Columnar Clouds and Internal Waves

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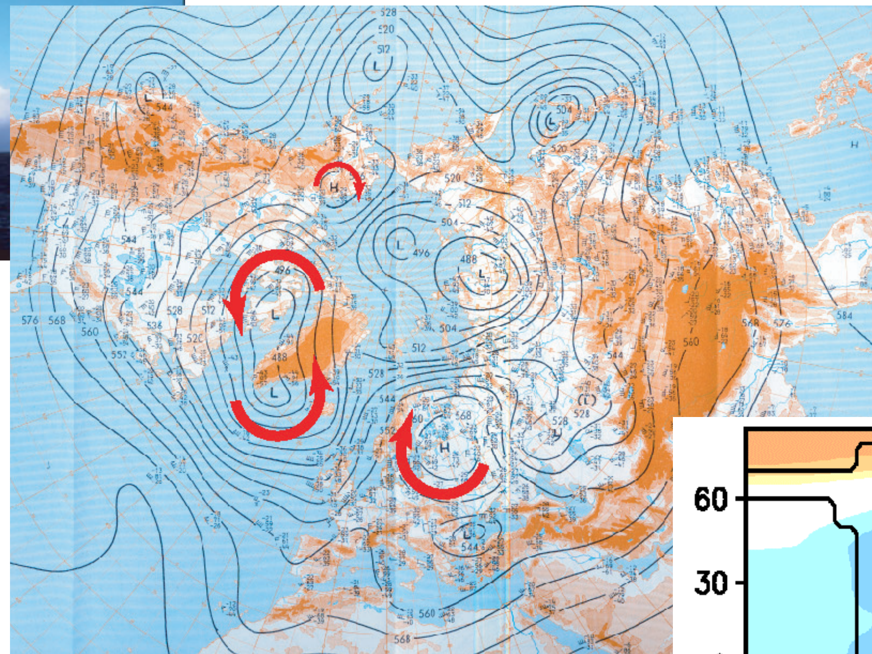
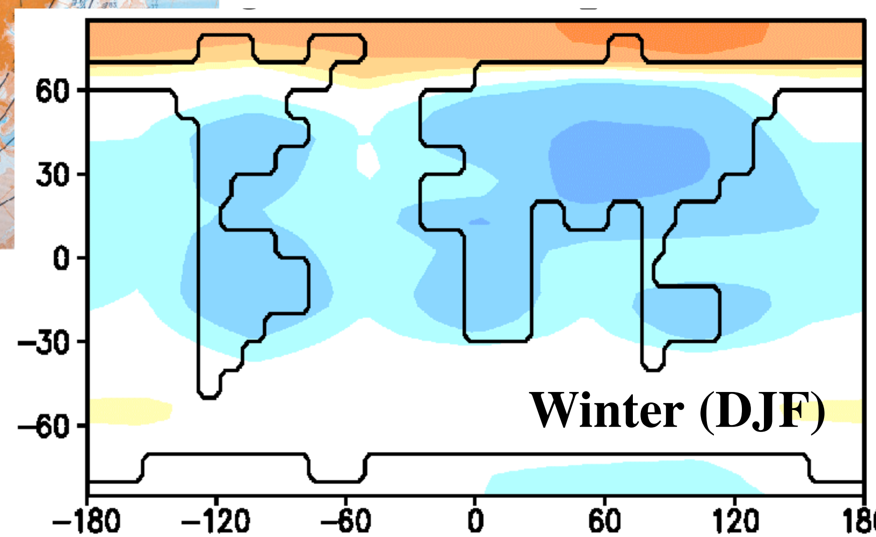
Oswald Knoth	(IfT, Leipzig)
Oliver Bühler	(Courant Institute, NYU)

10 km / 20 min



1000 km / 2 days



10000 km / 1 season

Scales

$$
\begin{aligned}
\boldsymbol{u}_t + \boldsymbol{u}\cdot\nabla\boldsymbol{u} + w\boldsymbol{u}_z + \nabla\pi &= \boldsymbol{S}_{\boldsymbol{u}} \\
w_t + \boldsymbol{u}\cdot\nabla w + w w_z + \pi_z &= -\theta' + S_w \\
{\theta'}_t + \boldsymbol{u}\cdot\nabla\theta' + w{\theta'}_z &= S'_\theta \\
\nabla\cdot(\rho_0\boldsymbol{u}) + (\rho_0 w)_z &= 0
\end{aligned}
$$

$$\theta = 1 + \varepsilon^4\theta'(\boldsymbol{x}, z, t) + o(\varepsilon^4)$$

Anelastic Boussinesque Model

10 km / 20 min

$$(\partial_\tau + \boldsymbol{u}^{(0)}\cdot\nabla)\, q = 0$$

$$q = \zeta^{(0)} + \Omega_0\beta\eta + \frac{\Omega_0}{\rho^{(0)}}\frac{\partial}{\partial z}\left(\frac{\rho^{(0)}}{d\Theta/dz}\theta^{(3)}\right)$$

$$\zeta^{(0)} = \nabla^2\pi^{(3)}, \qquad \theta^{(3)} = -\frac{\partial\pi^{(3)}}{\partial z}, \qquad \boldsymbol{u}^{(0)} = \frac{1}{\Omega}\boldsymbol{k}\times\nabla\pi^{(3)}$$

Quasi-geostrophic theory

1000 km / 2 days

$$
\begin{aligned}
\frac{\partial Q_T}{\partial t} + \nabla\cdot\boldsymbol{F}_T &= S_T \\
\frac{\partial Q_q}{\partial t} + \nabla\cdot\boldsymbol{F}_q &= S_q
\end{aligned}
$$

$$Q_\varphi = \int_{z_s}^{H_a}\rho\,\varphi\,dz\,, \qquad \boldsymbol{F}_\varphi = \int_{z_s}^{H_a}\rho\left(\boldsymbol{u}\,\varphi + (\widehat{\boldsymbol{u}'\varphi'}) + \boldsymbol{D}^\varphi\right)\,dz\,, \qquad \left(\varphi\in\{T,q\}\right)$$

$$T = T_s(t,\boldsymbol{x}) + \Gamma(t,\boldsymbol{x})\left(\min(z,H_T) - z_s\right), \qquad q = q_s(t,\boldsymbol{x})\exp\left(-\frac{z-z_s}{H_q}\right)$$

$$\rho = \rho_*\exp\left(-\frac{z}{h_{sc}}\right), \qquad p = p_*\exp\left(-\frac{\gamma z}{h_{sc}}\right) + p_0(t,\boldsymbol{x}) + g\rho_*\int_0^z\frac{T}{T_*}dz'$$

$$\boldsymbol{u} = \boldsymbol{u}_g + \boldsymbol{u}_a\,, \qquad f\rho_*\boldsymbol{k}\times\boldsymbol{u}_g = -\nabla_{\boldsymbol{x}}p \qquad \boldsymbol{u}_\alpha = \alpha\nabla p_0$$

V. Petoukhov et al., *CLIMBER-2* ..., Climate Dynamics, 16, (2000)

EMIC - equations (CLIMBER-2)

10000 km / 1 season

Scales

Three-dimensional compressible flow equations

$$
\begin{aligned}
\rho_t \quad &+ \nabla \cdot (\rho \boldsymbol{v}) &&= 0 \\
(\rho \boldsymbol{v})_t &+ \nabla \cdot (\rho \boldsymbol{v} \circ \boldsymbol{v}) + \nabla p + \boldsymbol{\Omega} \times \rho \boldsymbol{v} &&= \boldsymbol{S}_{\rho \boldsymbol{v}} - \rho g \, \boldsymbol{k} \\
(\rho e)_t &+ \nabla \cdot (\boldsymbol{v} \, [\rho e + p]) &&= S_{\rho e} \\
(\rho Y_j)_t &+ \nabla \cdot \big(\rho Y_j \boldsymbol{v}\big) &&= S_{\rho Y_j}
\end{aligned}
$$

$$
(\rho e) = \frac{p}{\gamma - 1} + \frac{1}{2} \rho \boldsymbol{v}^2 + \rho \sum_{j=1}^{N} Q_j Y_j
$$

How are the various reduced models related to this system ?

Key ingredients

1. **Identification of**
   - **uniformly valid system scales**
   - **non-dimensional parameters**
   - **distinguished limits**
2. **Specializations of a multiple scales ansatz**

$$
\begin{aligned}
a &= 6 \cdot 10^{6} \ \ \mathrm{m} \\
\Omega &= 10^{-4} \ \ 1/\,\mathrm{s} \\
g &= 9.81 \ \ \mathrm{m/s^2} \\
\\
p_{\mathrm{ref}} &= 10^{5} \ \ \mathrm{kg/ms^2} \\
T_{\mathrm{ref}} &= 300 \ \ \mathrm{K} \\
\Delta\theta &= 50 \ \ \mathrm{K} \\
\\
R &= 287 \ \ \mathrm{m^2/s^2K} \\
\gamma &= 1.4
\end{aligned}
$$

Key ingredients

1. **Identification of**

   - **uniformly valid system scales**
   - **non-dimensional parameters**
   - **distinguished limits**

2. **Specializations of a multiple scales ansatz**

$$\frac{c_{\mathrm{ref}}}{\Omega\, a} \sim 0.5$$

$$\frac{a\,\Omega^2}{g} \sim 6 \cdot 10^{-3}$$

$$\frac{\Delta\theta}{T_{\mathrm{ref}}} \sim 1.6 \cdot 10^{-1}$$

$$\left(c_{\mathrm{ref}} = \sqrt{\gamma R T_{\mathrm{ref}}}\right)$$

Key ingredients

1. **Identification of**
   - **uniformly valid system scales**
   - **non-dimensional parameters**
   - **distinguished limits**

$$\frac{c_{\mathrm{ref}}}{\Omega\, a} \sim \sqrt{\varepsilon}$$

$$\frac{h_{\mathrm{sc}}}{a} \sim \varepsilon^3$$

$$\frac{\Delta\theta}{T_{\mathrm{ref}}} \sim \varepsilon$$

$$(\varepsilon \to 0)$$

2. **Specializations of a multiple scales ansatz**

$$c_{\mathrm{ref}} = \sqrt{p_{\mathrm{ref}}/\rho_{\mathrm{ref}}}$$

$$h_{\mathrm{sc}} = p_{\mathrm{ref}}/g\rho_{\mathrm{ref}}$$

Scaled governing equations

$$
\begin{aligned}
\rho_t \quad &+ \nabla \cdot (\rho \boldsymbol{v}) &&= 0 \\
(\rho \boldsymbol{v})_t \quad &+ \nabla \cdot (\rho \boldsymbol{v} \circ \boldsymbol{v}) + \frac{1}{\varepsilon^4} \nabla p + \varepsilon \, \boldsymbol{\Omega} \times \rho \boldsymbol{v} &&= \boldsymbol{S}_{\rho \boldsymbol{v}} - \frac{1}{\varepsilon^4} \rho g \, \boldsymbol{k} \\
(\rho e)_t \quad &+ \nabla \cdot (\boldsymbol{v} \, [\rho e + p]) &&= S_{\rho e} \\
(\rho Y_j)_t \quad &+ \nabla \cdot \big(\rho Y_j \boldsymbol{v}\big) &&= \varepsilon^{\mu_i} S_{\rho Y_j}
\end{aligned}
$$

$$
(\rho e) = \frac{p}{\gamma - 1} + \frac{\varepsilon^4}{2} \rho \boldsymbol{v}^2 + \rho \sum_{j=1}^{N} \varepsilon^{\nu_j} \, Q_j Y_j
$$

