Synthetic focusing

in

Acousto-Electric Impedance Tomography

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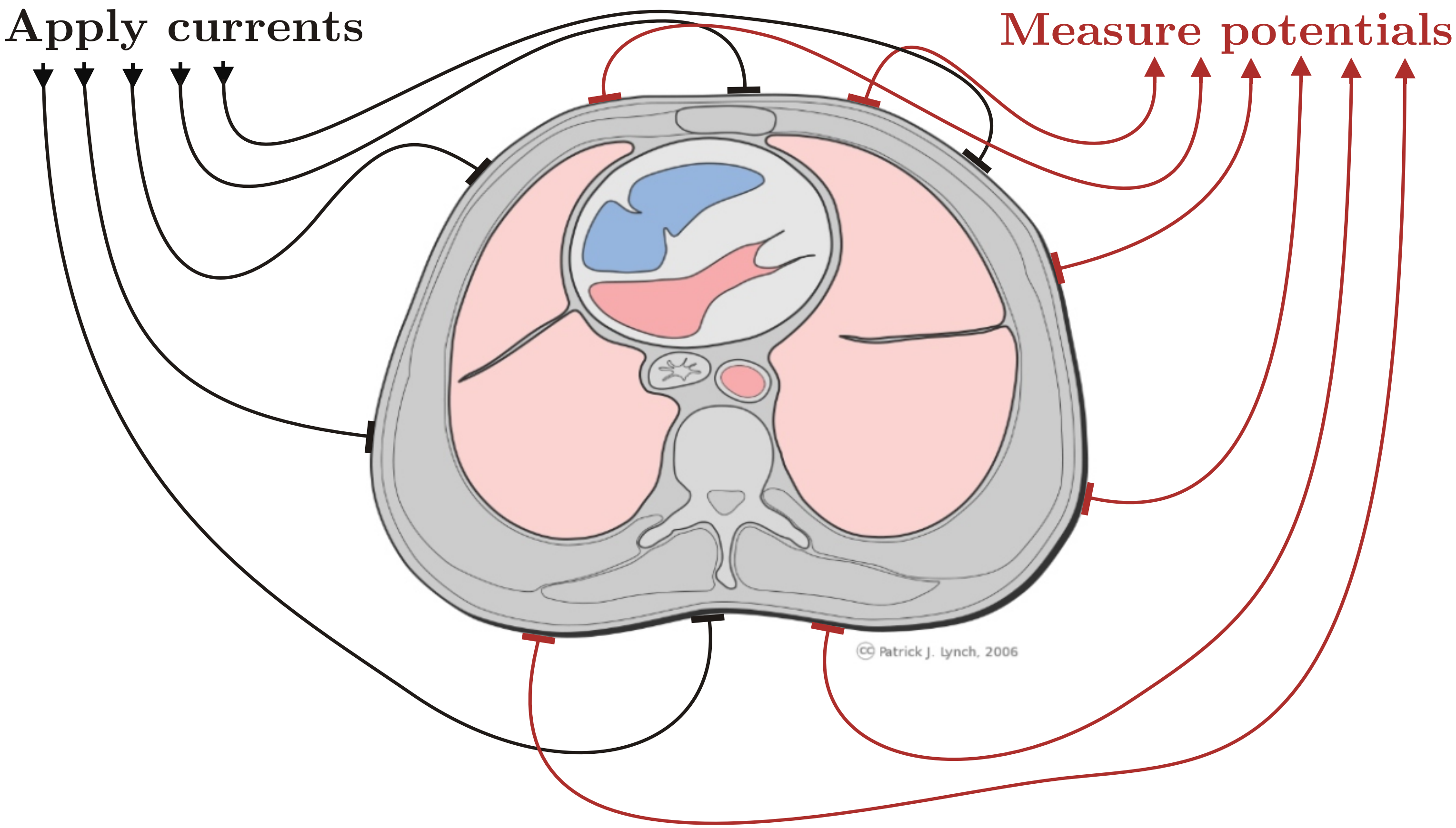
Suppported in part by
DOE grant DE-FG02-03ER25577
NSF grant DMS-0908243

Electrical Impedance Tomography (EIT)

We would like to reconstruct the electric conductivity $\sigma(x)$ within the body:



EIT: mathematics

Electric potential $u(x)$ satisfies the divergence equation

$$\nabla \cdot \sigma(x) \nabla u(x) = 0,$$

Electrical current at the point x equals $\sigma(x)\nabla u(x)$.

