Can One Hear the Heat of a Body?
A Survey of Mathematics of Thermo- and Photoacoustic Tomography

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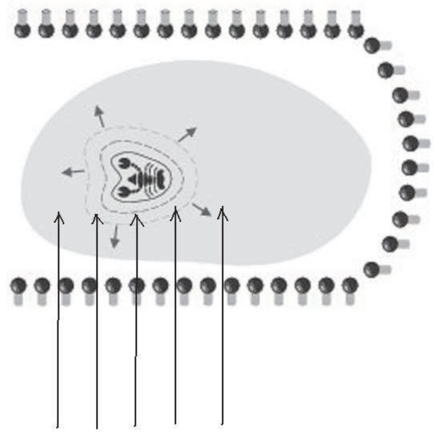

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An outline

- Mathematical model
- Uniqueness of reconstruction
- Inversion formulas and procedures
- Stability
- Range description
- Limited view reconstruction
- Speed of sound recovery

Surveys of mathematics of TAT

• M. Agranovsky, P. K., L. Kunyansky, On reconstruction formulas and algorithms for the thermoacoustic tomography, Ch. 8 in "Photoacoustic imaging and spectroscopy," CRC Press. 2009

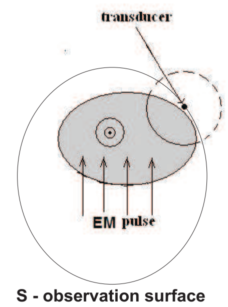
• D. Finch and Rakesh, The spherical mean value operator with centers on a sphere, Inv. Problems **23**(2007), No. 6, S37–S50.

• D. Finch and Rakesh, Recovering a function from its spherical mean values in two and three dimensions, Ch. 7 in "Photoacoustic imaging and spectroscopy," CRC Press. 2009

• P. Kuchment and L. Kunyansky, Mathematics of thermoacoustic tomography, European J. Appl. Math., **19** (2008), 191–224.

• S. Patch and O. Scherzer (Eds), special issue, Inverse Problems **23**(2007), No. 6.

1. **Thermoacoustic/Photoacoustic Tomography (TAT/PAT)**



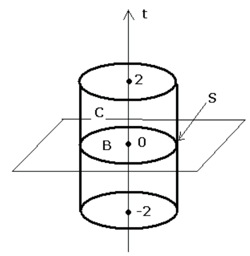
Goal: recover EM energy absorption $f(x)$. Cancerous cells absorb several times more energy than the healthy ones $\Rightarrow$ high contrast. Also high resolution of ultrasound.

Contributors present: Ambartsoumian, Arridge, Bal, Burgholzer, Finch, Haltmeier, Hristova, Kuchment, Kunyansky, Li, Nguyen, Palamodov, Quinto, Scherzer, Stefanov, Uhlmann, Xu.

2. **Mathematical model**

$$\begin{cases} p_{tt} = c^2(x)\Delta_x p, & t \geq 0, \quad x \in \mathbb{R}^3 \\ p(x,0) = f(x), & p_t(x,0) = 0, \\ p(y,t) = g(y,t) & \text{for } y \in S, \quad t \geq 0. \end{cases}$$

$c(x)$ - sound speed; $g(y,t)$ - data, S - *observation surface*.



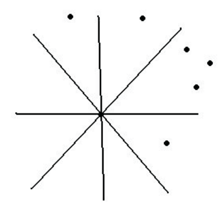
- Given g, find f. Why is this problem solvable?
- $c = 1, \Leftrightarrow$ inversion of *restricted spherical mean transform*:

$$R_S f(p,r) = \omega^{-1} \int_{|y-p|=r} f(y)\,d\sigma(y), \quad p \in S, \quad r \geq 0.$$

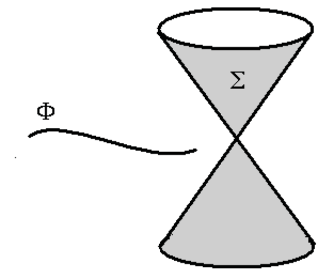
- $f(x)$ is NOT what's needed (hear talks by Arridge and Bal).
- Ultrasound attenuation is neglected (hear Burgholzer's talk).
- Detectors are assumed small omnidirectional. Other designs (Burgholzer, Haltmeier, Scherzer, et al) are integrating planar, line, and circular detectors. Mathematical models here are somewhat different.
- No free space outside S - see below.