Ready for asymptotics in ε

Recovered classical single-scale models:

$$\mathbf{U}^{(i)} = \mathbf{U}^{(i)}(\frac{t}{\varepsilon}, \boldsymbol{x}, \frac{z}{\varepsilon})$$ Linear small scale internal gravity waves

$$\mathbf{U}^{(i)} = \mathbf{U}^{(i)}(t, \boldsymbol{x}, z)$$ Anelastic & pseudo-incompressible models

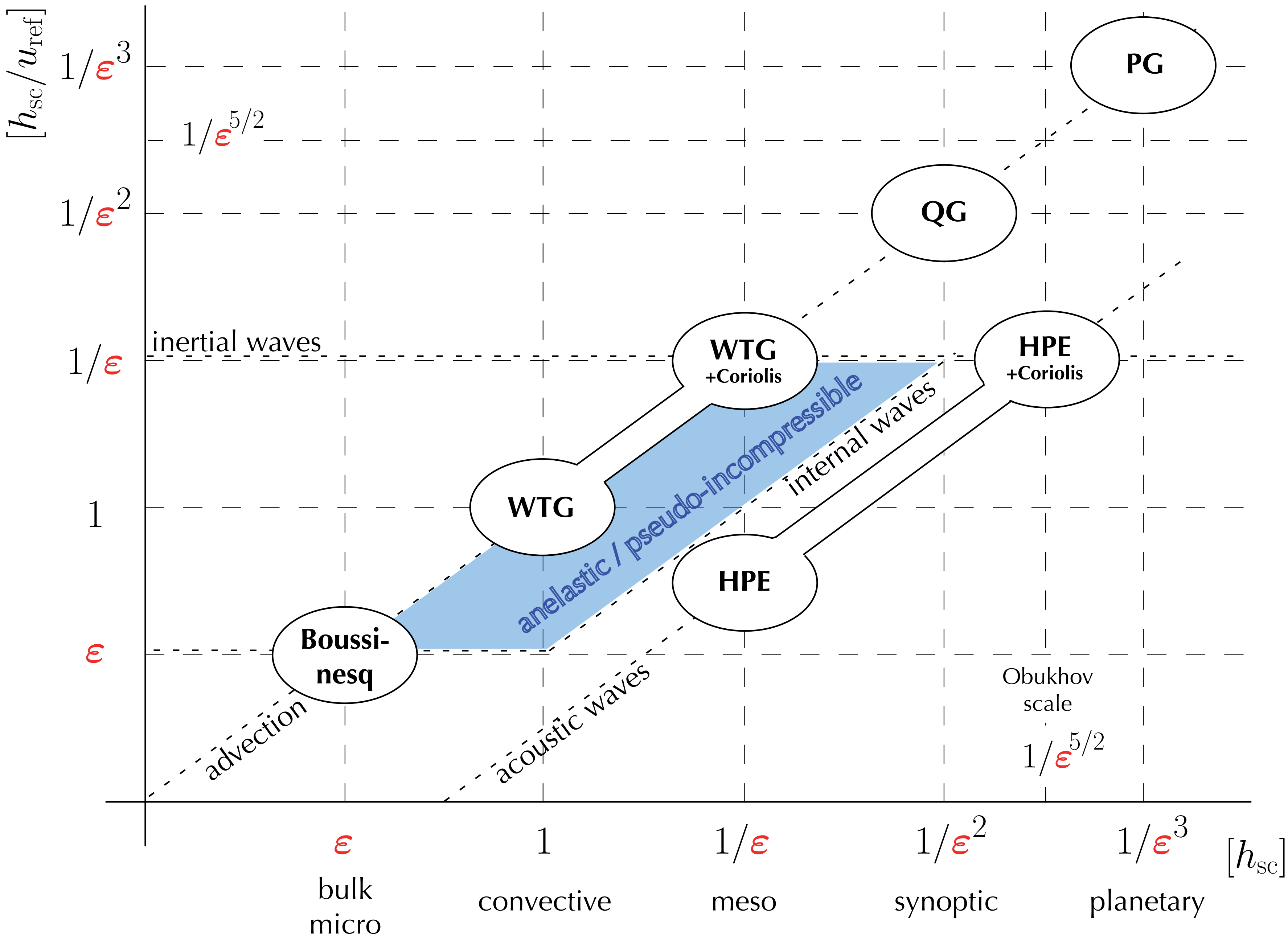
$$\mathbf{U}^{(i)} = \mathbf{U}^{(i)}(\varepsilon t, \varepsilon^2 \boldsymbol{x}, z)$$ Linear large scale internal gravity waves

$$\mathbf{U}^{(i)} = \mathbf{U}^{(i)}(\varepsilon^2 t, \varepsilon^2 \boldsymbol{x}, z)$$ Mid-latitude Quasi-Geostrophic Flow

$$\mathbf{U}^{(i)} = \mathbf{U}^{(i)}(\varepsilon^2 t, \varepsilon^2 \boldsymbol{x}, z)$$ Equatorial Weak Temperature Gradients

$$\mathbf{U}^{(i)} = \mathbf{U}^{(i)}(\varepsilon^2 t, \varepsilon^{-1} \xi(\varepsilon^2 \boldsymbol{x}), z)$$ Semi-geostrophic flow

$$\mathbf{U}^{(i)} = \mathbf{U}^{(i)}(\underline{\varepsilon^{3/2}} t, \underline{\varepsilon^{5/2}} x, \underline{\varepsilon^{5/2}} y, z)$$ Kelvin, Yanai, Rossby, and gravity Waves


$[h_{sc}/u_{ref}]$
$1/\varepsilon^3$
$1/\varepsilon^{5/2}$
$1/\varepsilon^2$
$1/\varepsilon$
1
ε
inertial waves
PG
QG
WTG
+Coriolis
HPE
+Coriolis
WTG
HPE
Boussi-
nesq
anelastic / pseudo-incompressible
internal waves
acoustic waves
advection
Obukhov
scale
$1/\varepsilon^{5/2}$
ε
bulk
micro
1
convective
$1/\varepsilon$
meso
$1/\varepsilon^2$
synoptic
$1/\varepsilon^3$
planetary
$[h_{sc}]$

Multiscale Modelling Framework

Scalings and Expansion Scheme

Exact Closure for the Small Scales

Results

Nonlinearity for Weak Undersaturation

Conclusions

Scalings

Characteristic (inverse) time scales

		dimensional	dimensionless
advection	:	$\dfrac{u_{\text{ref}}}{h_{\text{sc}}}$	1
sound	:	$\dfrac{\sqrt{g h_{\text{sc}}}}{h_{\text{sc}}}$	$\dfrac{\sqrt{g h_{\text{sc}}}}{u_{\text{ref}}} = \dfrac{1}{\varepsilon^2}$
internal waves	:	$N = \sqrt{\dfrac{g}{\overline{\theta}} \dfrac{d\overline{\theta}}{dz}}$	$\dfrac{\sqrt{g h_{\text{sc}}}}{u_{\text{ref}}} \sqrt{\dfrac{h_{\text{sc}}}{\overline{\theta}} \dfrac{d\overline{\theta}}{dz}} = \dfrac{1}{\varepsilon^2} \sqrt{\dfrac{h_{\text{sc}}}{\overline{\theta}} \varepsilon^2 \dfrac{d\overline{\theta}'}{dz}}$

Scaling for the equatorial region:*

$$\frac{h_{\text{sc}}}{\overline{\theta}} \frac{d\overline{\theta}}{dz} = O(\varepsilon^2) \qquad \text{implies} \qquad t_{\text{sound}} \sim \varepsilon t_{\text{internal}} \sim \varepsilon^2 t_{\text{adv}}$$

* Majda & Klein, JAS, (2003)

Clouds and internal waves

Columnar clouds / internal wave time scales*

general expansion scheme

$$\mathbf{U}(\boldsymbol{x}, z, t; \varepsilon) = \sum_i \varepsilon^i \mathbf{U}^{(i)} (\boldsymbol{\eta}, \boldsymbol{x}, z, \tau)$$

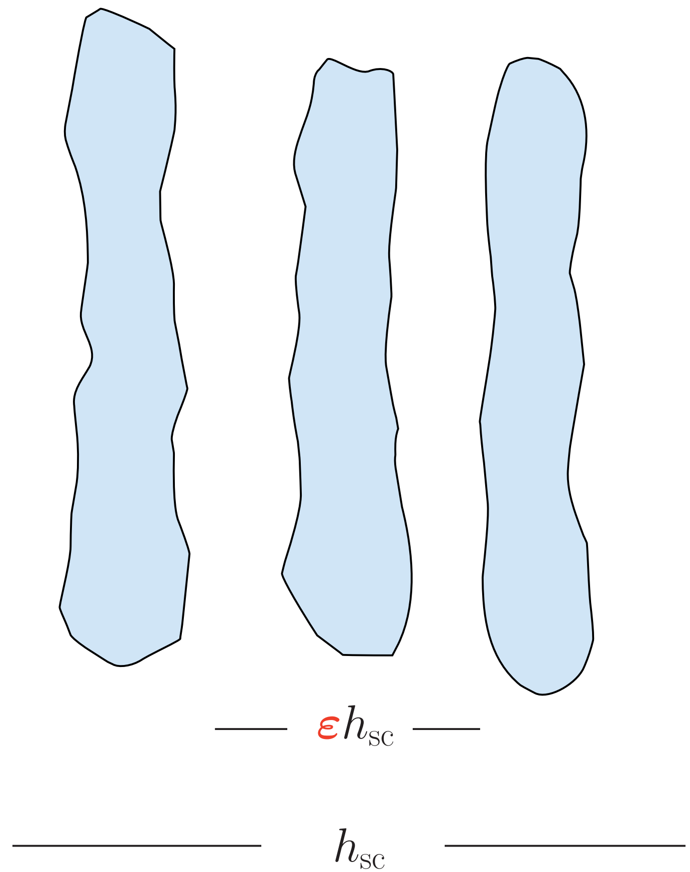
horizontal velocity scaling

$$\boldsymbol{u}^{(0)}(\boldsymbol{\eta}, \boldsymbol{x}, z, \tau) \equiv \boldsymbol{u}(\boldsymbol{x}, z, \tau)$$

$$\boldsymbol{\eta} = \boldsymbol{x}/\varepsilon$$

$$\tau = t/\varepsilon$$

$$\boldsymbol{x} = \frac{\boldsymbol{x}'}{h_{\rm sc}}, \qquad t = \frac{t' u_{\rm ref}}{h_{\rm sc}}$$



*Klein & Majda, TCFD, 2006

Clouds and internal waves

Convective scale

$$\boldsymbol{u}_\tau + \nabla_{\boldsymbol{x}}\pi = 0$$

$$\overline{w}_\tau + \pi_z = \overline{\theta}$$

$$\overline{\theta}_\tau + \overline{w}N^2 = \frac{\Gamma L^{**}}{p_0}\overline{\boldsymbol{C}}$$

$$\rho_0 \nabla_{\boldsymbol{x}} \cdot \boldsymbol{u} + (\rho_0 \overline{w})_z = 0$$

