Banach SSD spaces and classes of monotone sets

by

Stephen Simons

<simons@math.ucsb.edu>.

Abstract

We discuss the theory of SSD spaces and Banach SSD spaces. We explain why type (ED), dense type, type (D), type (NI) and strong representability are equivalent concepts for maximally monotone sets and how the known properties of strongly representable sets follow from known properties of sets of type (ED).

Downloads

You can download files containing these slides, and related papers from

<www.math.ucsb.edu/~simons/Banff.html>.

SSD spaces and Banach SSD spaces

$\big(B,\lfloor\cdot,\cdot\rfloor\big)$ is a *symmetrically self–dual space (SSD space)* if B is a nonzero real vector space and $\lfloor\cdot,\cdot\rfloor\colon B\times B\to\mathbb{R}$ is a symmetric bilinear form. $\big(B,\lfloor\cdot,\cdot\rfloor,\|\cdot\|\big)$ is a *Banach SSD space* if $\big(B,\lfloor\cdot,\cdot\rfloor\big)$ is an SSD space, $\big(B,\|\cdot\|\big)$ is a Banach space and,

$$\forall\, b,c\in B,\ \ \lfloor b,c\rfloor\le\|b\|\|c\|.$$

(⚷)

The quadratic form q

If $\big(B,\lfloor\cdot,\cdot\rfloor\big)$ is an SSD space. we define the quadratic form q on B by $q(b):=\frac{1}{2}\lfloor b,b\rfloor$. We have the parallelogram law:

$$b,c\in B\quad\Longrightarrow\quad \tfrac{1}{2}q(b-c)+\tfrac{1}{2}q(b+c)=q(b)+q(c).$$

Examples

(a) If B is a Hilbert space with inner product $(b,c)\mapsto\langle b,c\rangle$ then B is a Banach SSD space with $\lfloor b,c\rfloor:=\langle b,c\rangle$, and $q(b)=\frac{1}{2}\|b\|^2$.

(b) If B is a Hilbert space with inner product $(b,c)\mapsto\langle b,c\rangle$ then B is a Banach SSD space with $\lfloor b,c\rfloor:=-\langle b,c\rangle$, and $q(b)=-\frac{1}{2}\|b\|^2$.

(c) $\mathbb{R}^3$ is a Banach SSD space with $\big\lfloor(b_1,b_2,b_3),(c_1,c_2,c_3)\big\rfloor:=b_1c_2+b_2c_1+b_3c_3$. Then $q(b_1,b_2,b_3)=b_1b_2+\frac{1}{2}b_3^2$.

(d) $\mathbb{R}^3$ is **not** a Banach SSD space with $\big\lfloor(b_1,b_2,b_3),(c_1,c_2,c_3)\big\rfloor:=b_1c_2+b_2c_3+b_3c_1$. (The bilinear form $\lfloor\cdot,\cdot\rfloor$ is not symmetric.)

SSD spaces and Banach SSD spaces

$\big(B,\lfloor\cdot,\cdot\rfloor\big)$ is a *symmetrically self–dual space (SSD space)* if B is a nonzero real vector space and $\lfloor\cdot,\cdot\rfloor\colon B\times B\to\mathbb{R}$ is a symmetric bilinear form. $\big(B,\lfloor\cdot,\cdot\rfloor,\|\cdot\|\big)$ is a *Banach SSD space* if $\big(B,\lfloor\cdot,\cdot\rfloor\big)$ is an SSD space, $(B,\|\cdot\|)$ is a Banach space and,

$$\forall\ b,c\in B,\ \ \lfloor b,c\rfloor\le\|b\|\|c\|.$$

Another example

(e) Let E be a nonzero Banach space and $B:=E\times E^*$ under the norm

$$\big\|(x,x^*)\big\|:=\sqrt{\|x\|^2+\|x^*\|^2}.$$

$\forall\ b=(x,x^*),\ \ c=(y,y^*)\in B$, let

$$\lfloor b,c\rfloor:=\langle x,y^*\rangle+\langle y,x^*\rangle.$$

Then $\big(B,\lfloor\cdot,\cdot\rfloor,\|\cdot\|\big)$ is a Banach SSD space, and

$$q(b)=\langle x,x^*\rangle.$$

Any finite dimensional SSD space of this form must have even dimension. Thus odd dimensional cases of the examples considered on the previous slide cannot be of this form. This example uses two bilinear forms, and our later analysis will use three.