3. **Uniqueness of reconstruction**

Assume supp f - compact, S - observation surface.
Image to measured data operator $R_S : f \mapsto g$.
Q.: Can one uniquely reconstruct f from $g = R_S f$?
A.1: If S is closed - **yes**, even for a variable speed (PK '93, Agranovsky-Berenstein-PK '96, Agranovsky-PK '07).
A.2: $2D$ and constant speed - **yes** for S that does not fit into a Coxeter cross $\bigcup$ finite set (Lin-Pinkus' conjecture '03, proof Agranovsky-Quinto '96):



Open: Any dim. and constant speed - **conjectured yes**, unless $S \subset$ zeros of a homogeneous harmonic polynomial $\bigcup$ algebraic set of codim $\geq$ 2. (Partial results Agranovsky-Quinto, Ambartsoumian-P.K. '05 (using Finch-Patch-Rakesh))



Open: Analog (even in $2D$) of A.-Q. for decaying functions. (S closed - Agranovsky-Berenstein-P.K. '96)

Open: Hyperbolic plane analog (even for compactly supported functions).

4. Inversion formulas and procedures

- **Filtered backprojection (FBP)** (Finch-Patch-Rakesh '04, Xu-Wang '05, Finch-Haltmeier-Rakesh '06, Kunyansky '06, Nguyen '09)

$$f(y) = -\frac{1}{8\pi^2}\Delta_y \int\limits_S \frac{g(z,|z-y|)}{|z-y|} dA(z),$$

$$f(y) = -\frac{1}{8\pi^2}\int\limits_S \left(\frac{1}{t}\frac{d^2}{dt^2}g(z,t)\right)\Big|_{t=|z-y|} dA(z).$$

Known for **constant speed** and S - **sphere** (cylinder and plane analogs exist).

Various formulas disagree outside the range.
A unified approach to them is given by Nguyen '09.

• **Eigenfunctions series expansions** (Norton's precursors '80, '81, Kunyansky '06, Agranovsky-PK '07)
$S = \partial B$, $\psi_k(x), \lambda_k^2$ - eigenfunctions/eigenvalues of $A = -c^2(x)\Delta_D$ in B, speed c-non-trapping, g - data.

Series expansion

$$f(x)|_B = \sum_k f_k \psi_k(x),$$
$$f_k = -\lambda_k^{-1} \int_0^\infty \int_S \sin(\lambda_k t) g(x,t) \overline{\frac{\partial \psi_k}{\partial \nu}(x)} dx dt.$$

Works wonderfully when S - cube (Kunyansky '06).

- **Time reversal** (Fink, Finch-Patch-Rakesh '04, Xu-Wang '04, Burgholzer et al '07, Hristova-PK-Nguyen '08, Hristova '09, Stefanov-Uhlmann '09)

T – large, $p(x,T)$ – small inside S. Solve back in time:

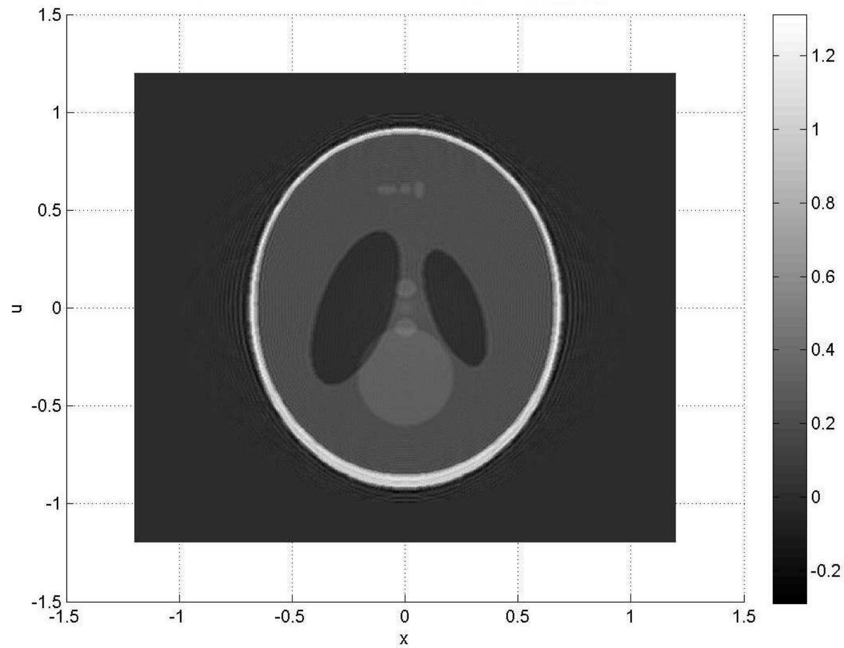
$$\begin{cases} p_{tt} = c^2(x)\Delta_x p, & t \geq 0, \quad x \in \mathbb{R}^3 \\ p(x,T) = 0, & p_t(x,T) = 0, \\ p(y,t) = g(y,t) & \text{for } y \in S, \quad t \geq 0. \end{cases}$$

Find at $t = 0$ approximation for $f(x) = p(x,0)$.
Exact when dimension $n > 1$ is odd, speed is constant.
Works approximately (estimates by Hristova '09), best in odd dimensions with non-trapping speed.
Works for any S, any speed; allows the support outside S.
Easy to implement.

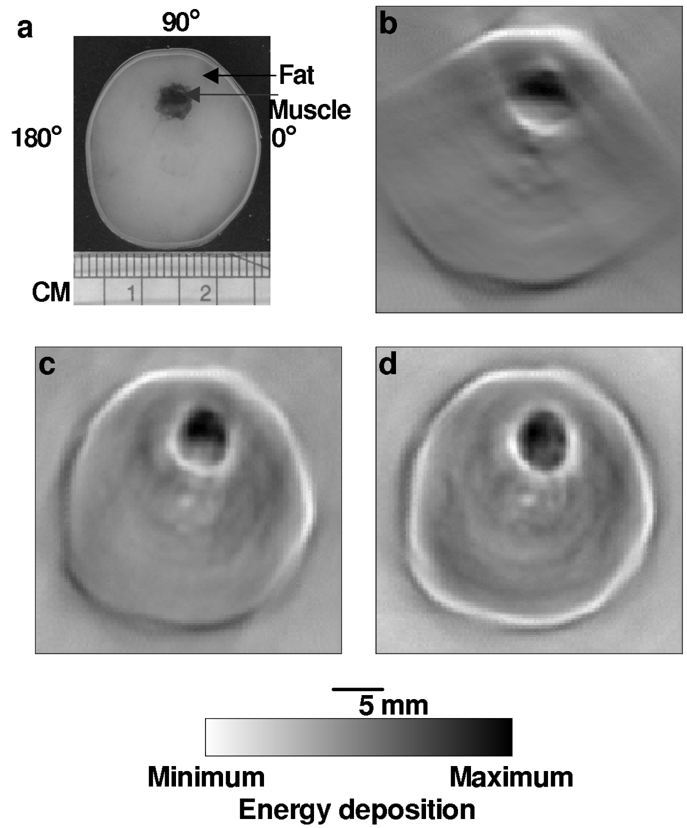
- **Parametrix** (Popov-Sushko, Xu-Wang, Burgholzer-Haltmeier-...)
- **Algebraic iterative procedures** (Anastasio, . . .)

