# Frobenius actions on the (co-)homology of Drinfeld modules

Ernst-Ulrich Gekeler

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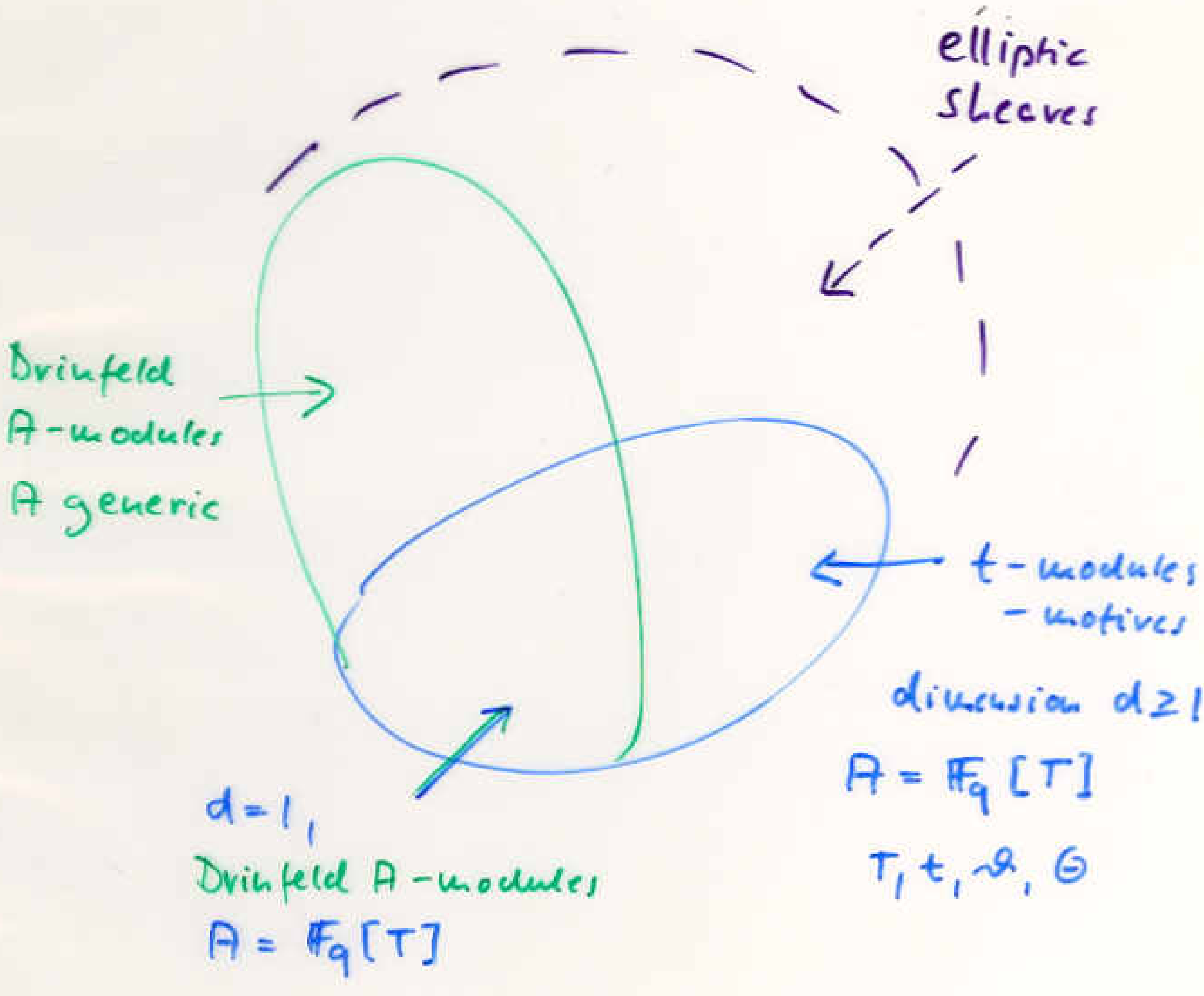

# Notation

$A = \Gamma(\mathcal{O}_X(X - \{\infty\}))$, $X$ = curve / $\mathbb{F}_q$

e.g. $A = \mathbb{F}_q[T]$ "Drinfeld ring"

$K = \mathrm{Quot}(A)$, $K_\infty$, $C_\infty = \widehat{\overline{K}}_\infty$

$\mathfrak{p}, \mathfrak{l}$ (maximal) primes of $A$, $\mathbb{F}_\mathfrak{p} := A/\mathfrak{p}$

$\gamma : A \longrightarrow L$ any $A$-field

$\gamma : A \hookrightarrow K \hookrightarrow L$ $\quad \mathrm{char}_A(L) = \infty$

$\gamma : A \longrightarrow \mathbb{F}_\mathfrak{p} \hookrightarrow L$ $\quad \mathrm{char}_A(L) = \mathfrak{p}$

$A_L := L \otimes A$ Dedekind ring ("$\otimes$" = $\otimes_{\mathbb{F}_q}$)

$I_L := \ker \left( \begin{array}{c} A_L \longrightarrow L \\ l \otimes a \mapsto l \cdot \gamma(a) \end{array} \right)$

$\hat{A}_L := \varprojlim A_L / I_L^n$ = ring of "A-Witt vectors of L"

$\tau = (x \mapsto x^q) \in \mathrm{End}_{L, \mathbb{F}_q}(\mathbb{G}_a) = L\{\tau\}$

$\tau c = c^q \tau$

$\phi/_L$ = Drinfeld $A$-module over $L$, of rank $r$

(co-)homology modules of $\phi / L$ ?

