On Mahler's method.

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After a paper of Mahler (1921)

Mahler's method is a technique to show the transcendency of values at algebraic complex numbers of transcendental solutions $f(x) \in L[[x]]$ of

$$f(x^d) = R(x, f(x)), \quad d > 1, \quad R \in \mathbb{Q}(X, Y)$$

The interest of the method is that it can be modified to produce algebraic independence

Example

Let us consider the formal series:

$$f(x)=\prod_{n=0}^{\infty}(1-x^{2^n})=\sum_{n=0}^{\infty}c_nx^n\in\mathbb{Z}[[x]],$$

converging in the open unit ball $B(0,1)\subset\mathbb{C}$ to an analytic function.

$$f(x^2)=\frac{f(x)}{1-x}.$$

f is transcendental over $\mathbb{C}(x)$.

Pólya-Carlson (1921): $f \in \mathbb{Z}[[x]]$ converging in $B(0,1)$ is either rational or transcendental

- f is irrational because of the functional equation
- $$f = \sum_{n=0}^{\infty} (-1)^{a_n} x^n$$
  with $(a_n)_{n\geq 0}$ the Thue-Morse sequence, is irrational because Thue-Morse sequence is known to be not ultimately periodic

Riemann-Hurwitz: if f is algebraic then f must be rational

Following Mahler's paper of 1929,

$f(\alpha)$ is transcendental for α algebraic, with $0 < |\alpha| < 1$.

Let $L \subset \mathbb{C}$ be a number field.
Absolute logarithmic height of $(\alpha_0 : \cdots : \alpha_n) \in \mathbb{P}_n(L)$:

$$h(\alpha_0 : \cdots : \alpha_n) = \frac{1}{[L : \mathbb{Q}]} \sum_{v \in M_L} d_v \log \max\{|\alpha_0|_v, \ldots, |\alpha_n|_v\}.$$

$|\cdot|_v$ chosen so that *product formula* holds:

$$\prod_{v \in M_L} |\alpha|_v^{d_v} = 1, \quad \alpha \in L^\times.$$

If $n = 1$ we also write $h(\alpha) := h(1 : \alpha)$.

For all $N \geq 0$, choose $P_N \in \mathbb{Z}[X, Y]$ non-zero of partial degrees $\leq N$, such that

$$F_N(x) := P_N(x, f(x)) = cx^{\nu(N)} + \cdots \quad \text{(auxiliary function)}$$

with $\nu(N) \geq N^2$.
For all n big enough depending on N and α, f,

$$-\infty < \log|F_N(\alpha^{2^{n+1}})| \leq c_1 \nu(N) 2^{n+1} \log|\alpha|.$$

Let us suppose by contradiction that $L = \mathbb{Q}(\alpha, f(\alpha)) \subset \mathbb{Q}^{alg.}$

$$F_N(\alpha^{2^{n+1}}) \quad = \quad P_N\left(\alpha^{2^{n+1}}, \frac{f(\alpha)}{(1-\alpha)\cdots(1-\alpha^{2^n})}\right) \in L$$

$d = [L : \mathbb{Q}]$

$$\begin{aligned}
\log|F_N(\alpha^{2^{n+1}})| \geq & \\
\geq & -d(L(P_N) + Nh(\alpha^{2^{n+1}}) + Nh(f(\alpha)/(1-\alpha)\cdots(1-\alpha^{2^n}))) \\
\geq & -d(L(P_N) + N2^{n+1}h(\alpha) + Nh(f(\alpha)) + \sum_{i=0}^{n} h(1-\alpha^{2^n})) \\
\geq & -d(L(P_N) + 2N2^{n+1}h(\alpha) + Nh(f(\alpha)) + (n+1)\log 2).
\end{aligned}$$

Therefore, dividing by $2^{n+1}N$,

$$c_2 N \log|\alpha| \geq -2dh(\alpha).$$

A good choice of N yields a contradiction.

For example, to prove that $f(1/2) \notin \mathbb{Q}^{\text{alg.}}$ it suffices to consider the polynomial

$$P(X, Y) = 2X^2 + XY + Y - 1$$

similarly, $f(2/3) \notin \mathbb{Q}^{\text{alg.}}$ is proved with

$$P = X^2Y^2 - 4X + 8X^2 + 4Y + 8XY - 12X^2Y - 3Y^2 - 6XY^2 - 1$$

Variant in positive characteristic

$$q = p^e, \quad A = \mathbb{F}_q[\theta], \quad K = \mathbb{F}_q(\theta), \quad K^{\text{alg.}}, \quad |\cdot|, \quad K_\infty, \quad \mathbb{C}_\infty$$

Consider the power series

$$\Pi(u) = \prod_{n=1}^{\infty}(1 - \theta u^{q^i}),$$

which converges for $u \in \mathbb{C}_\infty$ such that $|u| < 1$ and satisfies the functional equation:

$$\Pi(u^q) = \frac{\Pi(u)}{1 - \theta u^q}.$$

For $q = 2$, we notice that $\Pi(u) = \sum_{n=0}^{\infty} \theta^{b_n} u^{2n}$, where

$$(b_n)_{n\geq 0} = 0, 1, 1, 2, 1, 2, 2, 3, 1, 2, 2, \ldots$$

is the sequence with b_n which counts the number of 1's in the binary expansion of n