Name	closed form	exact	S	external sources	speed
FBP	+	+	sphere	-	constant
Series	-	+	any?	+	any?
Time reversal	-	-	any	+	any
Parametrix	+	-	any	-	const
Algebraic	-	-	any	+	any

- Open: Do closed form inversions exist for S not a sphere?
- Open: Do closed form inversions exist that do not react to external sources?



Time reversal reconstruction of Shepp-Logan phantom
(Y. Hristova)



Parametrix reconstruction of a physical phantom (Y. Xu)

5. **Stability** Stability of reconstruction with **full data** and non-trapping speed is comparable to MRI or X-ray CT scan. (Palamodov '07, Stefanov-Uhlmann '09)

6. **Range description**
S - sphere. *Range conditions* for R_S? *Moment conditions* (Lin& Pinkus '93, Agranovsky& Quinto '96, Patch '04) on data $g(p,r) = R_S f(p,r)$ (p - center, r - radius):
$\forall k \in \mathbb{Z}, k \geq 0$,

$$G_k(\omega) = \int_0^\infty r^{2k} g(p,r) dr$$

is a polynomial of degree at most k. **Incomplete set** of range conditions.

Complete (Ambartsoumian & P. K '05, D. Finch & Rakesh '05, Agranovsky & P. K.& Quinto '06, Palamodov '08, Agranovsky & Finch &P.K. '09, Agranovsky & Nguyen '09).

Data $g(y,t) = R_S f = \int_{\mathbb{S}} f(y + \omega t) d\omega$. $S = \partial B$ – unit sphere.
Theorem(AFK '09) *TFAE for a function $g \in C_0^\infty(S \times [0,2])$:*

(a) $g = R_S f$ for some $f \in C_0^\infty(B)$.

(b) $\forall (-\lambda^2, \psi_\lambda)$ – eigenv/eigenf of Δ_D, one has

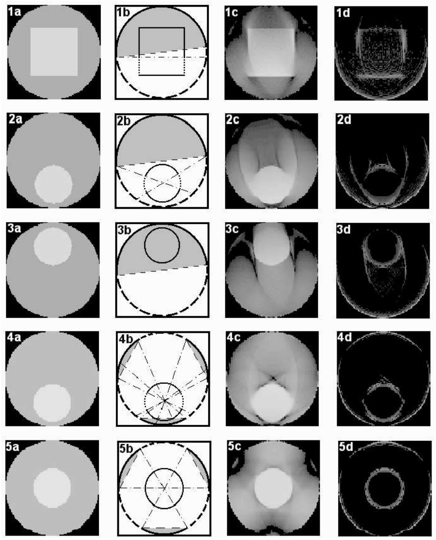
$$\int_{S \times [0,2]} g(x,t) \partial_\nu \psi_\lambda(x) j_{n/2-1}(\lambda t) t^{n-1} dx dt = 0.$$

(c) Let $\widehat{g}(x,\lambda) = \int g(x,t) j_{n/2-1}(\lambda t) t^{n-1} dt$. $\forall m \in \mathbb{Z}$, m^{th} spherical harmonic term $\widehat{g}_m(x,\lambda)$ of $\widehat{g}(x,\lambda)$ vanishes at non-zero zeros of Bessel function$J_{m+n/2-1}(\lambda)$.

7. **Limited view**

• Uniqueness (Agranovsky&Quinto '96, Finch-Patch-Rakesh '04, Stefanov&Uhlmann '09, Steinhauer '09)

• "Visible" singularities & instability. (Louis-Quinto '00, Xu-Wang-Ambartsoumian-PK '04, '09, Hristova-PK-Nguyen '08, Stefanov-Uhlmann '09, Nguyen '09)