Electrical on the boundary equals $\sigma \frac{\partial}{\partial n} u(x)$.

One measurement:
Apply on $\partial\Omega$ currents $\sigma \frac{\partial}{\partial n} u = g(x)$ and measure the potentials $U(x) = u\,|_{\partial\Omega}$.

$U(x)$ is one-dimensional, need more data:
Use different configuration of currents $g_k(x)$; measure the potentials $U_k(x)$.

Reconstruction problem: given $g_k(x)$, $U_k(x)$, $k = 1, \ldots N$, reconstruct $\sigma(x)$.

The instability of EIT

Suppose Ω is the unit disk and $\sigma(x) = 1$ in Ω. Then $u(r, \theta)$ is harmonic in Ω.

Suppose currents $g_k(\theta) = \exp(ik\theta)$, $k = 1, 2, \ldots$. Then

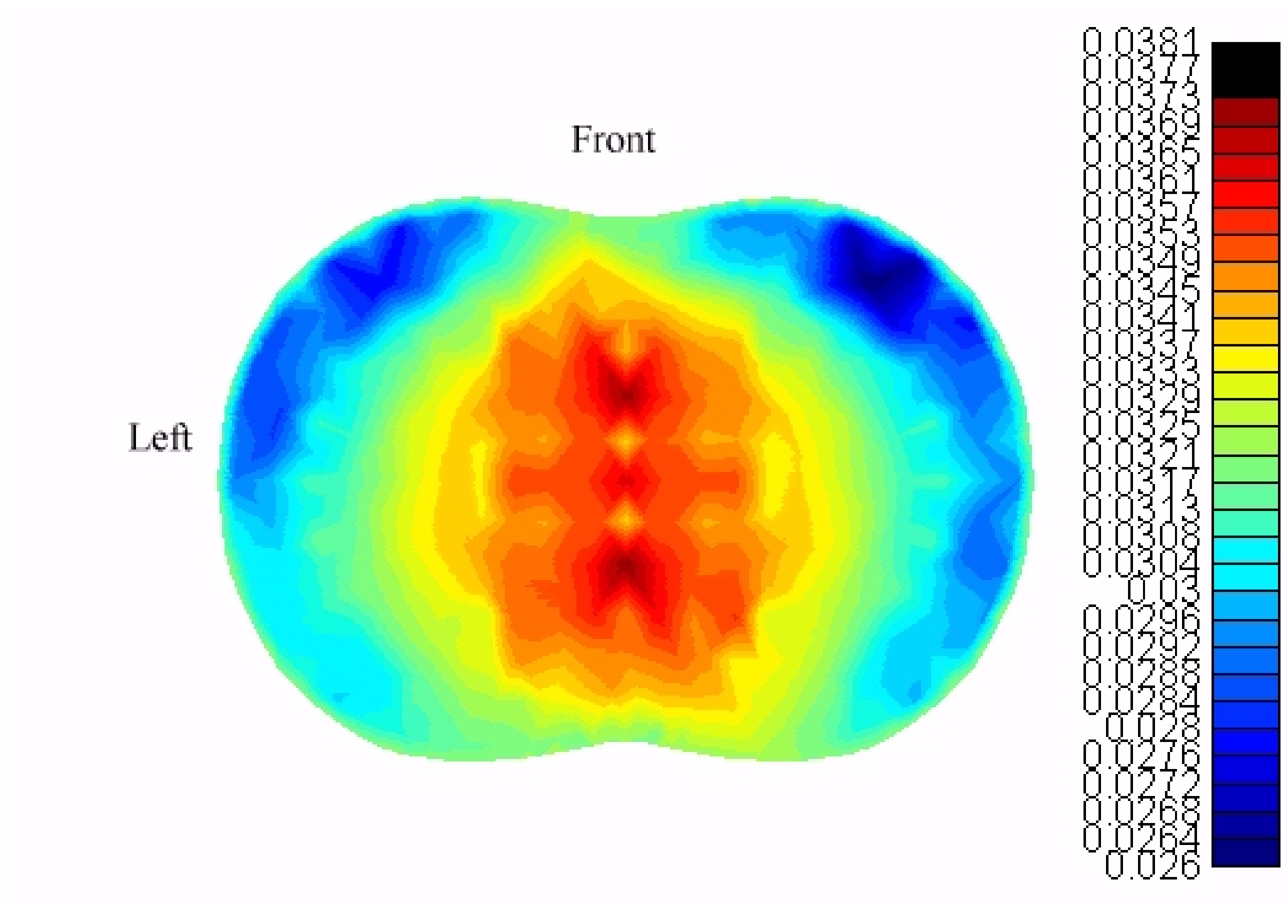
$$u_k(r, \theta) = \frac{1}{k} r^k \exp(ik\theta).$$