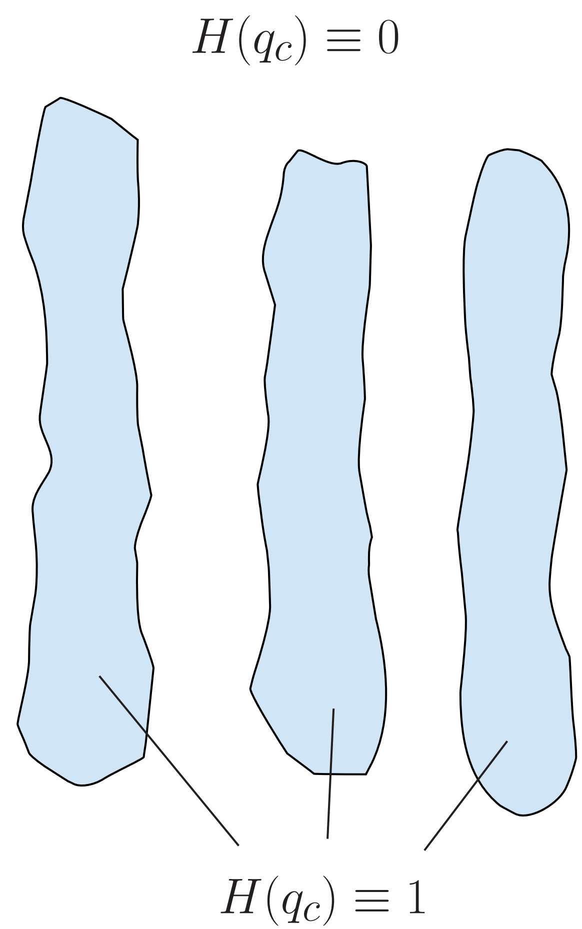
Cloud column scale

$$(\partial_\tau + \boldsymbol{u}\cdot\nabla_{\boldsymbol{\eta}})\,\widetilde{w} = \widetilde{\theta}$$

$$(\partial_\tau + \boldsymbol{u}\cdot\nabla_{\boldsymbol{\eta}})\,\theta + \widetilde{w}N^2 = \frac{\Gamma L^{**}}{p_0}\widetilde{\boldsymbol{C}}\,.$$

Moisture coupling

$$\boldsymbol{C} = H(q_c)\,\boldsymbol{C}_{\mathbf{d}} + [1 - H(q_c)]\,\boldsymbol{C}_{\mathbf{ev}}$$



Clouds and internal waves

Saturated Air

$$C_{\mathrm{d}} = C_{\mathrm{d}}^{**}\, \underline{\delta q_{\mathrm{v}}^{(n^*)}}\, q_{\mathrm{c}} \;=\; -\,(\widetilde{w} + \overline{w})\, \frac{dq_{\mathrm{vs}}}{dz}$$

$$(\partial_\tau + \boldsymbol{u} \cdot \nabla_{\boldsymbol{\eta}})\, q_{\mathrm{c}} \;=\; H(q_{\mathrm{c}})\, C_{\mathrm{d}} - C_{\mathrm{cr}}^{**} q_{\mathrm{r}}\, q_{\mathrm{c}}$$

$$(\partial_\tau + \boldsymbol{u} \cdot \nabla_{\boldsymbol{\eta}})\, q_{\mathrm{r}} \;=\; 0$$

Undersaturated Air

$$C_{\mathrm{ev}} \;=\; -C_{\mathrm{ev}}^{**}\, \left(q_{\mathrm{vs}}(z) - q_{\mathrm{v}}\right)\, q_{\mathrm{r}}^{1/2}$$

$$(\partial_\tau + \boldsymbol{u} \cdot \nabla_{\boldsymbol{\eta}})\, q_{\mathrm{v}} \;=\; 0$$

$$(\partial_\tau + \boldsymbol{u} \cdot \nabla_{\boldsymbol{\eta}})\, q_{\mathrm{r}} \;=\; 0$$

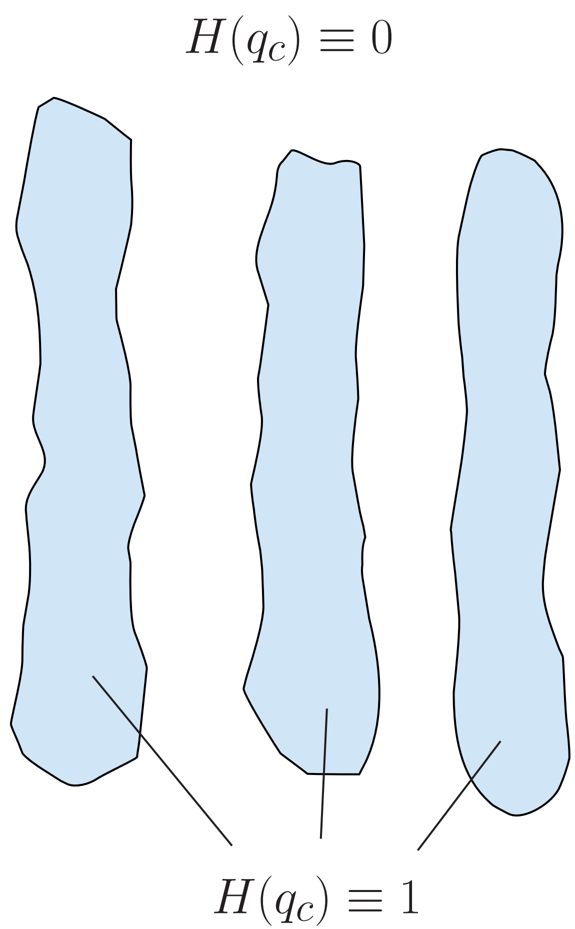
Clouds and internal waves

Convective scale $(\boldsymbol{x}, z, \tau)$

$$
\begin{aligned}
\boldsymbol{u}_\tau + \nabla_{\boldsymbol{x}}\pi &= 0 \\
\overline{w}_\tau + \pi_z &= \overline{\theta} \\
\overline{\theta}_\tau + \overline{w}N^2 &= \frac{\Gamma L^{**}}{p_0}\overline{\boldsymbol{C}} \\
\rho_0 \nabla_{\boldsymbol{x}} \cdot \boldsymbol{u} + \left(\rho_0 \overline{w}\right)_z &= 0
\end{aligned}
$$

Cloud column scale $(\boldsymbol{\eta}, \boldsymbol{x}, z, \tau)$

$$
\begin{aligned}
\left(\partial_\tau + \boldsymbol{u}\cdot\nabla_{\boldsymbol{\eta}}\right)\widetilde{w} &= \widetilde{\theta} \\
\left(\partial_\tau + \boldsymbol{u}\cdot\nabla_{\boldsymbol{\eta}}\right)\theta + \widetilde{w}N^2 &= \frac{\Gamma L^{**}}{p_0}\widetilde{\boldsymbol{C}}\,.
\end{aligned}
$$



Moisture coupling

$$
\boldsymbol{C} = H(q_{\rm c})\, \boldsymbol{C}_{\rm d} + [1 - H(q_{\rm c})]\, \boldsymbol{C}_{\rm ev}
$$

Clouds and internal waves

Convective scale

$$
\begin{aligned}
\boldsymbol{u}_\tau + \nabla_{\boldsymbol{x}}\pi &= 0 \\
\overline{w}_\tau + \pi_z &= \overline{\theta} \\
\overline{\theta}_\tau + \overline{w}N^2 &= \frac{\Gamma L^{**}}{p_0}\overline{\boldsymbol{C}} \\
\rho_0 \nabla_{\boldsymbol{x}} \cdot \boldsymbol{u} + \left(\rho_0 \overline{w}\right)_z &= 0
\end{aligned}
$$

Cloud column scale

$$
\begin{aligned}
\left(\partial_\tau + \boldsymbol{u}\cdot\nabla_{\boldsymbol{\eta}}\right)\widetilde{w} &= \widetilde{\theta} \\
\left(\partial_\tau + \boldsymbol{u}\cdot\nabla_{\boldsymbol{\eta}}\right)\theta + \widetilde{w}N^2 &= \frac{\Gamma L^{**}}{p_0}\widetilde{\boldsymbol{C}}\,.
\end{aligned}
$$

Moisture coupling

$$
\boldsymbol{C} = H(q_{\mathrm{c}})\,\boldsymbol{C}_{\mathbf{d}} + \left[1 - H(q_{\mathrm{c}})\right]\,\boldsymbol{C}_{\mathbf{ev}}
$$

Analytical microscale closure I

$$
\begin{aligned}
\boldsymbol{C}_{\mathbf{ev}} &= -C_{\mathrm{ev}}^{**}\,\left(q_{\mathrm{vs}}(z) - q_{\mathrm{v}}\right)\,{q_{\mathrm{r}}}^{1/2} \\
\overline{\left[1 - H(q_{\mathrm{c}})\right]\,\boldsymbol{C}_{\mathbf{ev}}} &= -\overline{C}(\boldsymbol{x}, z)
\end{aligned}
$$

Clouds and internal waves

Convective scale

$$\boldsymbol{u}_\tau + \nabla_{\boldsymbol{x}} \pi = 0$$

$$\overline{w}_\tau + \pi_z = \overline{\theta}$$

$$\overline{\theta}_\tau + \overline{w} N^2 = \frac{\Gamma L^{**}}{p_0} \overline{\boldsymbol{C}}$$

$$\rho_0 \nabla_{\boldsymbol{x}} \cdot \boldsymbol{u} + \left(\rho_0 \overline{w}\right)_z = 0$$

Cloud column scale

$$\left(\partial_\tau + \boldsymbol{u} \cdot \nabla_{\boldsymbol{\eta}}\right) \widetilde{w} = \widetilde{\theta}$$

$$\left(\partial_\tau + \boldsymbol{u} \cdot \nabla_{\boldsymbol{\eta}}\right) \theta + \widetilde{w} N^2 = \frac{\Gamma L^{**}}{p_0} \widetilde{\boldsymbol{C}} \,.$$

Moisture coupling

$$\boldsymbol{C} = H(q_c)\, \boldsymbol{C}_{\mathrm{d}} + [1 - H(q_c)]\, \boldsymbol{C}_{\mathrm{ev}}$$

Analytical microscale closure I

$$\boldsymbol{C}_{\mathrm{ev}} = -C_{\mathrm{ev}}^{**} \left(q_{\mathrm{vs}}(z) - q_{\mathrm{v}}\right) {q_{\mathrm{r}}}^{1/2}$$

$$\overline{[1 - H(q_c)]\, \boldsymbol{C}_{\mathrm{ev}}} = -\overline{C}(\boldsymbol{x}, z)$$

$$\boldsymbol{C}_{\mathrm{d}} = -\left(\widetilde{w} + \overline{w}\right) \frac{dq_{\mathrm{vs}}}{dz}$$

Clouds and internal waves

Convective scale

$$\boldsymbol{u}_\tau + \nabla_{\boldsymbol{x}} \pi = 0$$

$$\overline{w}_\tau + \pi_z = \overline{\theta}$$

$$\overline{\theta}_\tau + \overline{w} N^2 = \frac{\Gamma L^{**}}{p_0} \overline{\boldsymbol{C}}$$

$$\rho_0 \nabla_{\boldsymbol{x}} \cdot \boldsymbol{u} + \left(\rho_0 \overline{w}\right)_z = 0$$

Cloud column scale

$$\left(\partial_\tau + \boldsymbol{u} \cdot \nabla_{\boldsymbol{\eta}}\right) \widetilde{w} = \widetilde{\theta}$$

$$\left(\partial_\tau + \boldsymbol{u} \cdot \nabla_{\boldsymbol{\eta}}\right) \theta + \widetilde{w} N^2 = \frac{\Gamma L^{**}}{p_0} \widetilde{\boldsymbol{C}} \,.$$

Moisture coupling

$$\boldsymbol{C} = H(q_c)\, \boldsymbol{C}_d + [1 - H(q_c)]\, \boldsymbol{C}_{ev}$$

Analytical microscale closure I

$$\boldsymbol{C}_{ev} = -C_{ev}^{**} \left(q_{vs}(z) - q_v\right) {q_r}^{1/2}$$

$$\overline{[1 - H(q_c)]\, \boldsymbol{C}_{ev}} = -\overline{C}(\boldsymbol{x}, z)$$

$$\boldsymbol{C}_d = -\left(\widetilde{w} + \overline{w}\right) \frac{dq_{vs}}{dz}$$

$$\overline{H(q_c)\, \boldsymbol{C}_d} = -\left(\overline{H(q_c)\, \widetilde{w}} + \overline{H(q_c)} \overline{w}\right) \frac{dq_{vs}}{dz}$$