• To clarify matters, we introduce the following more precise notation: $\big(B,\lfloor\cdot,\cdot\rfloor\big)$ and $\big(D,\lceil\cdot,\cdot\rceil\big)$ will always be SSD spaces and $\big(B,\lfloor\cdot,\cdot\rfloor,\|\cdot\|\big)$ and $\big(D,\lceil\cdot,\cdot\rceil,\|\cdot\|\big)$ will always be Banach SSD spaces. We will call $\lfloor\cdot,\cdot\rfloor$ "floor" and $\lceil\cdot,\cdot\rceil$ "ceiling", and sometimes B the "floor space" and D the "ceiling space".

q–positive sets

Let $\big(B, \lfloor\cdot,\cdot\rfloor\big)$ be a SSD space and $A \subset B$. We say that A is *q–positive* if $A \ne \emptyset$ and

$$b, c \in A \Longrightarrow q(b-c) \ge 0.$$

Examples

(a) B is a Hilbert space with $q(b) = \frac{1}{2}\|b\|^2$: every nonempty subset of B is q–positive.

(b) B is a Hilbert space with $q(b) = -\frac{1}{2}\|b\|^2$: the q–positive subsets of B are the singletons.

(e) E is a nonzero Banach space, $B := E \times E^*$, $\forall\ b = (x, x^*) \in B$, $q(b) = \langle x, x^*\rangle$. Let $\emptyset \ne A \subset B$. Then A is q–positive when

$$(x, x^*), (y, y^*) \in A \quad \Longrightarrow \quad \langle x - y, x^* - y^*\rangle \ge 0.$$

That is to say,

$$A \text{ is } q\text{–positive} \iff A \text{ is a monotone subset of } E \times E^*.$$

General notation

- Let X be a vector space and $f\colon X \to]{-\infty}, \infty]$. Then $\operatorname{dom} f := \{x \in X\colon\ f(x) \in \mathbb{R}\}$.
- f is *proper* if $\operatorname{dom} f \ne \emptyset$.
- $\mathcal{PC}(X)$ is the set of all proper convex functions $f\colon X \to]{-\infty}, \infty]$.
- If X is a Banach space, $\mathcal{PCLSC}(X) := \{f \in \mathcal{PC}(X)\colon\ f \text{ is lower semicontinuous}\}$.
- If $\big(B, \lfloor\cdot,\cdot\rfloor\big)$ is a SSD space, A will always denote a q–positive subset of B.

The q–positive set given by a convex function

Let $f \in \mathcal{PC}(B)$ and $f \ge q$ on B. Let $\mathcal{P}_q(f) := \big\{b \in B\colon\ f(b) = q(b)\big\}$. If $\mathcal{P}_q(f) \ne \emptyset$ then $\mathcal{P}_q(f)$ is a q–positive subset of B.

Proof. Let $b, c \in \mathcal{P}_q(f)$. Then, from the parallelogram law, the quadraticity of q, and the convexity of f,

$$\begin{aligned}\tfrac{1}{2}q(b-c) = q(b) + q(c) - \tfrac{1}{2}q(b+c) &= q(b) + q(c) - 2q\big(\tfrac{1}{2}(b+c)\big)\\ &\ge f(b) + f(c) - 2f\big(\tfrac{1}{2}(b+c)\big) \ge 0.\end{aligned}$$

□

- If $f \in \mathcal{PC}(B)$, we write $f^{@}$ for the intrinsic conjugate of f with respect to the pairing $\lfloor\cdot,\cdot\rfloor$. That is to say, $\forall\ c \in B$,

$$f^{@}(c) := \sup\nolimits_B \big[\lfloor\cdot, c\rfloor - f\big].$$

- Let $f \in \mathcal{PC}(B)$. f is a BC–function if

$$b \in B \quad \Longrightarrow \quad f^{@}(b) \ge f(b) \ge q(b). \tag{$\star$}$$

"BC" stands for "bigger conjugate".

Surprise result

Let $f \in \mathcal{PC}(B)$ be a BC–function. Then $\mathcal{P}_q(f^{@}) = \mathcal{P}_q(f)$.

- Let $f \in \mathcal{PC}(B)$. f is a BC–function if

$$b \in B \quad \Longrightarrow \quad f^{@}(b) \geq f(b) \geq q(b). \qquad (\star)$$

Surprise result

Let $f \in \mathcal{PC}(B)$ *be a BC–function. Then* $\mathcal{P}_q(f^{@}) = \mathcal{P}_q(f)$.