Within a smaller disk Ω_1 of radius 1/2 potentials decrease exponentially:

$$|u_k(r, \theta)| \leq \frac{1}{k2^k} \underset{k \to \infty}{\longrightarrow} 0$$

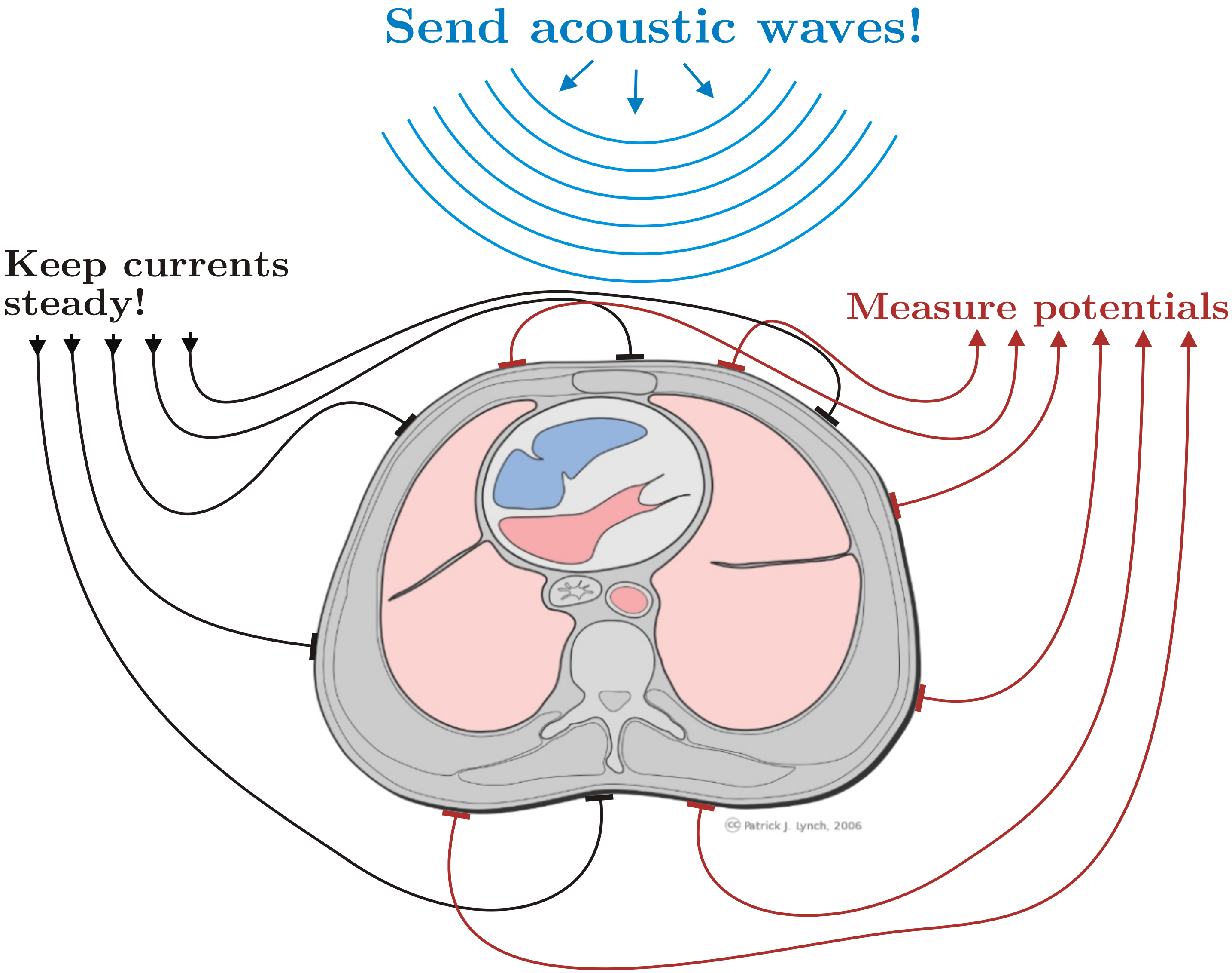
Currents just do not want to go inside! EIT is exponentially unstable!