Clouds and internal waves

Convective scale

$$\boldsymbol{u}_\tau + \nabla_{\boldsymbol{x}}\pi = 0$$

$$\overline{w}_\tau + \pi_z = \overline{\theta}$$

$$\overline{\theta}_\tau + \overline{w}N^2 = \frac{\Gamma L^{**}}{p_0}\overline{\boldsymbol{C}}$$

$$\rho_0\nabla_{\boldsymbol{x}}\cdot\boldsymbol{u} + (\rho_0\overline{w})_z = 0$$

Cloud column scale

$$(\partial_\tau + \boldsymbol{u}\cdot\nabla_{\boldsymbol{\eta}})\,\widetilde{w} = \widetilde{\theta}$$

$$(\partial_\tau + \boldsymbol{u}\cdot\nabla_{\boldsymbol{\eta}})\,\theta + \widetilde{w}N^2 = \frac{\Gamma L^{**}}{p_0}\widetilde{\boldsymbol{C}}\,.$$

Moisture coupling

$$\boldsymbol{C} = H(q_{\mathrm{c}})\,\boldsymbol{C}_{\mathrm{d}} + [1 - H(q_{\mathrm{c}})]\,\boldsymbol{C}_{\mathrm{ev}}$$

Analytical microscale closure I

$$\boldsymbol{C}_{\mathrm{ev}} = -C^{**}_{\mathrm{ev}}\,(q_{\mathrm{vs}}(z) - q_{\mathrm{v}})\,{q_{\mathrm{r}}}^{1/2}$$

$$\overline{[1 - H(q_{\mathrm{c}})]\,\boldsymbol{C}_{\mathrm{ev}}} = -\overline{C}(\boldsymbol{x}, z)$$

$$\boldsymbol{C}_{\mathrm{d}} = -\,(\widetilde{w} + \overline{w})\,\frac{dq_{\mathrm{vs}}}{dz}$$

$$\overline{H(q_{\mathrm{c}})\,\boldsymbol{C}_{\mathrm{d}}} = -\left(\overline{H(q_{\mathrm{c}})\,\widetilde{w}} + \overline{H(q_{\mathrm{c}})}\overline{w}\right)\frac{dq_{\mathrm{vs}}}{dz}$$

$$\overline{H(q_{\mathrm{c}})\,\boldsymbol{C}_{\mathrm{d}}} = -\,(\;w' \;+ \sigma(\boldsymbol{x}, z)\,\overline{w}\;)\,\frac{dq_{\mathrm{vs}}}{dz}$$

Clouds and internal waves

Convective scale

$$\boldsymbol{u}_\tau + \nabla_{\boldsymbol{x}} \pi \;=\; 0$$

$$\overline{w}_\tau + \pi_z \;=\; \overline{\theta}$$

$$\overline{\theta}_\tau + \overline{w} N^2 \;=\; \frac{\Gamma L^{**}}{p_0} \overline{\boldsymbol{C}}$$

$$\rho_0 \nabla_{\boldsymbol{x}} \cdot \boldsymbol{u} + \left(\rho_0 \overline{w}\right)_z \;=\; 0$$

Cloud column scale

$$\left(\partial_\tau + \boldsymbol{u} \cdot \nabla_{\boldsymbol{\eta}}\right) \widetilde{w} \;=\; \widetilde{\theta}$$

$$\left(\partial_\tau + \boldsymbol{u} \cdot \nabla_{\boldsymbol{\eta}}\right) \theta + \widetilde{w} N^2 \;=\; \frac{\Gamma L^{**}}{p_0} \widetilde{\boldsymbol{C}} \,.$$

Moisture coupling

$$\boldsymbol{C} = H(q_{\rm c})\, \boldsymbol{C}_{\rm d} + [1 - H(q_{\rm c})]\, \boldsymbol{C}_{\rm ev}$$

Analytical microscale closure I

$$\boldsymbol{C}_{\rm ev} \;=\; -C_{\rm ev}^{**} \left(q_{\rm vs}(z) - q_{\rm v}\right) \, {q_{\rm r}}^{1/2}$$

$$\overline{[1 - H(q_{\rm c})]\, \boldsymbol{C}_{\rm ev}} \;=\; -\overline{C}(\boldsymbol{x}, z)$$

$$\boldsymbol{C}_{\rm d} \;=\; -\left(\widetilde{w} + \overline{w}\right) \frac{dq_{\rm vs}}{dz}$$

$$\overline{H(q_{\rm c})\, \boldsymbol{C}_{\rm d}} \;=\; -\left(\overline{H(q_{\rm c})\, \widetilde{w}} + \overline{H(q_{\rm c})}\overline{w}\right) \frac{dq_{\rm vs}}{dz}$$

$$\overline{H(q_{\rm c})\, \boldsymbol{C}_{\rm d}} \;=\; -\left(\; w' \; + \sigma(\boldsymbol{x}, z)\, \overline{w}\;\right) \frac{dq_{\rm vs}}{dz}$$

New averaged microscale variables

$$w'(\boldsymbol{x}, z, \tau) \;=\; \overline{H(q_{\rm c})\, \widetilde{w}}$$

$$\sigma(\boldsymbol{x}, z) \;=\; \overline{H(q_{\rm c})}$$

Clouds and internal waves

Convective scale

$$\boldsymbol{u}_\tau + \nabla_{\boldsymbol{x}}\pi = 0$$

$$\overline{w}_\tau + \pi_z = \overline{\theta}$$

$$\overline{\theta}_\tau + \overline{w}N^2 = \frac{\Gamma L^{**}}{p_0}\overline{\boldsymbol{C}}$$

$$\rho_0\nabla_{\boldsymbol{x}}\cdot\boldsymbol{u} + (\rho_0\overline{w})_z = 0$$

Cloud column scale

$$(\partial_\tau + \boldsymbol{u}\cdot\nabla_{\boldsymbol{\eta}})\,\widetilde{w} = \widetilde{\theta}$$

$$(\partial_\tau + \boldsymbol{u}\cdot\nabla_{\boldsymbol{\eta}})\,\theta + \widetilde{w}N^2 = \frac{\Gamma L^{**}}{p_0}\widetilde{\boldsymbol{C}}\,.$$

Moisture coupling

$$\boldsymbol{C} = H(q_{\rm c})\,\boldsymbol{C}_{\rm d} + [1 - H(q_{\rm c})]\,\boldsymbol{C}_{\rm ev}$$

Analytical microscale closure II

$$(\partial_\tau + \boldsymbol{u}\cdot\nabla_{\boldsymbol{\eta}})\,H(q_{\rm c}) = 0$$

$$(\partial_\tau + \boldsymbol{u}\cdot\nabla_{\boldsymbol{\eta}})\,H(q_{\rm c})\widetilde{w} = H(q_{\rm c})\widetilde{\theta}$$

$$(\overline{H(q_{\rm c})\,\widetilde{w}})_\tau = \overline{H(q_{\rm c})\,\widetilde{\theta}}$$

$$w'_\tau = \theta'$$

Clouds and internal waves

Convective scale

$$
\begin{aligned}
\boldsymbol{u}_\tau + \nabla_{\boldsymbol{x}} \pi &= 0 \\
\overline{w}_\tau + \pi_z &= \overline{\theta} \\
\overline{\theta}_\tau + \overline{w} N^2 &= \frac{\Gamma L^{**}}{p_0} \overline{\boldsymbol{C}} \\
\rho_0 \nabla_{\boldsymbol{x}} \cdot \boldsymbol{u} + (\rho_0 \overline{w})_z &= 0
\end{aligned}
$$

Cloud column scale

$$
\begin{aligned}
(\partial_\tau + \boldsymbol{u} \cdot \nabla_{\boldsymbol{\eta}})\, \widetilde{w} &= \widetilde{\theta} \\
(\partial_\tau + \boldsymbol{u} \cdot \nabla_{\boldsymbol{\eta}})\, \theta + \widetilde{w} N^2 &= \frac{\Gamma L^{**}}{p_0} \widetilde{\boldsymbol{C}} \,.
\end{aligned}
$$

Moisture coupling

$$
\boldsymbol{C} = H(q_\mathrm{c})\, \boldsymbol{C}_\mathrm{d} + [1 - H(q_\mathrm{c})]\, \boldsymbol{C}_\mathrm{ev}
$$

Analytical microscale closure II

$$
\begin{aligned}
(\partial_\tau + \boldsymbol{u} \cdot \nabla_{\boldsymbol{\eta}})\, H(q_\mathrm{c}) &= 0 \\
(\partial_\tau + \boldsymbol{u} \cdot \nabla_{\boldsymbol{\eta}})\, H(q_\mathrm{c}) \widetilde{w} &= H(q_\mathrm{c}) \widetilde{\theta} \\
(\overline{H(q_\mathrm{c})\, \widetilde{w}})_\tau &= \overline{H(q_\mathrm{c})\, \widetilde{\theta}} \\
w'_\tau &= \theta'
\end{aligned}
$$

analogously

$$
\theta'_\tau + \sigma w'\, N^2 = \sigma \left[(1 - \sigma) \overline{w}\, N^2 + \overline{C} \right]
$$

where

$$
\begin{aligned}
\theta'(\boldsymbol{x}, z, \tau) &= \overline{H(q_\mathrm{c})\, \widetilde{\theta}} \\
w'(\boldsymbol{x}, z, \tau) &= \overline{H(q_\mathrm{c})\, \widetilde{w}}
\end{aligned}
$$

Clouds and internal waves

Coupled micro-macro dynamics on convective scales

$$
\begin{aligned}
\boldsymbol{u}_\tau \;\; + \nabla_{\boldsymbol{x}} \pi &= 0 \\
\overline{w}_\tau \;\; + \pi_z &= \overline{\theta} \\
\overline{\theta}_\tau \;\; + (1 - \boldsymbol{\sigma}) \overline{w} N^2 &= \boldsymbol{w'} N^2 - \overline{C} \\
\rho_0 \nabla_{\boldsymbol{x}} \cdot \boldsymbol{u} + (\rho_0 \overline{w})_z &= 0 \\
\\
w'_\tau &= \theta' \\
\theta'_\tau \;\; + \sigma w' \, N^2 &= \sigma (1 - \sigma) \overline{\boldsymbol{w}} N^2 + \sigma \overline{C} \, .
\end{aligned}
$$

where

$\sigma(\boldsymbol{x}, z), \overline{C}(\boldsymbol{x}, z), N(z)$ are prescribed

Clouds and internal waves

Coupled micro-macro dynamics on convective scales (with mean advection)