Betti

$\ell$-adic

de Rham

no proper cohomology theories, only analogues of the classical $H_1$ and $H^1$

Betti $\quad \phi / C_\infty \quad \phi \leftrightarrow \Lambda =$ A-lattice in $C_\infty$

$H_{Betti}(\phi, A) := \Lambda$ projective A-module, rank $r$

$H_{Betti}(\phi, B) := \Lambda \otimes_A B$

$H^*_{Betti}(\phi, B) := \mathrm{Hom}_A(\Lambda, B)$

B any A-algebra (or even A-module)

## $\ell$- adic

$H_\ell(\phi) := T_\ell(\phi) \leftarrow$ Tate module,

constructed from $\ell^\infty$-torsion points

$A_\ell$-free of rank $r$ $(\ell \neq \mathrm{char}_A(L))$

of rank $r-h < r$

$(\ell = p = \mathrm{char}_A(L))$

endowed with a $\mathrm{Gal}(L)$- action

Coefficients in $K_\ell = \mathrm{Quot}(A_\ell)$, $\overline{K}_\ell, \ldots$ :

apply $\otimes_{A_\ell} K_\ell$

"Cohomology": apply $\mathrm{Hom}_{A_\ell}(\ , A_\ell)$

$H_{Betti}$ and $H_\ell$ have obvious functorial properties

Comparison isomorphism:

$$\phi / C_\infty : \quad \Lambda \otimes_A A_\ell \xrightarrow{\cong} T_\ell(\phi)$$

de Rham (various definitions)

$$H^*_{DR}(\phi, L) = \mathrm{Ext}^{\#}(\phi, \mathbb{G}_a)$$

(Deligne, Anderson)

via biderivations (Jing Yu)

$\phi : A \longrightarrow L\{\tau\}$ Drinfeld module
$a \longmapsto \phi_a$

$M(\phi, L) := L\{\tau\}$ as $L \otimes A$ - module

$l \otimes 1$ acts through left multiplication by $l \in L$

$1 \otimes a$ " right " $\phi_a$

$N(\phi, L) = A_L$ - submodule $L\{\tau\}\tau$

$\eta : A \longrightarrow N(\phi, L)$ $\mathbb{F}_q$ - linear biderivation iff
$a \longmapsto \eta_a$ "derivation"

- $\eta$ $\mathbb{F}_q$ - linear
- $\eta_{ab} = \gamma(a)\eta_b + \eta_a \cdot \phi_b$, $a, b \in A$

$\eta$ strictly inner $:\Longleftrightarrow \exists\, n \in N(\phi, L)$ s.t.

$$\eta_a = \gamma(a) n - n \cdot \phi_a$$

$D(\phi, L) :=$ $A_L$-module of all derivations

$\uparrow$

$D_{si}(\phi, L) :=$ " strictly inner "

$\|$

$I_L D(\phi, L)$

$$H^*_{DR}(\phi, L) := D(\phi, L)/D_{si}(\phi, L)$$
$$= D(\phi, L) \otimes_{A_L} L$$

<u>Theorem</u> $H_{Betti}(\phi, A) \times H^*_{DR}(\phi, C_\infty) \longrightarrow C_\infty$

$$(\lambda, [\eta]) \mapsto \int_\lambda \eta$$

is a perfect duality

( Anderson for $A = \mathbb{F}_q[T]$, 1987 ?
G. for general $A$, 1988 )

yields $H^*_{DR}(\phi, C_\infty) \xrightarrow{\cong} H^*_{Betti}(\phi, C_\infty)$

comparison of different $L$-structures if

$\phi/L$, $K \subset L \subset C_\infty$ : transcendence problems!

Anderson - Brownawell - Papanikolas - Yu ...