Example of EIT (Courtesy Wikipedia)



Stabilizing EIT with the help of acoustic waves

Acoustic pressure changes the conductivity of the tissues!



Modeling AEIT

The change is proportional to $\sigma(x)$:

$$\ln \sigma^{new}(x, x_0) = \ln \sigma(x) + \ln \xi(x)$$

Factor $\ln\xi(x)$ is small and proportional to the change in acoustic pressure.

Simplest approach: assume that the pressure can be localized

$$\ln \xi(x) = \eta_{x_0}(x)$$

where $\eta_{x_0}(x) \ll 1$ is a radial function (of $|x - x_0|$) centered at x_0, with a narrow support

First results

Perturbation $\eta_{x_0}(x) \Longrightarrow$ solution is $u(x) + w_{x_0}(x)$

"Electrical Impedance Tomography By Elastic Deformation" (2008)
H. Ammari, E. Bonnetier, Y. Capdeboscq, M. Tanter, and M. Fink:

$$\int_{\partial\Omega} w_{x_0}(x)dA(x) \approx \sigma(x_0)|\nabla u(x_0)|^2 = S(x_0)$$

(1) Find $S(x_0)$ for x_0 scanning Ω.

(2) Repeat for a second set of different BC.

(3) Solve a non-linear optimization problem, find $\sigma(x)$ and ∇u from $S(x)$.

Major drawback: non-linearity — as we will see...

A linear approach (new)

Re-write the divergence equation:

$$\Delta u(x) + \nabla u(x) \cdot \nabla \ln \sigma(x) = 0$$

Perturbation $w(x)$ satisfies:

$$\Delta w + \nabla w \cdot \nabla \ln \sigma = -\nabla u(x) \cdot \nabla \eta(x - x_0)$$

Assume $\eta_{x_0}(x)$ approximates the Dirac's δ-function $\delta(x - x_0)$:

$$\Delta w + \nabla w \cdot \nabla \ln \sigma(x) \approx -\nabla u(x_0) \cdot \nabla \delta(x - x_0) + \Delta u(x_0) \cdot \delta(x - x_0)$$

$G(x, x_0)$ is the (unknown) Green's function:

$$w(x) \approx -\nabla u(x_0) \cdot \nabla G(x, x_0) + \Delta u(x_0) G(x, x_0),$$

$w(x)$ is measured on $\partial\Omega$. If some approximation to $G(x, x_0)$ is known, find $\nabla u(x_0)$ and $\Delta u(x_0)$ by matching the boundary values.