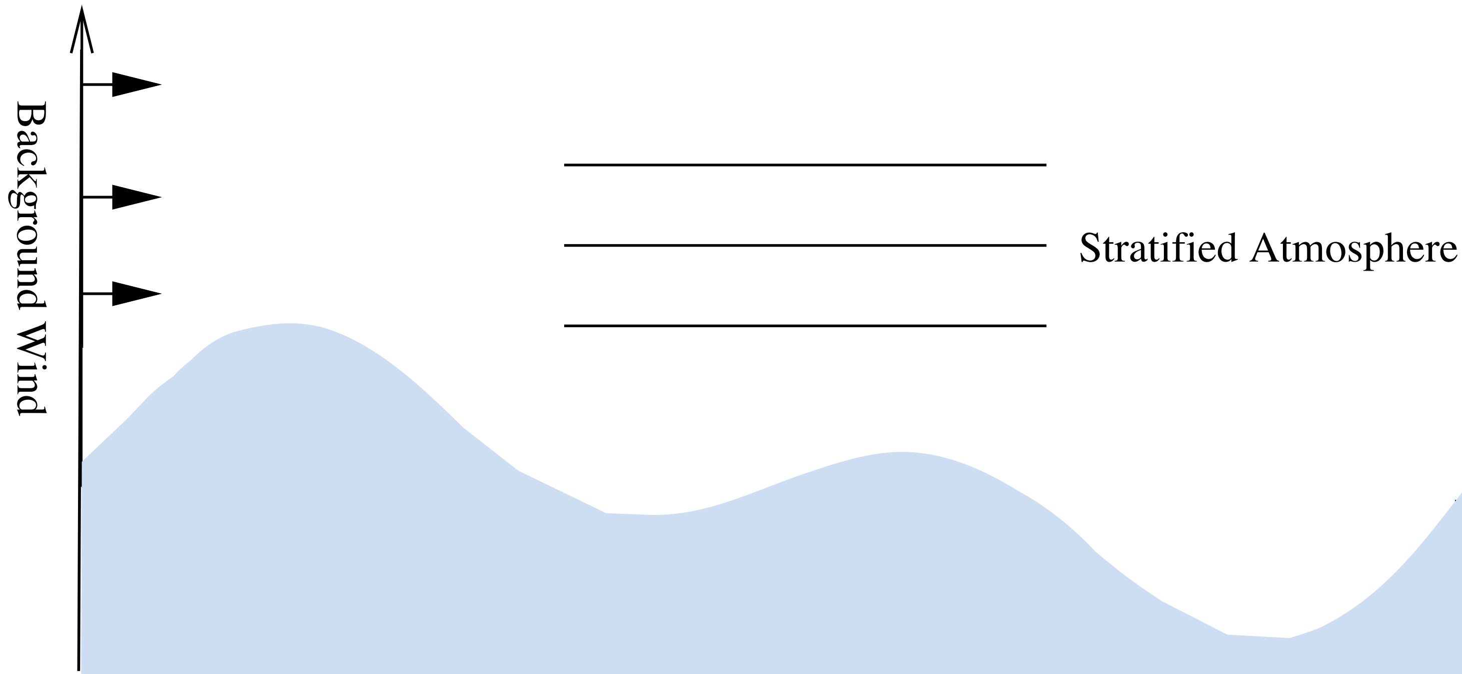
$$
\begin{aligned}
D_\tau \, \boldsymbol{u} + \nabla_{\boldsymbol{x}} \pi &= 0 \\
D_\tau \, \overline{w} + \pi_z &= \overline{\theta} \\
D_\tau \, \overline{\theta} + (1 - \boldsymbol{\sigma}) \overline{w} N^2 &= \boldsymbol{w'} N^2 - \overline{C} \\
\rho_0 \nabla_{\boldsymbol{x}} \cdot \boldsymbol{u} + \left(\rho_0 \overline{w}\right)_z &= 0
\end{aligned}
$$

$$
\begin{aligned}
D_\tau \, w' &= \theta' \\
D_\tau \, \theta' + \sigma w' \, N^2 &= \sigma (1 - \sigma) \overline{\boldsymbol{w}} N^2 + \sigma \overline{C} \, .
\end{aligned}
$$

where

$D_\tau = \partial_\tau + \boldsymbol{u}^\infty \cdot \nabla_{\boldsymbol{x}}$ and $\sigma(\boldsymbol{x}, z), \overline{C}(\boldsymbol{x}, z), N(z)$ are prescribed

Clouds may narrow the spectrum of lee waves



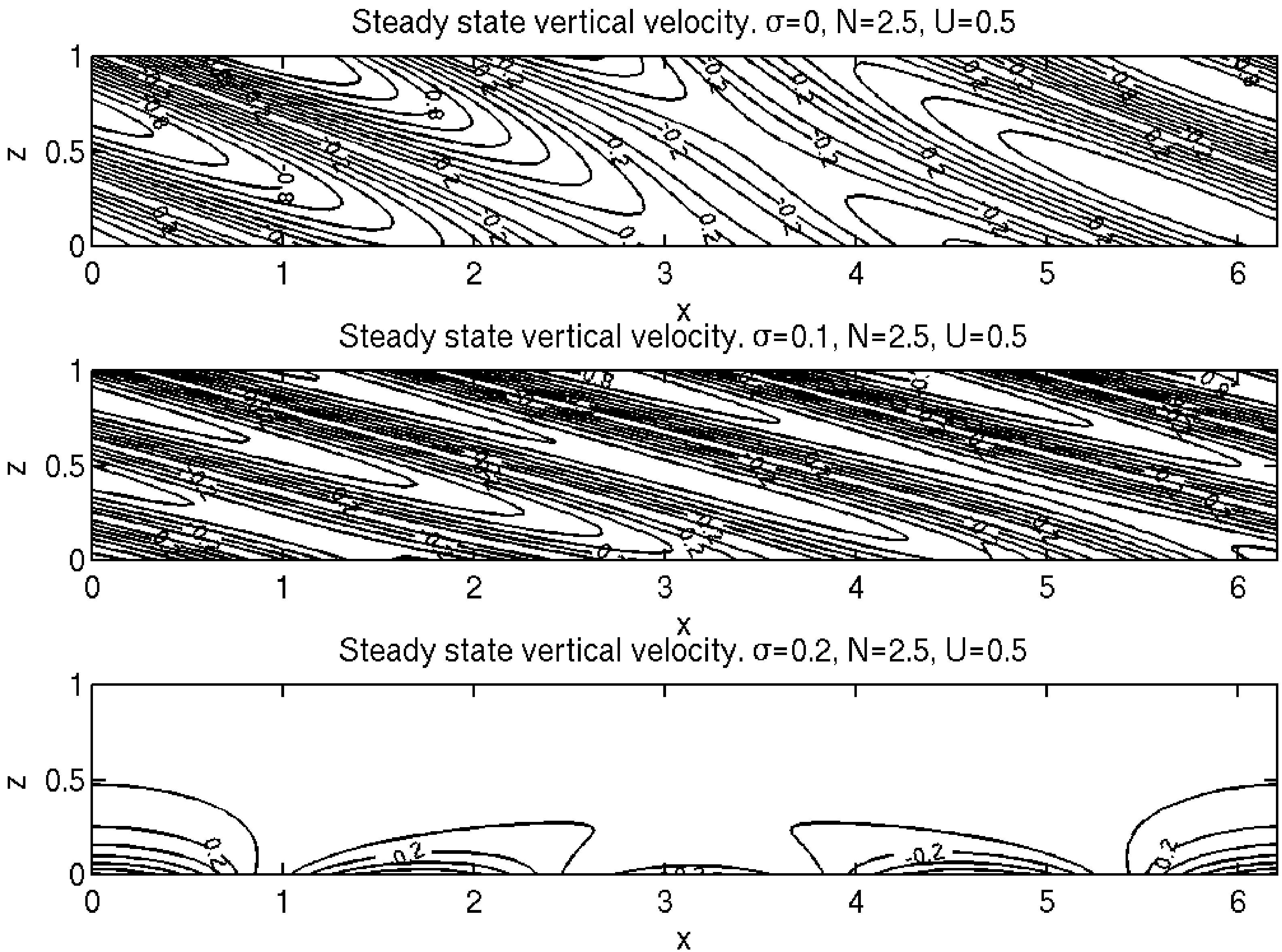
without cloud

$$k_{\text{up}} = \frac{N}{u^{\infty}}$$

with cloud

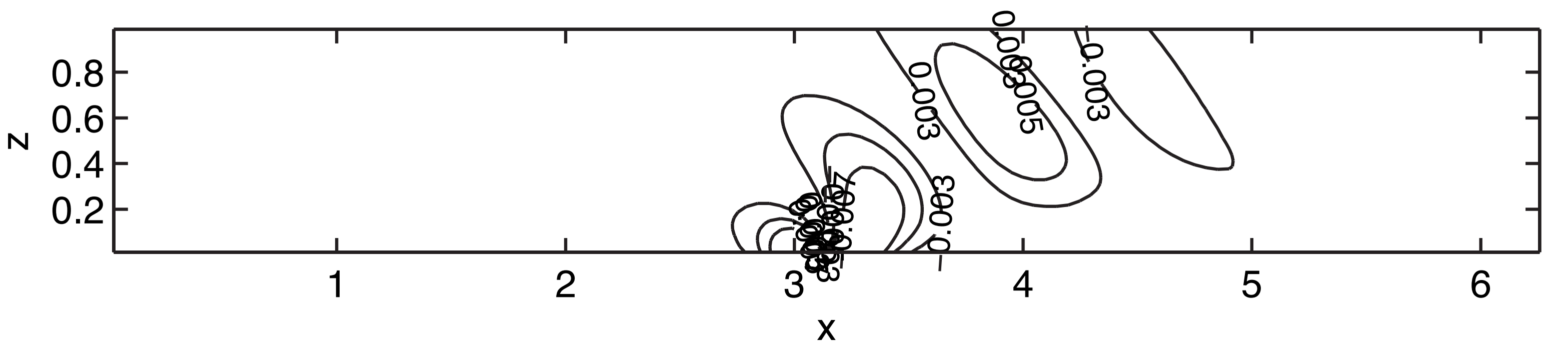
$$k_{\text{up}} = \frac{N}{u^{\infty}} \quad \text{and} \quad k_{\text{low}} = \sqrt{\sigma}\,\frac{N}{u^{\infty}}$$

