(x,y) + T_{M/e}(x,y), & \text{otherwise.} \end{cases}$$

The extremal doubly-even, self-dual binary codes:

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$1 \times [48, 24, 12]$ code

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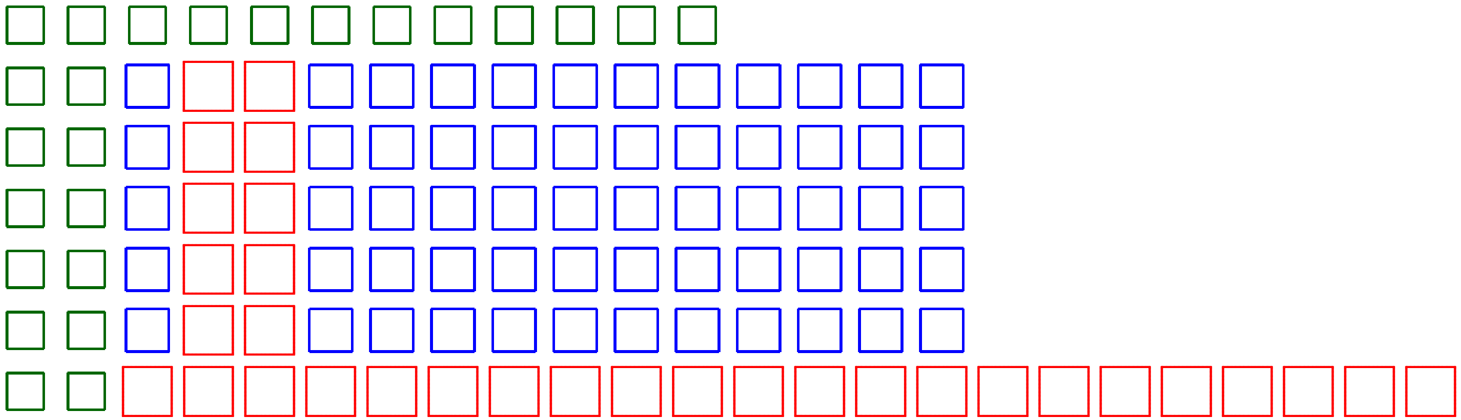