Repeat while varying $x_0 \Longrightarrow$ reconstruct $\nabla u(x_0)$ and $\nabla u(x_0)$ on the computational grid.

Reconstructing the conductivity

Re-write the original equation

$$\Delta u(x) + \nabla u(x) \cdot \nabla \ln \sigma(x) = 0$$

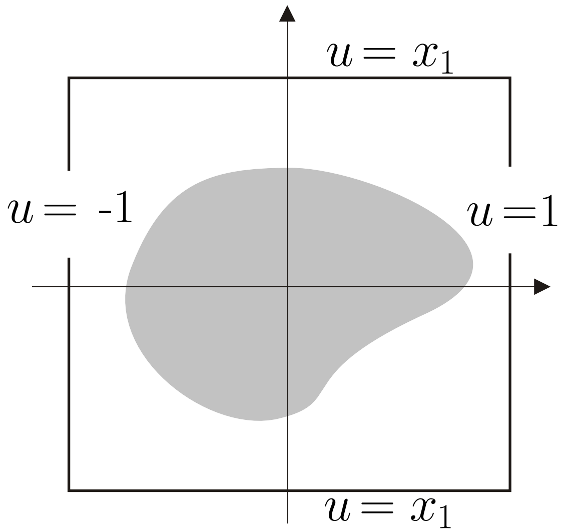
as

$$\nabla u(x) \cdot \nabla \ln \sigma(x) = -\Delta u(x).$$

Now since $\nabla u(x)$ is known, this a first order PDE (transport equation).

It can be solved for $\ln \sigma(x)$.

Numerical simulations: the details



$$u(x) = x_1 + v(x)$$

$$\Delta v + \nabla v \cdot \nabla \ln \sigma(x) = -\frac{\partial}{\partial x_1}\sigma(x)$$

$$v\frac{\partial v}{\partial n} = 0 \text{ on } \partial\Omega$$

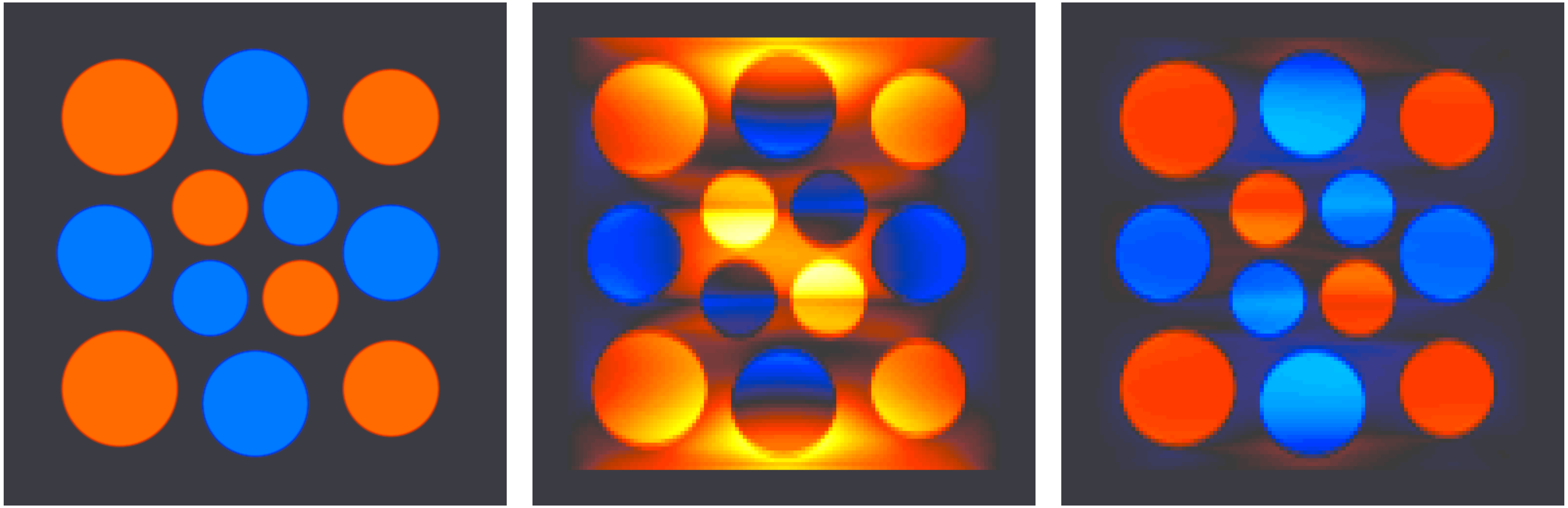
Need to solve for $v(x)$ efficiently. Importantly, v extends to a C^∞ double periodic function on $\mathbb{R}^2 \Longrightarrow$ approximations by cosine Fourier series are spectrally accurate and fast.