Lee waves over $sin(x) + sin(2x)$–topography

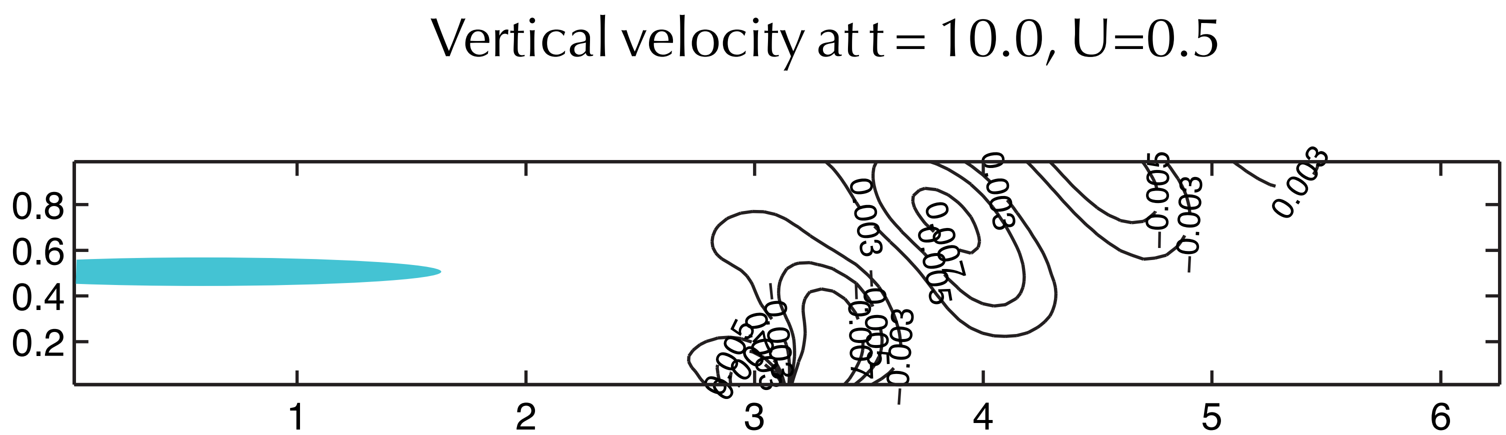


Cloud meets lee wave

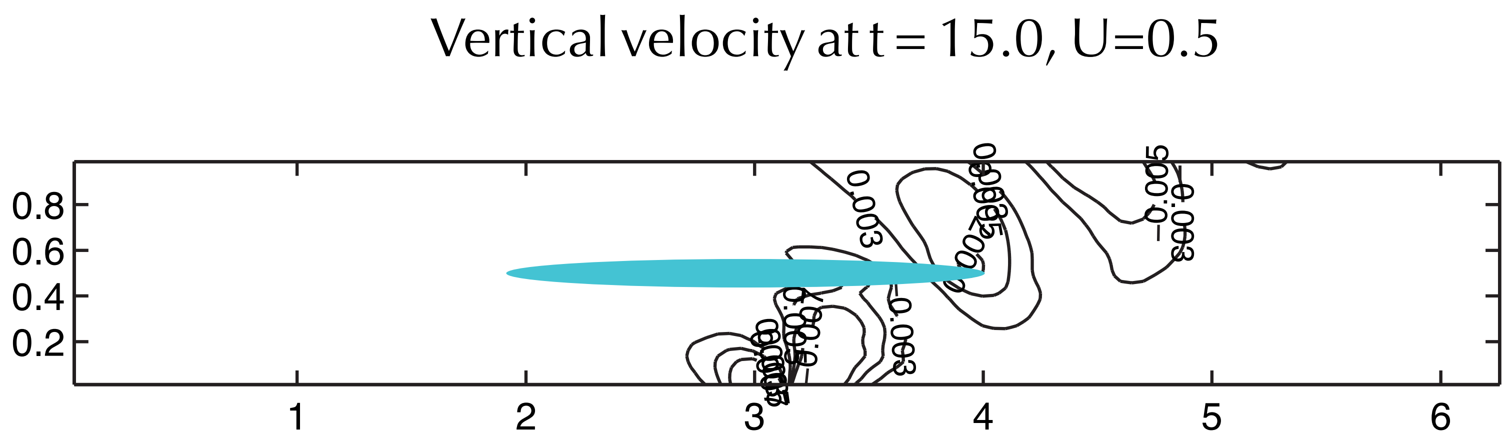
Vertical velocity at t = 5.0, U=0.5



Cloud meets lee wave

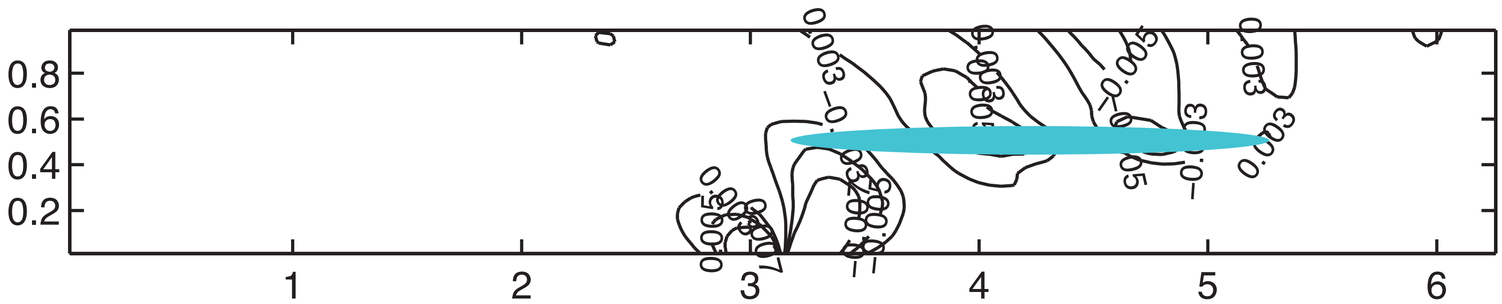


Cloud meets lee wave



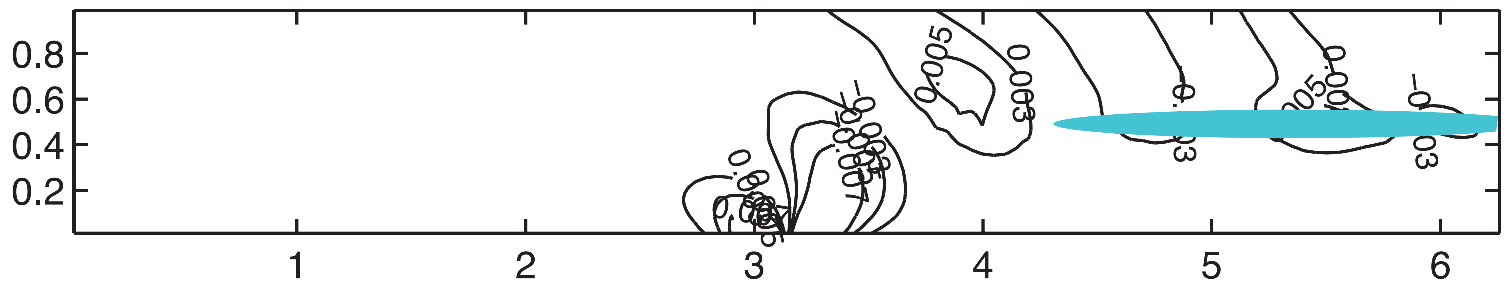
Cloud meets lee wave

Vertical velocity at t = 17.5, U=0.5



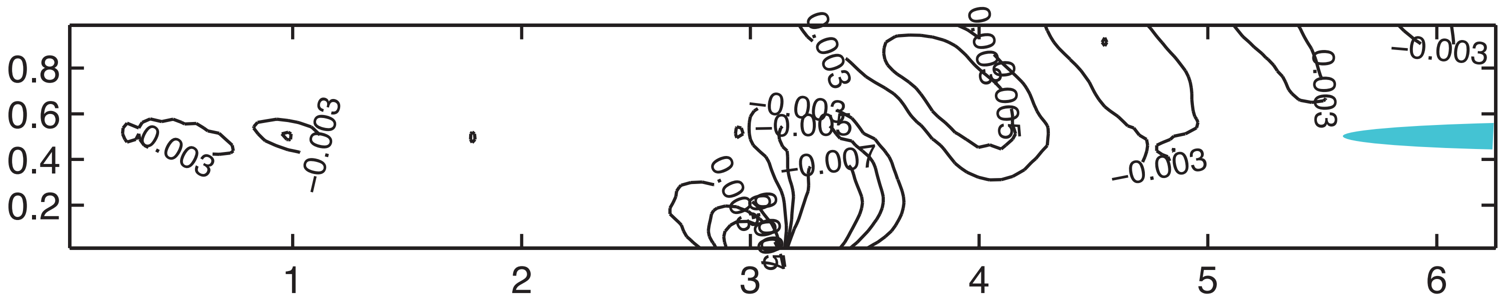
Cloud meets lee wave

Vertical velocity at t = 20.0, U=0.5



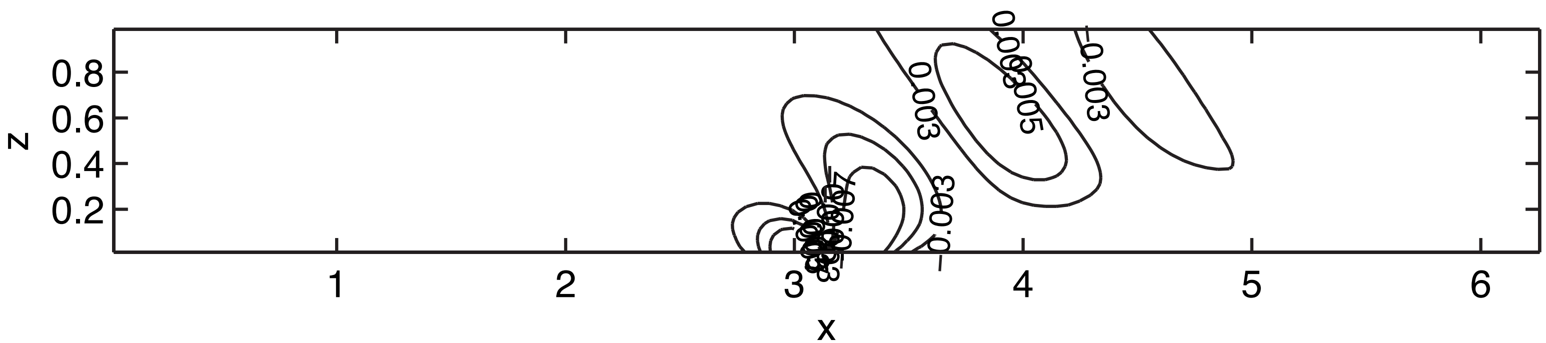
Cloud meets lee wave

Vertical velocity at t = 22.5, U=0.5



Cloud meets lee wave

Vertical velocity at t = 5.0, U=0.5



Dry flow over a hill, w

Clouds and internal waves

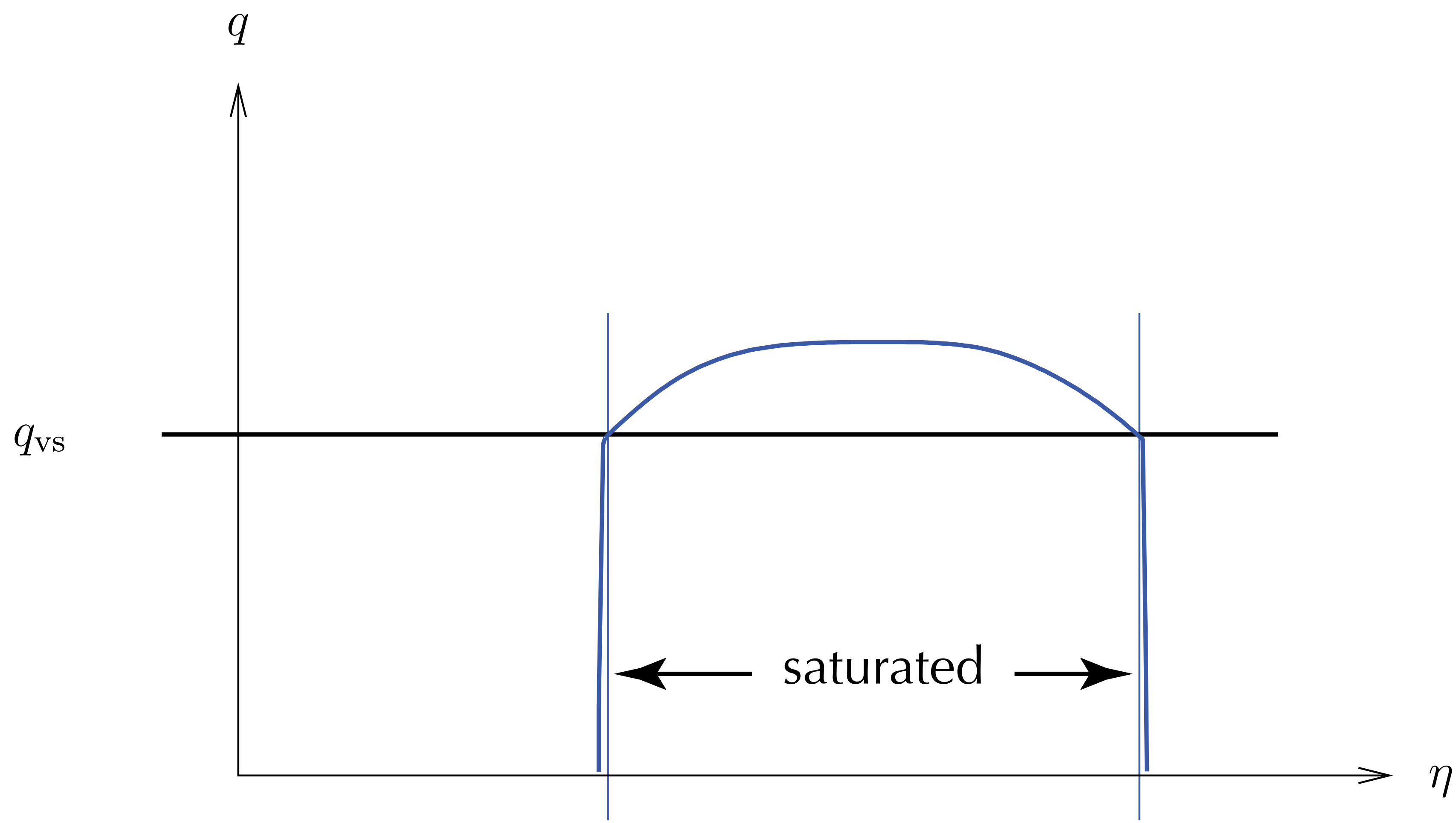
Moist flow over a hill, q_c

Clouds and internal waves

Moist flow over a hill, w