Π is transcendental over $\mathbb{C}_\infty(x)$

- Because it has infinitely many zeroes
- By a criterion of Sharif and Woodcock generalising part of Christol's theorem (exercise, using that b has infinite image)

For α algebraic over K with $0 < |\alpha| < 1$, $\Pi(\alpha)$ is transcendental over K

"Same" proof as before

- Construction of an auxiliary function with multiplicity in 0
- Extrapolation on $\{\alpha^{q^n}\}$
- Absolute logarithmic height $h : \mathbb{P}_n(K^{\text{alg.}}) \longrightarrow \mathbb{R}_{\geq 0}$
- An analogue of Liouville's inequality

More generally, assume that we are in one of the following two cases.

- $\mathcal{C} = \mathbb{C}$, $\mathcal{Q} = \mathbb{Q}$, $|\cdot|$ archimedean absolute value
- $\mathcal{C} = \mathbb{C}_\infty$, $\mathcal{Q} = K$, $|\cdot|$ ultrametric absolute value $|\theta| = q$

L finite extension of $\mathcal{Q}$

Let $f \in L[[x]]$, $R \in L(X, Y)$, $h_Y(R) < d$ and $\alpha \in L$ be such that:

- f is transcendental over $\mathcal{C}(x)$
- f converges for $x \in \mathcal{C}$, $|x| < 1$
- There exists $d > 1$ such that $f(x^d) = R(x, f(x))$

Then, for all $n \gg 0$, $f(\alpha^{d^n}) \in \mathcal{C}$ is transcendental over $\mathcal{Q}$.

Application (Denis, $\mathcal{C} = \mathbb{C}_\infty$): the following "number" is transcendental

$$\widetilde{\pi} = \theta(-\theta)^{1/(q-1)} \prod_{i=1}^{\infty} (1 - \theta^{1-q^i})^{-1} = \theta(-\theta)^{1/(q-1)} \Pi(\theta^{-1})^{-1}$$

$(\alpha = \theta^{-1})$

A result of Corvaja and Zannier 2002 (deduced from Schmidt's Subspace Theorem).

L number field

- $f \in \mathbb{Q}^{\text{alg.}}[[x]]$ not a polynomial
- $\alpha \in L$
- S a finite set of places containing the archimedean ones
- $\mathcal{A} \subset \mathbb{N}$ an infinite set
- $f(\alpha^n) \in L$ is an S-integer for all $n \in \mathcal{A}$

Then,

$$\liminf_{n \in \mathcal{A}} \frac{h(f(\alpha^n))}{n} = \infty$$

Application with $\mathcal{A} = \{d, d^2, d^3, \ldots\}$:

- $f \in \mathbb{Q}^{\text{alg.}}[[x]]$
- f not a polynomial
- $f(x^d) = R(x, f(x))$ with $R \in \mathbb{Q}^{\text{alg.}}(X, Y)$
- $\deg_Y R < d$

Then, $f(\alpha^{d^n})$ is transcendental for α algebraic and for all $n \gg 0$

Algebraic independence (Loxton-van der Poorten, Denis,...)

L finite extension of $\mathcal{Q}$

Let $f_1,\ldots,f_m\in L[[x]]$ and $\alpha\in L$, $0<|\alpha|<1$ be such that:

- $f_1,\ldots,f_m$ are algebraically independent over $\mathcal{C}(x)$
- f_i converges for $x\in\mathcal{C}$, $|x|<1$ for all i
- There exists $d>1$ such that $f_i(x^d)=a_i(x)f_i(x)+b_i(x)$ with $a_i,b_i\in L(x)$ for all i

Then, for $n\gg 0$, $f_1(\alpha^{d^n}),\ldots,f_m(\alpha^{d^n})\in\mathcal{C}$ are algebraically independent over $\mathcal{Q}$.

- If $\mathcal{C} = \mathbb{C}$, Loxton-van der Poorten (1977), generalised by Nishioka, Becker, Töpfer,...
- In both cases $\mathcal{C} = \mathbb{C}, \mathbb{C}_\infty$, it can be deduced from a criterion of Philippon (1992).
- Denis (2000) used this criterion in the case $\mathcal{C} = \mathbb{C}_\infty$.

More is true when $\mathcal{C} = \mathbb{C}$ (Philippon)

Let

- $f_1, \ldots, f_m \in L[[x]]$
- $\mathcal{A} \in \mathbf{Mat}_{n\times n}(L(x)), \quad \mathcal{B} \in \mathbf{Mat}_{n\times 1}(L(x))$
- $\alpha \in \mathbb{C}$

be such that:

- $f_1, \ldots, f_m$ are algebraically independent
- f_i converges for $x \in \mathcal{C}$, $|x| < 1$
- $\underline{f}(x^d) = \mathcal{A}(x) \cdot \underline{f}(x) + \mathcal{B}(x)$

Then, for $n \gg 0$, $\alpha, f_1(\alpha^{d^n}), \ldots, f_m(\alpha^{d^n}) \in \mathcal{C}$ generate a subfield of $\mathbb{C}$ of transcendence degree $\geq m$.