Reduce to Fredholm second kind and solve

$$v + \Delta^{-1}(\nabla v \cdot \nabla \ln \sigma) = -\Delta^{-1}\left(\frac{\partial \sigma}{\partial x_1}\right)$$

Reconstructions with delta-like perturbations

Phantom $\ln \sigma(x)$	... currents	... potentials
$\max \sigma(x) = 1.05$	measured	measured
$\min \sigma(x) = 0.95$		



New idea: synthesizing the measurements

Problem: There is no way to apply the delta-like pressure $\eta_{x_0}(x) \approx \delta(x - x_0)$ inside the body.

Solution: Send spherical waves instead, and synthesize the necessary measurements!

The mathematics of synthesis

New representation for the Bessel function (almost Helmholtz):

$$J_0(\lambda|x-y|) = c\ \mathrm{Im} \int_{\partial B} \Big[\Phi(\lambda|z-x|) \frac{\partial}{\partial n_z} J_0(\lambda|z-y|) - J_0(\lambda|z-x|) \frac{\partial}{\partial n_z} \Phi(\lambda|z-y|) \Big]\, dl_z$$

where $\Phi(\lambda|x|)$ is the free-space Green's function for the Helmholtz equation.
Or,

$$J_0(\lambda|x-y|) = c_1\, \mathrm{Im} \int_{\partial B} \Phi(\lambda|z-x|) \frac{\partial}{\partial n_{\mathbf{z}}} \overline{\Phi(\lambda|z-y|)}\, dl_{\mathbf{z}}$$

$J_0(\lambda|x-y|)$ is expressed in terms of outgoing cylindrical waves $\Phi(\lambda|z-x|)$.

Works in all dimensions!

The mathematics of synthesis, continued

On the other hand

$$\exp\left(-\frac{|x-y|^2}{a^2}\right) = \int_0^\infty J_0(\lambda|x-y|)\exp\left(-\frac{\lambda^2}{b^2}\right)\lambda d\lambda$$

So

$$\eta_{x_0}(x) = \exp\left(-\frac{|x-x_0|^2}{a^2}\right) \approx \operatorname{Im}\sum_{k,m=0}^{\infty}\alpha_k(x_0)\Phi(\lambda_k|z_m - x|)$$