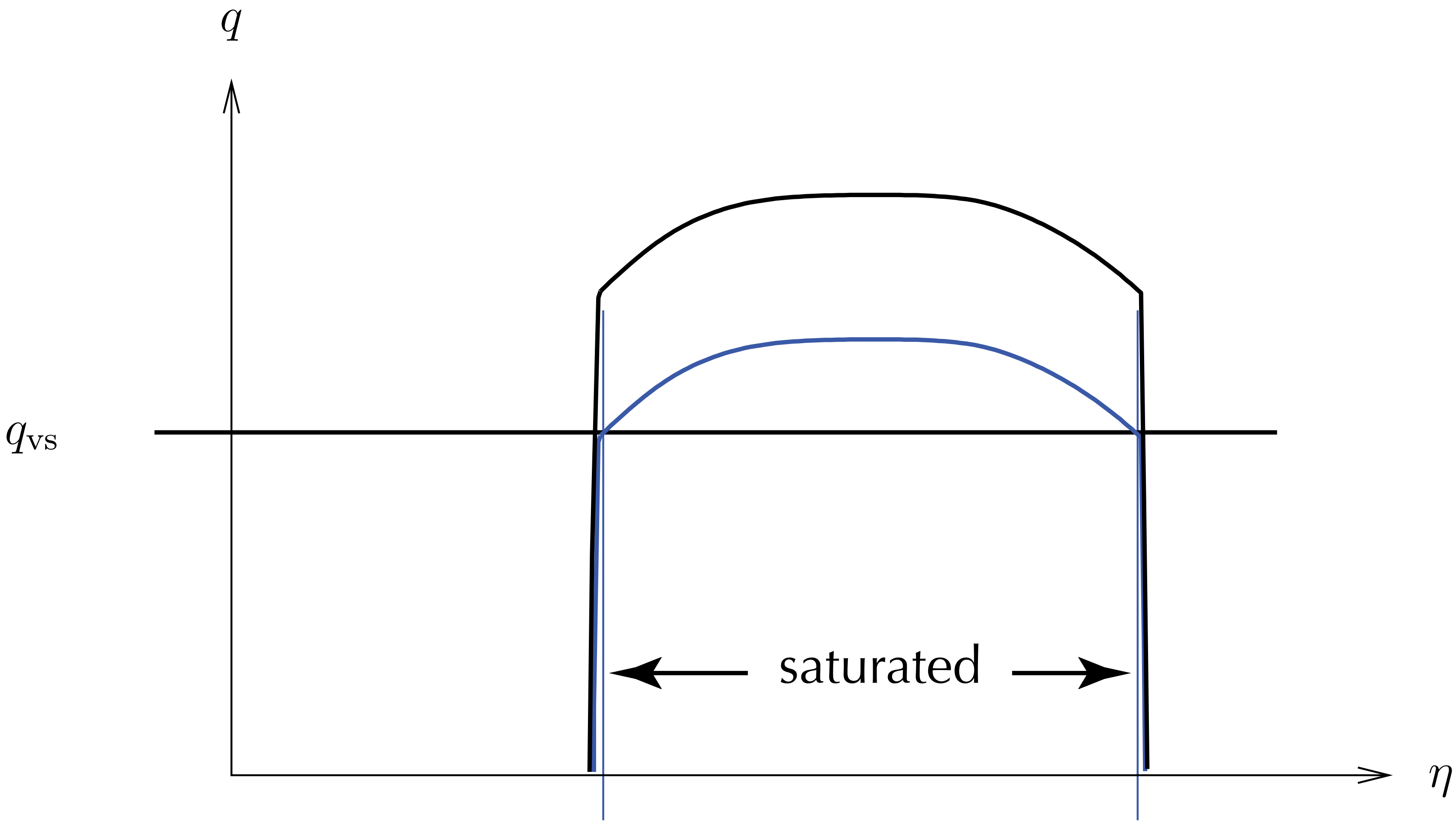
Clouds and nonlinear internal waves

Original regime: subsaturation $O(1)$



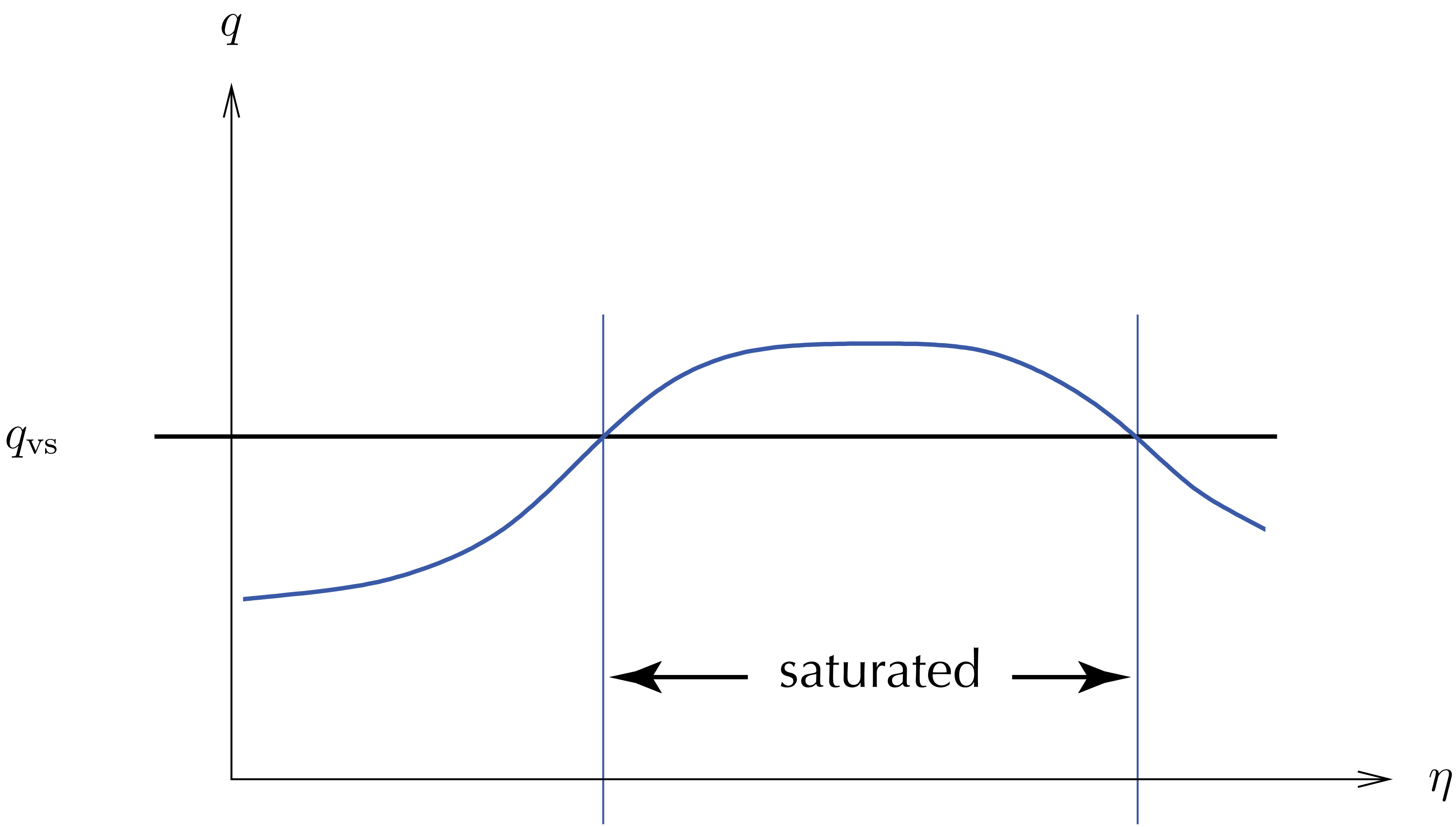
Clouds and nonlinear internal waves

Original regime: subsaturation $O(1)$



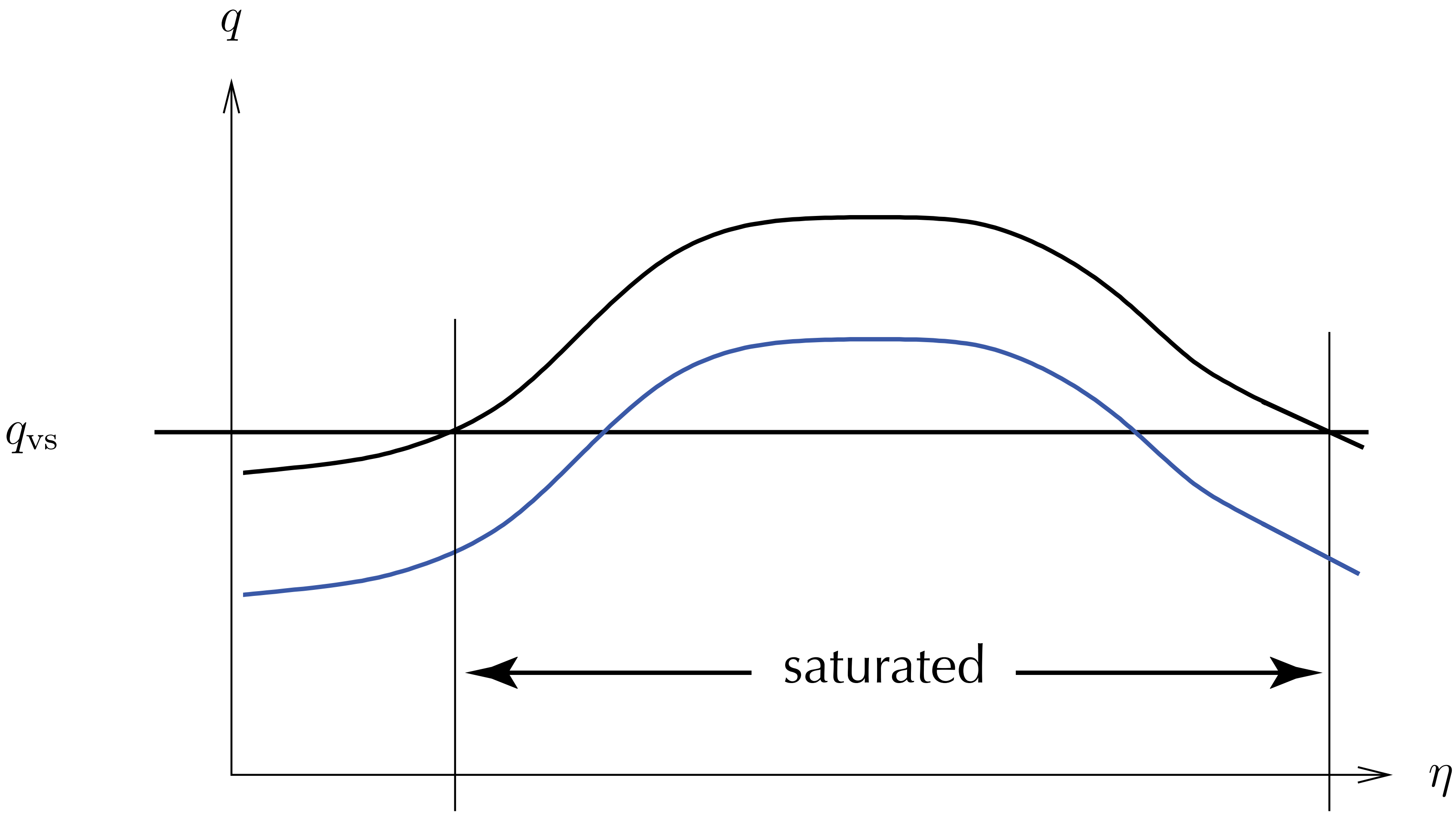
Clouds and nonlinear internal waves

New regime: subsaturation $O(\varepsilon)$



Clouds and nonlinear internal waves

New regime: subsaturation $O(\varepsilon)$



Clouds and nonlinear internal waves

Convective scale

$$\boldsymbol{u}_\tau + \nabla_{\boldsymbol{x}}\pi = 0$$

$$\overline{w}_\tau + \pi_z = \overline{\theta}$$

$$\overline{\theta}_\tau + \overline{w}N^2 = \frac{\Gamma L^{**}}{p_0}\overline{\boldsymbol{C}}$$

$$\rho_0\nabla_{\boldsymbol{x}}\cdot\boldsymbol{u} + \left(\rho_0\overline{w}\right)_z = 0$$