Uses:

- A criterion for algebraic independence by Philippon (1998)
- Construction at $x = 0$ with Siegel's lemma and extrapolation on $\{\alpha^{d^n}\}$
- A multiplicity estimate by Nishioka (1990).

Some results with $C = \mathbb{C}_\infty$ (by Denis method)

- $\beta_1, \ldots, \beta_m \in K$. $\log_{\text{Carlitz}}(\beta_i)$ K-linearly independent $\Rightarrow$ algebraically independent
- The first $p-1$ "divided derivatives" of $\widetilde{\pi}$ are algebraically independent
- $\widetilde{\pi}$ and "odd" values of Carlitz-Goss zeta function are algebraically independent
- Various $\widetilde{\pi}$'s are algebraically independent

Denis deformation of Carlitz's logarithms ($\alpha = \theta$)

$\beta = \beta(\theta) \in K$, $|\beta| < q^{q/(q-1)}$.

$$F_\beta(x) = \beta(x) + \sum_{n \geq 1} (-1)^n \frac{\beta(x^{q^n})}{\prod_{j=1}^n (x^{q^j} - \theta)}$$

1. converges for $|x| > q^{1/q}$
2. $\log_{\text{Carlitz}}(\beta) = F_\beta(\theta)$,
3. $F_\beta(x^q) = (\theta - x^q)(F_\beta(x) - \beta(x))$,
4. $F_{\beta_1+\beta_2} = F_{\beta_1} + F_{\beta_2}$,
5. $F_{\Phi_{\text{Carlitz}}(\theta)\beta}(x) = \theta F_\beta(x) + (x - \theta)\beta(x)$

$\beta_1, \ldots, \beta_m \in K$, $|\beta_i| < q^{q/(q-1)}$

$f_1 := F_{\beta_1}, \ldots, f_m := F_{\beta_m}$

$a = \theta - x^q$

$b_i = -\frac{\beta_i(x)}{\theta - x^q}$

If $f_1, \ldots, f_m$ are algebraically dependent over $K^{\text{alg.}}(x)$, then, there is a non-trivial linear dependence relation

$$c_1 f_1 + \cdots + c_m f_m + c_0 = 0$$

with

- $c_1, \ldots, c_m \in K^{\text{alg.}}$
- $c_0 \in K^{\text{alg.}}(x)$, $c_0(x^q) = ac_0(x) - \sum_i c_i b_i$

- $\beta_1 =$ root of $X^q - X - \theta = 0$
- $\beta_2 = \theta$

$\log_{\text{Carlitz}}(\beta_1), \log_{\text{Carlitz}}(\beta_2)$ are algebraically independent (after Papanikolas Theorem).

Is it possible to prove it with Mahler's method?

Probably not with $\alpha = \theta$ but it can be done with $\alpha = \beta_1$

Define, for $\beta \in K$,

$$\widetilde{F}_\beta(x) = \beta(x) + \sum_{n=1}^{\infty}(-1)^n \frac{\beta(x^{q^n})}{\prod_{j=1}^{n}(x^{q^{j+1}} - x^{q^j} - \theta)}$$

We have:

1. $\widetilde{F}_\beta(x^q) = (\theta - x^{q^2} + x^q)(\widetilde{F}_\beta(x) - \beta(x))$
2. $\widetilde{F}_{(\theta^q - \theta)\beta + \beta^q}(x) = \theta \widetilde{F}_\beta(x) + (x^q - x - \theta)\beta(x)$
3. $\widetilde{F}_\beta(\alpha) = \log_{\text{Carlitz}}(\beta(\alpha))$

In particular

- $\widetilde{F}_\theta(\alpha) = \log_{\text{Carlitz}}(\alpha)$
- $\widetilde{F}_{\theta^q - \theta}(\alpha) = \log_{\text{Carlitz}}(\theta)$

Mahler's method gives measures of algebraic independence:
$Q \in \mathbb{Z}[X_1, \ldots, X_m] \setminus \{0\}$

- $\deg Q \leq D$,
- $H(Q) \leq H$

$$|Q(f_1(\alpha), \ldots, f_m(\alpha))| \geq \exp\{-c_1 D^m (D^{m+2} + \log H)\}$$

(Töpfer, 1995)

Recent result by Denis:

$$|Q(\widetilde{\pi})| \geq \exp\{-c_2 D^4(D + \log H)\}$$

(extends to Carlitz's logarithms of rationals and to certain ζ-values)

Mahler's method (for $\mathcal{C} = \mathbb{C}$) extends to:

- Functions of several variables, linear functional equations (Loxton-van der Poorten, Nishioka,...)
- Non-linear functional equations: $P(x, f(x), f(x^d)) = 0$ (Greuel, 2000)