Now, if $w(x, \eta_{x_0})$ is the perturbation due to $\eta_{x_0}(x)$, and

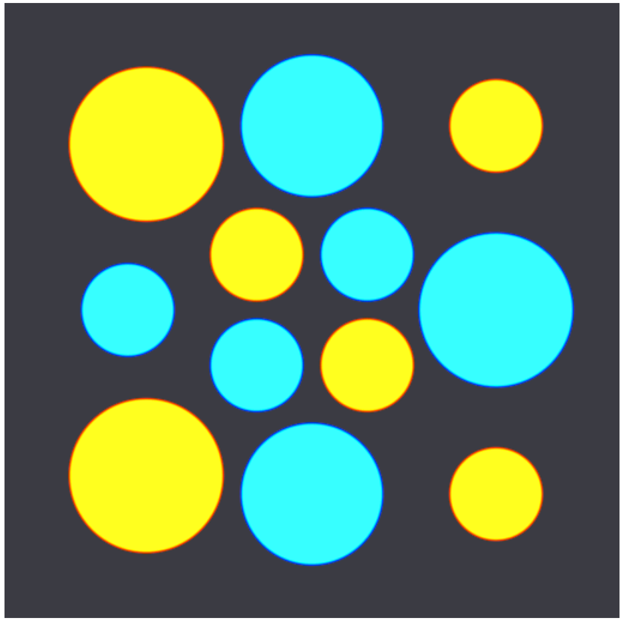
if $w(x, \Phi(\lambda_k|z_m - x|))$ is the perturbation due to $\Phi(\lambda_k|z_m - x|)$,

then, **by linearity**:

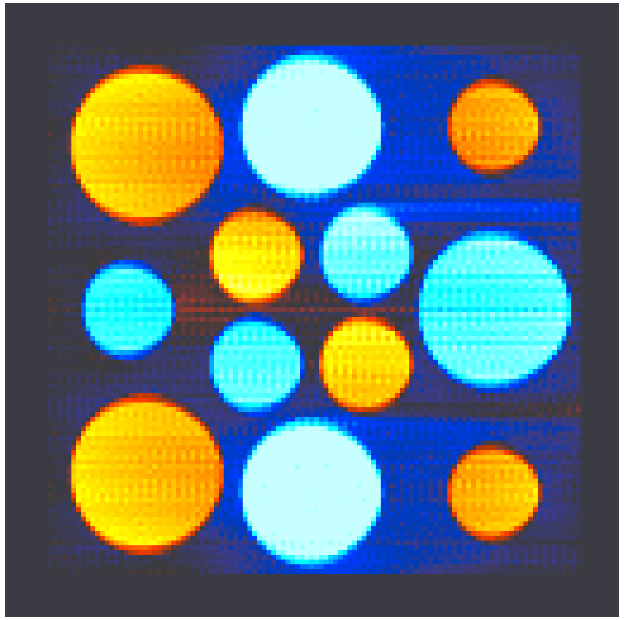
$$w(x, \eta_{x_0}) \approx \operatorname{Im}\sum_{k=0}^{\infty}\alpha_k(x_0)w(x, \Phi(\lambda_k|z_m - x|))$$

Example: conductivity almost constant

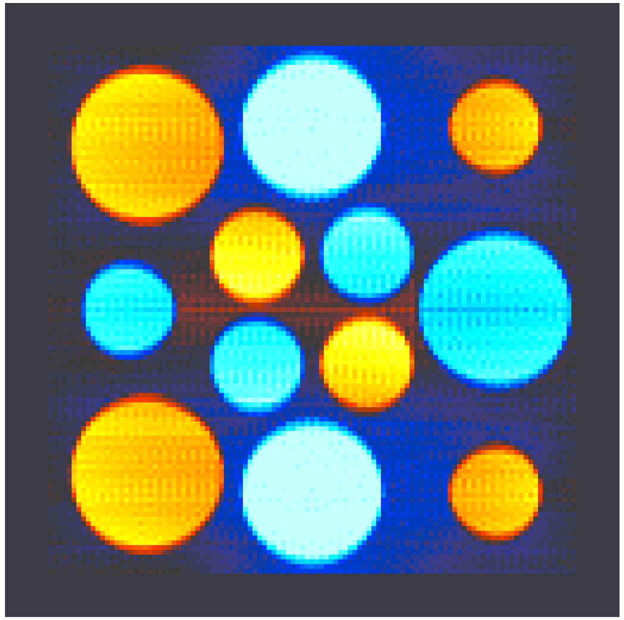
Phantom $\ln \sigma(x)$
$\max \sigma(x) = 1.05$
$\min \sigma(x) = 0.95$