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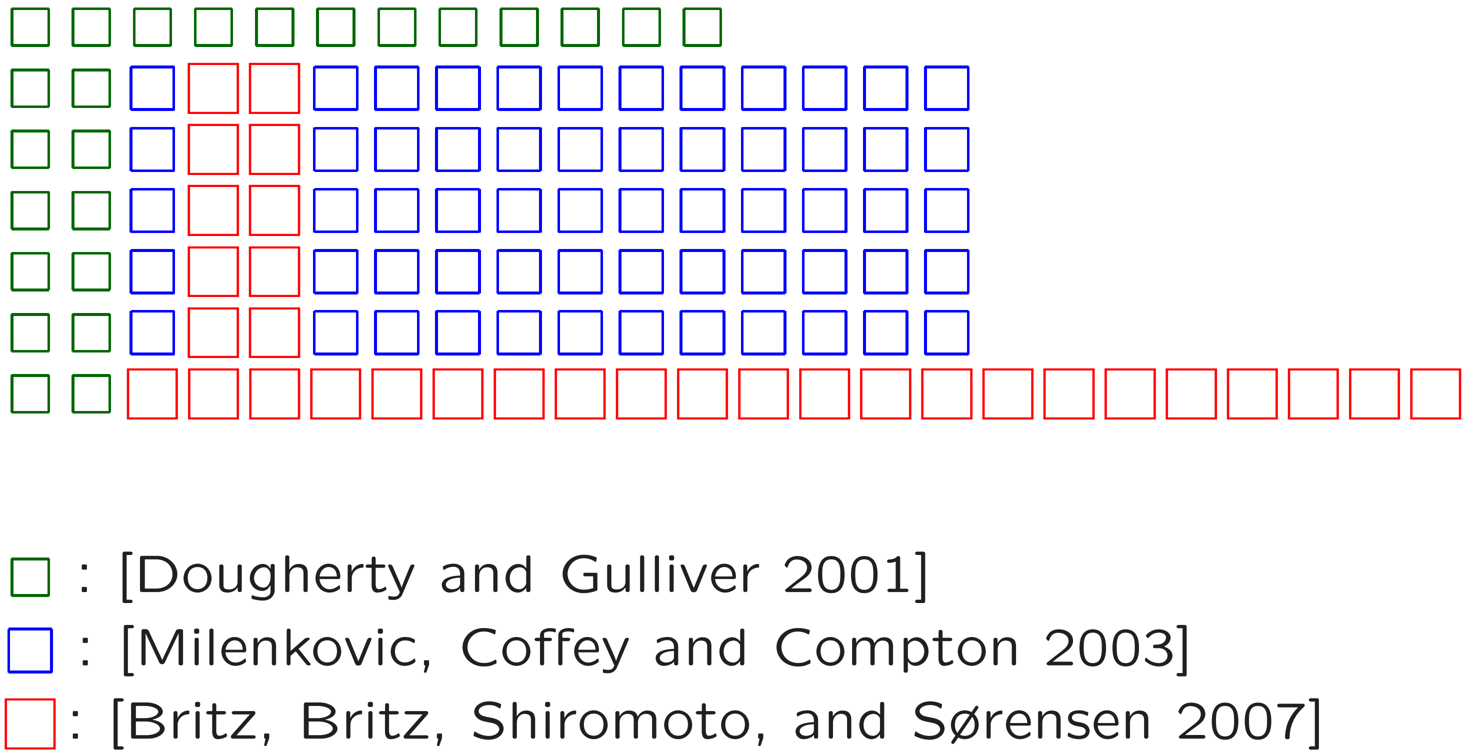
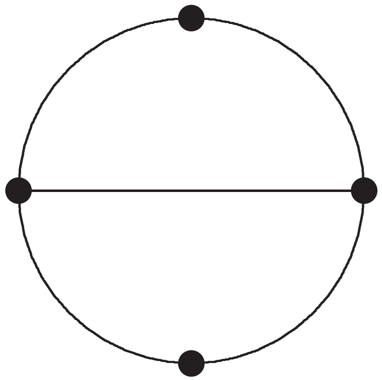
: [Britz, Britz, Shiromoto, and Sørensen 2007]

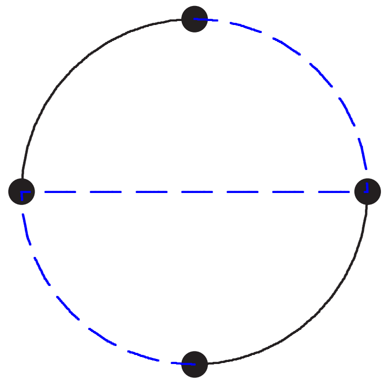
The extremal doubly-even, self-dual binary codes:

1 sec	$1 \times [24, 12, 8]$ code
40 sec	$5 \times [32, 16, 8]$ codes
800 hours	$1 \times [48, 24, 12]$ code

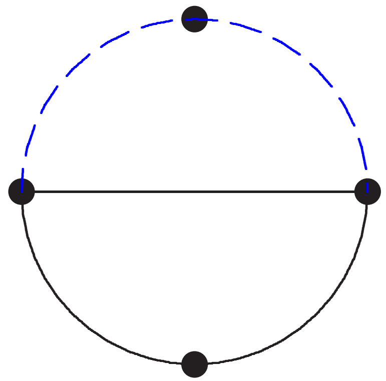
$W_C^{(r)}(z)$:







Bond = minimal cutset



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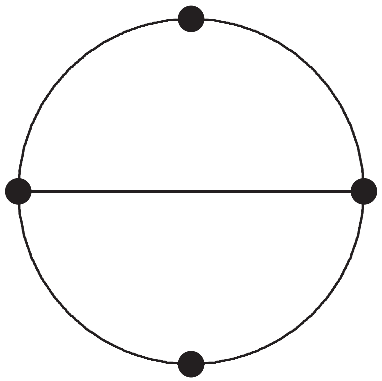
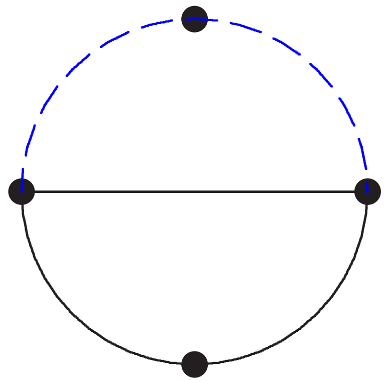
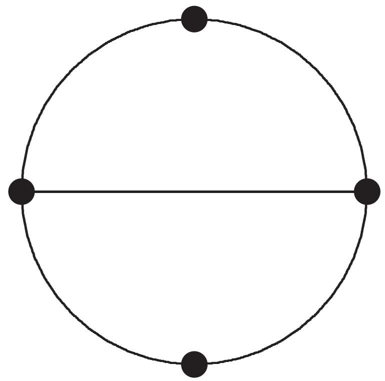


b_1 = minimal size of a bond

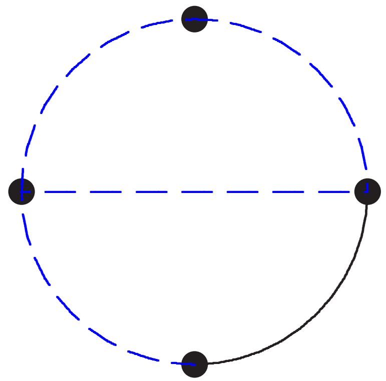


b_1 = minimal size of a bond = 2



b_1 = minimal size of a bond = 2

b_2 = min. # edges in 2 distinct bonds =



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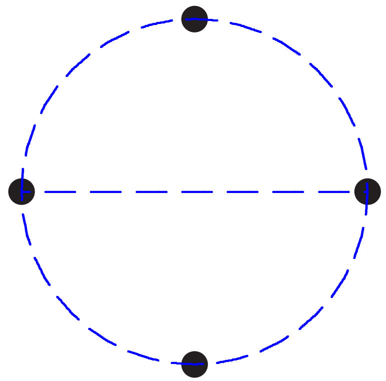

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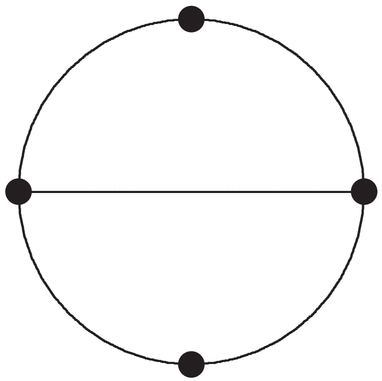

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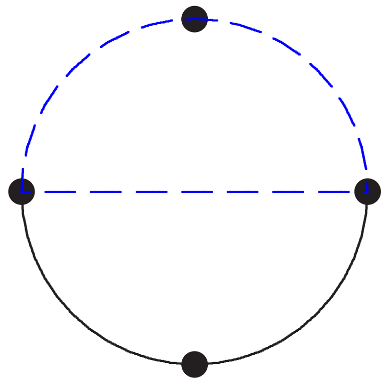

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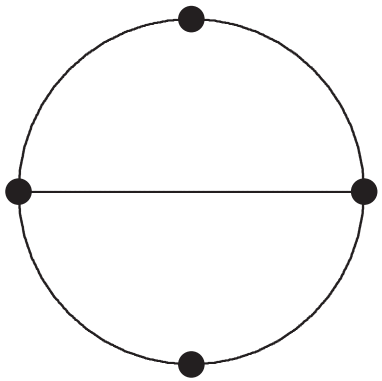
c_1 = minimal size of a cycle = 3