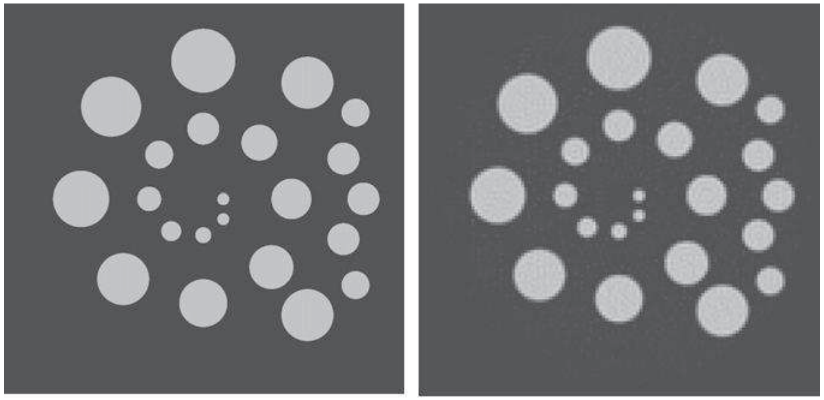


The "invisible" parts are blurred.

8. **Quality reconstructions from partial data**

Q.: Can one obtain quality reconstructions if the whole object is in "visible" zone?

A.: Yes, at least for constant sound speed (Kunyansky 2008; Patch '04 used range conditions with less satisfactory results)



Phantom (left) in the visible zone and its reconstruction (right) (Kunyansky '08).

Open: Variable speed case.

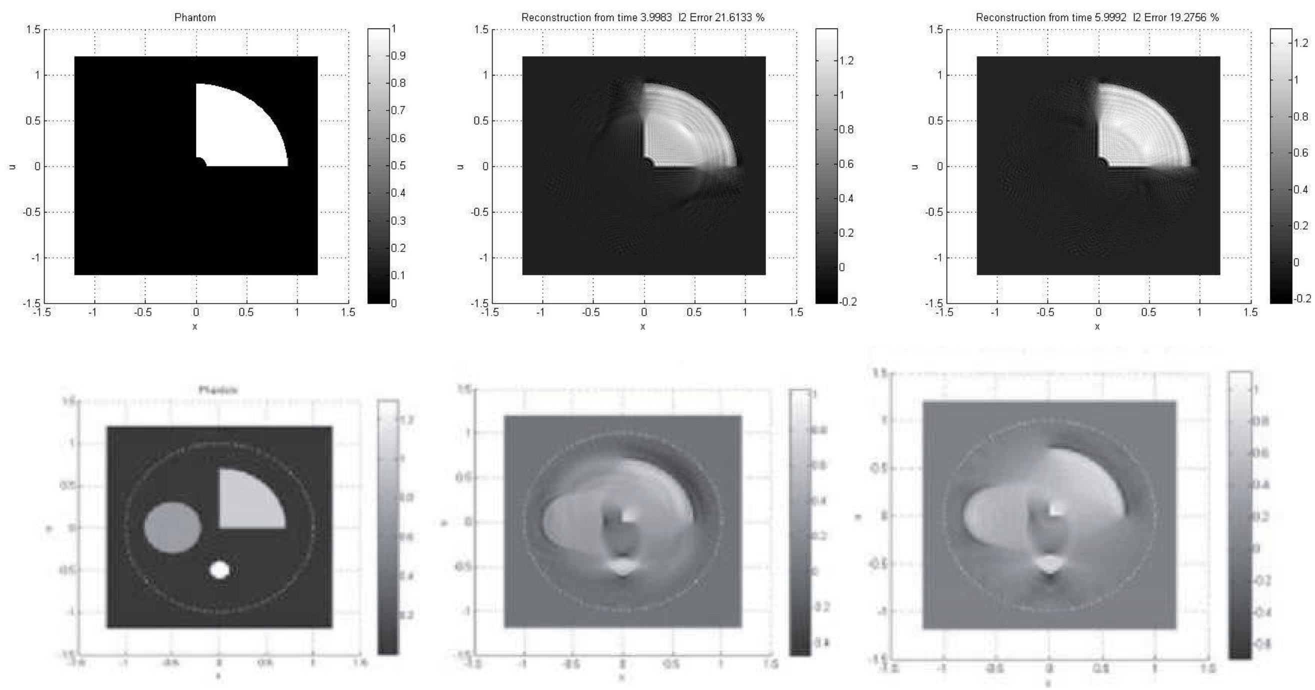
9. **Trapping:**
$H = \frac{c^2(x)}{2}|\xi|^2$, bicharacteristics:

$$\begin{cases} x'_t = \frac{\partial H}{\partial \xi} = c^2(x)\xi \\ \xi'_t = -\frac{\partial H}{\partial x} = -\frac{1}{2}\nabla \left(c^2(x)\right)|\xi|^2 \\ x|_{t=0} = x_0, \xi|_{t=0} = \xi_0. \end{cases}$$

Their projections to $\mathbb{R}^n_x$ - *rays*.
Non-trapping condition: Rays (with $\xi_0 \neq 0$) tend to ∞ when $t \to \infty$. Non-trapping $\Rightarrow$ decay and eventual smoothing in any compact region.
Trapping "crater" speed $c(x) = |x|$ for $r_1 < |x| < r_2$.

Worse is a parabolic crater.

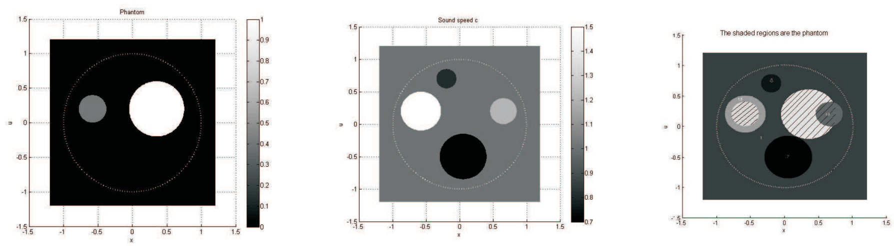
Variable speed and “full view” (Hristova-PK-Nguyen ’08)



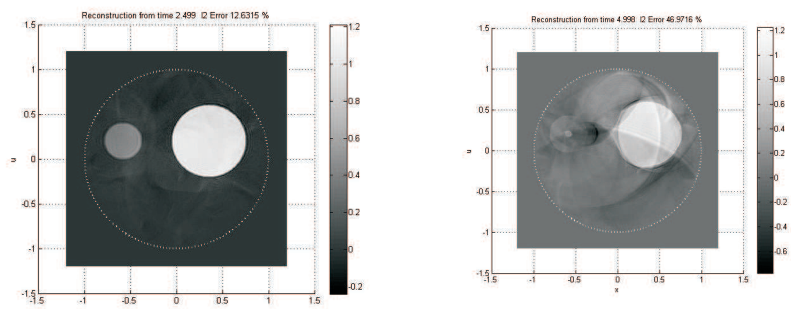
“Limited view” blurring effect due to trapping (crater (top) and parabolic (bottom) speeds).

10. **Finding the sound speed**

What if we get the speed wrong?



Phantom, sound speed, and their overlap.



Reconstructions: correct (left) and average (right) speed.

Can one find the speed?

Transmission ultrasound tomography before TAT (Xu-Wang).
Open: Is the speed c uniquely determined by TAT data?
Can one recover it? (analog of a SPECT problem)
Successful numerical experiments (Anastasio-Zhang '06, Yuan&Jiang '05-...).
f is supported strictly inside $S \Rightarrow$ **constant** speed is determined uniquely (PK-Nguyen '08).
f is supported inside $S \Rightarrow$ range conditions **locally** uniquely determine coefficient α in $\alpha c(x)$ (PK-Nguyen '08).
Relation to the transmission eigenvalue problem (Finch '08).
If $c_1(x) \geq c_2(x)$, TAT data coincide only if $c_1(x) = c_2(x)$ (Finch&Hickmann, Agranovsky '09).
Linearized uniqueness result in $1D$ (Nguyen '09), detection of constant speed in $1D$ (Finch&Hickman).