Cloud column scale

$$\left(\partial_\tau + \boldsymbol{u}\cdot\nabla_{\boldsymbol{\eta}}\right)\widetilde{w} = \widetilde{\theta}$$

$$\left(\partial_\tau + \boldsymbol{u}\cdot\nabla_{\boldsymbol{\eta}}\right)\theta + \widetilde{w}N^2 = \frac{\Gamma L^{**}}{p_0}\widetilde{\boldsymbol{C}}\,.$$

Moisture coupling

$$\boldsymbol{C} = H(q_{\mathrm{c}})\,\boldsymbol{C}_{\mathbf{d}} + [1 - H(q_{\mathrm{c}})]\,\boldsymbol{C}_{\mathbf{ev}}$$

Analytical microscale closure I

$$\boldsymbol{C}_{\mathbf{ev}} = -C_{\mathrm{ev}}^{**}\,\left(q_{\mathrm{vs}}(z) - q_{\mathrm{v}}\right)\,{q_{\mathrm{r}}}^{1/2}$$

$$\overline{[1 - H(q_{\mathrm{c}})]\,\boldsymbol{C}_{\mathbf{ev}}} = -\overline{C}(\boldsymbol{x}, z)$$

$$\boldsymbol{C}_{\mathbf{d}} = -\left(\widetilde{w} + \overline{w}\right)\frac{dq_{\mathrm{vs}}}{dz}$$

$$\overline{H(q_{\mathrm{c}})\,\boldsymbol{C}_{\mathbf{d}}} = -\left(\overline{H(q_{\mathrm{c}})\,\widetilde{w}} + \overline{H(q_{\mathrm{c}})}\overline{w}\right)\frac{dq_{\mathrm{vs}}}{dz}$$

$$\overline{H(q_{\mathrm{c}})\,\boldsymbol{C}_{\mathbf{d}}} = -\left(\;w' \;+ \sigma(\boldsymbol{x}, z)\;\overline{w}\;\right)\frac{dq_{\mathrm{vs}}}{dz}$$

New averaged microscale variables

$$w'(\boldsymbol{x}, z, \tau) = \overline{H(q_{\mathrm{c}})\,\widetilde{w}}$$

$$\sigma(\boldsymbol{x}, z) = \overline{H(q_{\mathrm{c}})}$$

Clouds and nonlinear internal waves

Convective scale

$$\boldsymbol{u}_\tau + \nabla_{\boldsymbol{x}}\pi = 0$$

$$\overline{w}_\tau + \pi_z = \overline{\theta}$$

$$\overline{\theta}_\tau + \overline{w}N^2 = \frac{\Gamma L^{**}}{p_0}\overline{\boldsymbol{C}}$$

$$\rho_0 \nabla_{\boldsymbol{x}} \cdot \boldsymbol{u} + \left(\rho_0 \overline{w}\right)_z = 0$$

Cloud column scale

$$\left(\partial_\tau + \boldsymbol{u}\cdot\nabla_{\boldsymbol{\eta}}\right)\widetilde{w} = \widetilde{\theta}$$

$$\left(\partial_\tau + \boldsymbol{u}\cdot\nabla_{\boldsymbol{\eta}}\right)\theta + \widetilde{w}N^2 = \frac{\Gamma L^{**}}{p_0}\widetilde{\boldsymbol{C}}\,.$$

Moisture coupling

$$\boldsymbol{C} = H(q_\mathrm{c})\,\boldsymbol{C}_\mathrm{d} + \left[1 - H(q_\mathrm{c})\right]\,\boldsymbol{C}_\mathrm{ev}$$

Analytical microscale closure II

$$\left(\partial_\tau + \boldsymbol{u}\cdot\nabla_{\boldsymbol{\eta}}\right) H(q_\mathrm{c}) \neq 0$$

$$\left(\partial_\tau + \boldsymbol{u}\cdot\nabla_{\boldsymbol{\eta}}\right) H(q_\mathrm{c})\widetilde{w} = ??$$

$$\left(\overline{H(q_\mathrm{c})\,\widetilde{w}}\right)_\tau = !!$$

Clouds and nonlinear internal waves

New formulation for the saturation indicator function $H(q_c)$:

Total moisture conservation

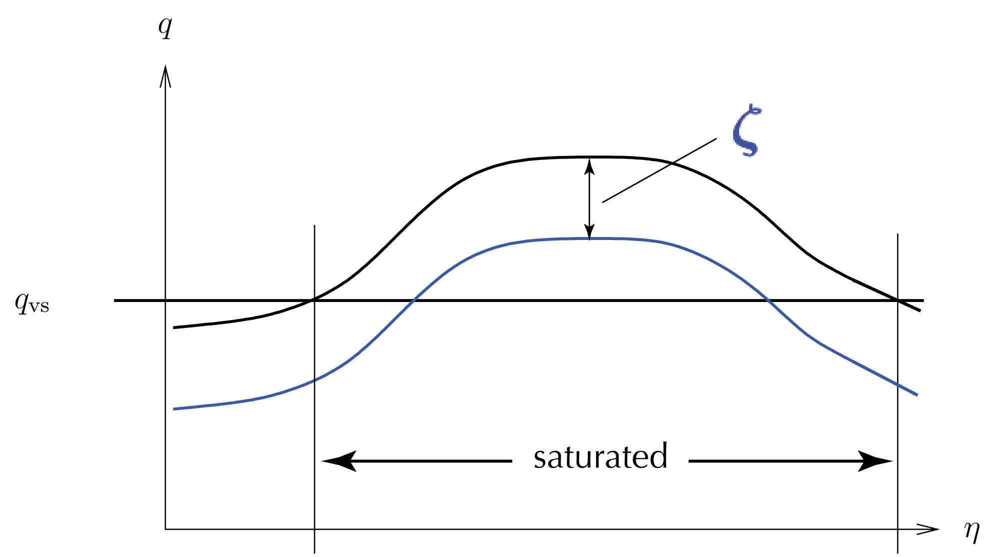
$$(\partial_\tau + \boldsymbol{u} \cdot \nabla_{\boldsymbol{\eta}}) \left[H(q_{\text{c}}) q_{\text{c}}^{(1)} + (1 - H(q_{\text{c}})) q_{\text{v}}^{(1)} \right] + w^{(0)} \frac{dq_{\text{vs}}^{(0)}}{dz} = 0$$

ζ: First-order vertical displacement with

$$(\partial_\tau + \boldsymbol{u} \cdot \nabla_{\boldsymbol{\eta}}) \, \zeta = w^{(0)}$$

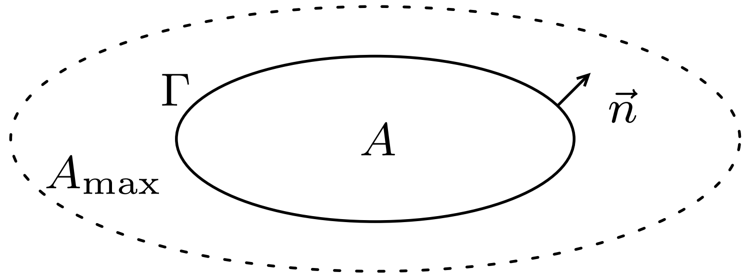
By choice of initial data:

$$H(q_{\text{c}}) \equiv H(\zeta) \qquad \text{and} \qquad \sigma \equiv \overline{H(\zeta)}$$



Clouds and nonlinear internal waves

$\overline{H(q_c)\,\widetilde{w}}$ = area-integral of $\widetilde{w}$ over saturated domain.



As in finite volumes with moving boundary:

$$\left(\overline{H(q_c)\,\widetilde{w}}\right)_\tau = \overline{H(q_c)\,\widetilde{\theta}} + \int_{\partial A} \widetilde{w}\, v_n\, d\sigma$$

Observation for undersaturated regions

$$\begin{aligned}(\partial_\tau + \boldsymbol{u}\cdot\nabla_{\boldsymbol{\eta}})\,\widetilde{w} \;&=\; \widetilde{\theta}\\ (\partial_\tau + \boldsymbol{u}\cdot\nabla_{\boldsymbol{\eta}})\,\widetilde{\theta} + \widetilde{w}N^2 \;&=\; -\overline{C_d}(\underline{\tau, \boldsymbol{x}, z})\end{aligned}$$

Suggestive assumption:

$$\underline{\widetilde{w}|_{\partial A} \equiv \widetilde{w}_{us}(\tau, \boldsymbol{x}, z)}$$

Clouds and nonlinear internal waves

Coupled micro-macro dynamics on convective scales

$$
\begin{aligned}
\boldsymbol{u}_\tau \ + \nabla_{\boldsymbol{x}} \pi &= 0 \\
\overline{w}_\tau \ + \pi_z &= \overline{\theta} \\
\overline{\theta}_\tau \ + (1 - \boldsymbol{\sigma}) \overline{w} N^2 &= \boldsymbol{w'} N^2 \\
\rho_0 \nabla_{\boldsymbol{x}} \cdot \boldsymbol{u} + \left(\rho_0 \overline{w}\right)_z &= 0
\end{aligned}
$$

$$
\begin{aligned}
w'_\tau &= \theta' + \widetilde{w}_{\rm us} \frac{\partial \boldsymbol{\sigma}}{\partial \tau} \\
\theta'_\tau \ + \boldsymbol{\sigma} w' N^2 &= \boldsymbol{\sigma} (1 - \boldsymbol{\sigma}) \overline{\boldsymbol{w}} N^2 + \widetilde{\theta}_{\rm us} \frac{\partial \boldsymbol{\sigma}}{\partial \tau} \, .
\end{aligned}
$$

$$
\frac{\partial \boldsymbol{\sigma}}{\partial \tau} = (\overline{w} + \widetilde{w}_{\rm us}) \frac{\partial \sigma_0}{\partial \zeta} (\boldsymbol{\zeta}, x, z)
$$

$\widetilde{w}_{\rm us}, \widetilde{\theta}_{\rm us}$ are particular solutions of

$$
\begin{aligned}
\widetilde{w}_\tau &= \widetilde{\theta} \\
\widetilde{\theta}_\tau + \widetilde{w} N^2 &= -\boldsymbol{\sigma} \, w' \, \tfrac{dq_{\rm vs}}{dz}
\end{aligned}
$$