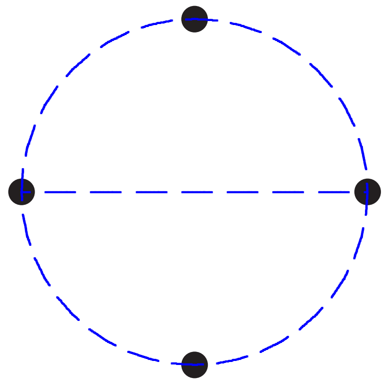
b_1 = minimal size of a bond = 2

b_2 = min. # edges in 2 distinct bonds = 4

b_3 = min. # edges in 3 distinct bonds $B_1, B_2, B_3,\ \ B_3 \not\subseteq B_1 \cup B_2$ = 5

c_1 = minimal size of a cycle = 3

c_2 = min. # edges in 2 distinct cycles =



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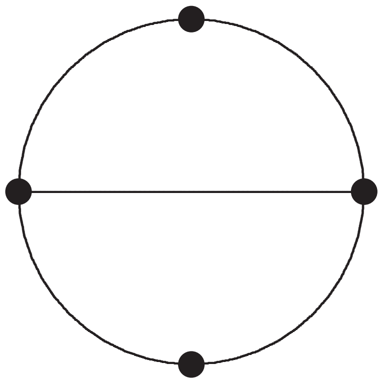

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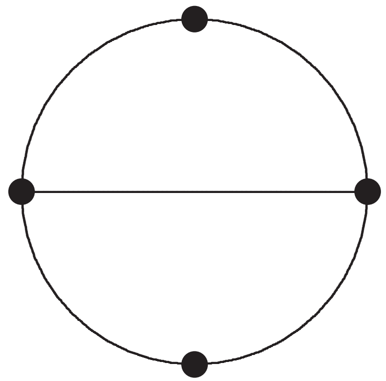
c_1 = minimal size of a cycle = 3

c_2 = min. # edges in 2 distinct cycles = 5

Set $U = \{b_1, b_2, b_3\} = \{2, 4, 5\}$

and $V = \{5 + 1 - c_2, 5 + 1 - c_1\} = \{1, 3\}$.



$b_1 =$ minimal size of a bond $= 2$

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Set $U = \{b_1, b_2, b_3\} = \{2, 4, 5\}$

and $V = \{5 + 1 - c_2, 5 + 1 - c_1\} = \{1, 3\}$.

$U \cup V = \{1, 2, 3, 4, 5\}$ and $U \cap V = \emptyset$

G = a multigraph on n edges

Define

k = # edges in a spanning forest of G

b_i = min. # edges in i bonds, none contained in the union of the others

c_j = min. # edges in j cycles, none contained in the union of the others

$U = \{b_1, \ldots, b_k\}$

$V = \{n+1-c_{n-k}, \ldots, n+1-c_1\}$.

Britz 2007: $U \cup V = \{1, \ldots, n\}$ and $U \cap V = \emptyset$.

$M =$ a matroid of rank k on n elements

Define

$f_i =$ maximal size of an i-rank set in M

$f_j^* =$ maximal size of an j-rank set in M^*

$U = \{f_0 + 1, \ldots, f_{k-1} + 1\}$

$V = \{n - f_{n-k-1}^*, \ldots, n - f_0^*\}.$

Britz, Mayhew, Shiromoto: $U \cup V = \{1, \ldots, n\}$ and $U \cap V = \emptyset$.

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Britz, Mayhew, Shiromoto: Further generalizations.

Britz, Mayhew, Shiromoto: Applications to graphs, codes, modules, matchings.

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Britz, Mayhew, Shiromoto: $U \cup V = \{1, \ldots, n\}$ and $U \cap V = \emptyset$.

Proof. Assume that the theorem is false.

Then $f_i + 1 = n - f_j^*$ for some i, j.

Let $A \subseteq E$ satisfy $|A| = f_j^*$ and $r_{M^*}(A) = j$.

Then $|E - A| = f_i + 1$, so $r_M(E - A) \geq i + 1$.

Since $|E - A| + r_{M^*}(A) - r(M^*) = r_M(E - A)$,

$$-f_j^* + j + r \geq i + 1.$$

Similarly,

$$n - f_i + i - r \geq j + 1.$$

Hence, $1 = n - f_i - f_j^* \geq 2$, a contradiction.

Thank you.