Reconstruction
Left-to-right



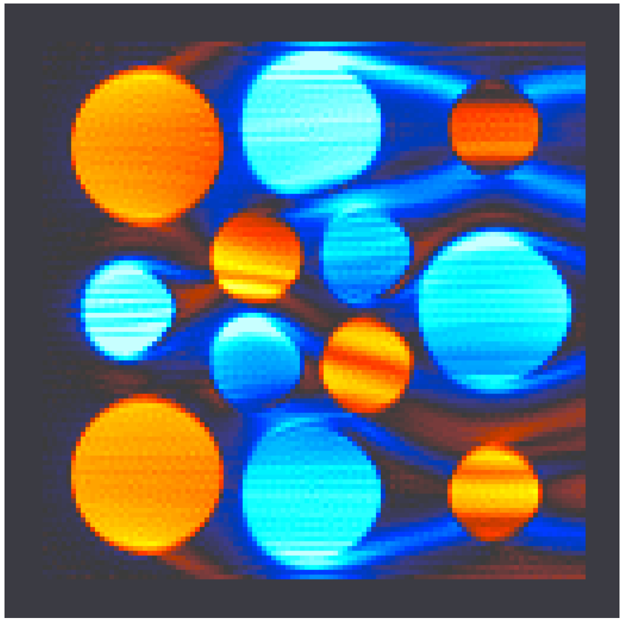
Reconstruction
Average



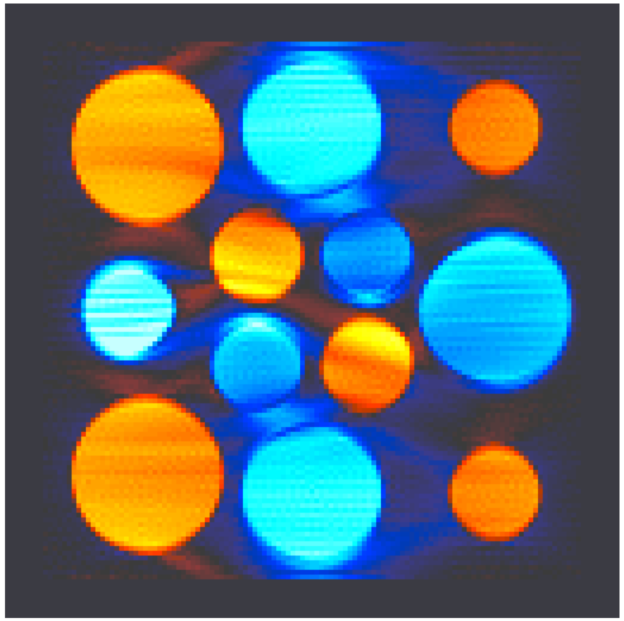
Example: conductivity varies a lot

Phantom $\ln \sigma(x)$
$\max \sigma(x) = 2.0$
$\min \sigma(x) = 0.5$

Reconstruction
Left-to-right



Reconstruction
Average



Using two measurements to reduce error?

(1) Use one set of currents, recover $\nabla u^{(1)}$.

(2) Use another set of currents, recover $\nabla u^{(2)}$.

(3) Now

$$\begin{cases} \nabla u^{(1)} \cdot \nabla \ln \sigma = -\Delta u^{(1)} \\ \nabla u^{(2)} \cdot \nabla \ln \sigma = -\Delta u^{(2)} \end{cases}$$

Solve this 2×2 linear system at each x, find $\nabla \ln \sigma(x)$.

Will this help to avoid error propagation along characteristics?

Work in progress, we'll see...