- $H^*_{DR}$ may be described in a coordinate-free language and generalized to $\phi/S$, where $S$ is an arbitrary $A$-scheme (instead of $S = \operatorname{Spec} L$ as before). Contravariant functor in $\phi$ ($S$ fixed) and in $S$ (pullback)

- locally free sheaf $\mathcal{H}_{DR}(\phi, S)$ of constant rank $r$ if $\phi/S$ has constant rank $r$

- base change, formalism of vanishing cycles

- related to $T_x(M^r)$ $\qquad x \longleftrightarrow \phi$

tangent space at $x$

moduli space for rank-$r$ Drinfeld modules

(G. 1989)

## Frobenius actions

$\phi / L$ , $L \supset \mathbb{F}_p$ finite, $\ell \neq p$

$\alpha \in \mathrm{End}(\phi) \rightsquigarrow i_\ell(\alpha) \in \mathrm{End}_{A_\ell}(T_\ell(\phi))$

$P_\alpha(X) :=$ char. polynomial of $i_\ell(\alpha)$

coefficients in $A$ independent of $\ell \neq p$

take $\alpha = F_L$ = Frobenius associated with $L$

$$P_{\phi/L}(X) := P_{F_L}(X)$$

<u>Thm</u> (G. 1990) (i) $\mathrm{cok}(i_\ell(F_L)) \xrightarrow{\cong} \ell$-part of $(L, \phi)$

(ii) $(P_{\phi/L}(1)) = \chi(L, \phi)$

($\chi(M)$ = Euler-Poincaré char. of the finite $A$-module $M$, $\chi(A/\mathfrak{m}) = \mathfrak{m}$, mult. in short exact sequences)

<u>Corollary</u> : $L$ carries a Drinfeld module $\Rightarrow p^{[L:\mathbb{F}_p]}$ principal ideal of $A$.

$$Z_{\phi/L}(X) := \prod_{0 \le i \le r} \Lambda^i \det(1 - X i_\ell(F))^{(-1)^{i+1}}$$

<u>Thm.</u> (G. 1990) Write

$$X \frac{d}{dX} \log Z_{\phi/L}(X) = \sum_{n \ge 1} a_n X^n \in A[[X]]$$

then $(a_n) = \chi(L^{(n)}; \phi)$

<u>Example:</u> $A = \mathbb{F}_q[T]$, $r = 2$

$$P_{\phi/L}(X) = X^2 - aX + b, \quad a, b \in A$$

$$|a|^2 \le |L|$$

$$(b) = \mathfrak{p}^{[L : \mathbb{F}_\mathfrak{p}]}$$

$$Z_{\phi/L}(X) = \frac{1 - aX + bX^2}{(1-X)(1-bX)}$$

How do we calculate $a, b$?

$b$: formula of Hsia-Yu (2000)

$a$: relation with Eisenstein series (G. 1988, 2004)

# Frobenius on $H^*_{DR}$

$\phi/L$ , $L \supset \mathbb{F}_\mathfrak{p}$ (later: $L$ perfect)

$|\mathbb{F}_\mathfrak{p}| = q^d$ , $\varphi = \varphi_\mathfrak{p} = \tau^d$

$f = \sum a_i X^i \in L[X] \rightsquigarrow f^{(\varphi)} := \sum a_i^{q^d} X^i$

$$F_\mathfrak{p} = F : \phi \longrightarrow \phi^{(\varphi)}$$
$$x \longmapsto x^{q^d}$$

geometric Frobenius

by functoriality: $F_D : D(\phi^{(\varphi)}, L) \to D(\phi, L)$

$F_{DR} : H^*_{DR}(\phi^{(\varphi)}, L) \to H^*_{DR}(\phi, L)$

(L-linear, in general not bijective)

$$\varphi_D : D(\phi, L) \longrightarrow D(\phi^{(\varphi)}, L)$$
$$\eta \longmapsto \eta^{(\varphi)}$$

arithmetic Frobenius

(not L-linear, but $\varphi$-semilinear; bijective if L perfect)

$$\begin{tikzcd} Fr_D : D(\phi, L) \arrow[r, "\varphi_D"] \arrow[rr, bend right] & D(\phi^{(\varphi)}, L) \arrow[r, "F_D"] & D(\phi, L) \end{tikzcd}$$

total Frobenius

ditto: $Fr_{DR}$

Hasse-Witt:

$$H^*_{DR}(\phi, L) = H^*_{DR}(\phi, L)_0 \oplus H^*_{DR}(\phi, L)_1$$

unique decomposition

$Fr_{DR}$ acts nilpotently     $Fr_{DR}$ acts bijectively

<u>Thm</u>: $\dim H^*_{DR}(\phi, L)_0 = h = \text{height}(\phi)$

$\dim \ker (Fr_{DR}^i) = i \quad (0 \leq i \leq h)$

(B. Anglès 1996 in case L finite, G. 2008 general case)

<u>Thm</u> (G. 2008) $\exists$ (quasi-)canonical bilinear map

$$({}_{\mathfrak{p}}\phi)(\bar{L}) \times H^*_{DR}(\phi, L)/H^*_{DR}(\phi, L)_0 \longrightarrow \bar{L}$$

$$(x, [\eta]) \longmapsto \langle x, [\eta] \rangle$$

that after $\otimes \bar{L}$ is a perfect duality

"quasi": well-defined up to $\mathbb{F}_{\mathfrak{p}}^*$

In case $\mathfrak{p} = (p)$ is principal:

$$\langle x, [\eta] \rangle = \eta_p(x)$$