Proof. Let c be an arbitrary element of $\mathcal{P}_q(f)$. Let $b \in B$ and $\lambda \in]0,1[$ be arbitrary. For simplicity, let $\mu := 1 - \lambda \in]0,1[$. Then, from the quadraticity of q, the convexity of f and $(\star)$,

$$\lambda^2 q(b) + \lambda\mu\lfloor b,c\rfloor + \mu^2 q(c) = q\big(\lambda b + \mu c\big) \leq f(\lambda b + \mu c)$$
$$\leq \lambda f(b) + \mu f(c) = \lambda f(b) + \mu q(c).$$

Thus

$$\lambda\mu\lfloor b,c\rfloor - \lambda f(b) \leq \lambda\mu q(c) - \lambda^2 q(b).$$

Dividing by λ and letting $\lambda \to 0$,

$$\lfloor b,c\rfloor - f(b) \leq q(c).$$

Taking the supremum over $b \in B$,

$$f^{@}(c) \leq q(c),$$

and $(\star)$ implies that $c \in \mathcal{P}_q(f^{@})$. Thus we have proved that $\mathcal{P}_q(f) \subset \mathcal{P}_q(f^{@})$. The opposite inclusion is obvious from $(\star)$. □

The convex function given by a q–positive subset, A, of $\big(B, \lfloor\cdot,\cdot\rfloor\big)$

We define $\Phi_A\colon\ B \to \,]{-\infty}, \infty\,]$ by

$$\Phi_A(b) := \sup\nolimits_A \big[\lfloor b,\cdot\rfloor - q\big] = q(b) - \inf q(A - b).$$

- $\Phi_A = q$ on A and $\Phi_A \in \mathcal{PC}(B)$.
- $\forall\ c \in B$ and $a \in A$, $\lfloor c, a\rfloor - q(a) \le \Phi_A(c)$, and so $\lfloor c, a\rfloor - \Phi_A(c) \le q(a)$. Thus $${\Phi_A}^{@}(a) \le q(a).$$
- Let $c \in B$. Then

$${\Phi_A}^{@}(c) = \sup\nolimits_B \big[\lfloor\cdot, c\rfloor - \Phi_A\big] \ge \sup\nolimits_A \big[\lfloor c,\cdot\rfloor - \Phi_A\big] = \sup\nolimits_A \big[\lfloor c,\cdot\rfloor - q\big] = \Phi_A(c)$$

and

$${\Phi_A}^{@@}(c) = \sup\nolimits_B \big[\lfloor\cdot, c\rfloor - {\Phi_A}^{@}\big] \ge \sup\nolimits_A \big[\lfloor c,\cdot\rfloor - {\Phi_A}^{@}\big] \ge \sup\nolimits_A \big[\lfloor c,\cdot\rfloor - q\big] = \Phi_A(c).$$

It is easy to see that ${\Phi_A}^{@@}(c) \le \Phi_A(c)$.

Properties of Φ_A

$\Phi_A(b) = q(b) - \inf q(A - b)$, $\Phi_A = q$ on A, ${\Phi_A}^{@} \ge \Phi_A$ on B and ${\Phi_A}^{@@} = \Phi_A$.

- Let $f \in \mathcal{PC}(B)$. f is a BC–function if

$$b \in B \quad \Longrightarrow \quad f^{@}(b) \geq f(b) \geq q(b). \qquad (\star)$$

Surprise result

Let $f \in \mathcal{PC}(B)$ *be a BC–function. Then* $\mathcal{P}_q(f^{@}) = \mathcal{P}_q(f)$.

Properties of Φ_A

$\Phi_A(b) = q(b) - \inf q(A - b)$, $\Phi_A = q$ on A, ${\Phi_A}^{@} \geq \Phi_A$ on B and ${\Phi_A}^{@@} = \Phi_A$.

$\mathcal{P}_q({\Phi_A}^{@})$ theorem

Let $\Phi_A \geq q$ *on* B. *Then* Φ_A *is a BC–function, and so* $\mathcal{P}_q({\Phi_A}^{@}) = \mathcal{P}_q(\Phi_A)$.

- Let $b \in B$ and $\Phi_A(b) \leq q(b)$. Then $\inf q(A - b) \geq 0$, and so $A \cup \{b\}$ is q–positive. So if A is maximally q–positive then $b \in B \setminus A \Longrightarrow \Phi(b) > q(b)$.
- To sum up:

$$A \text{ maximally } q\text{–positive} \quad \Longrightarrow \quad \Phi_A \geq q \text{ on } B \quad \text{and} \quad \mathcal{P}_q(\Phi_A) = A.$$

The convex function given by a maximally q–positive set

Let A be a maximally q–positive subset of B. Then Φ_A is a BC–function, and so

$$\mathcal{P}_q({\Phi_A}^{@}) = \mathcal{P}_q(\Phi_A) = A.$$

SSD–homomorphisms

Let $\big(B, \lfloor\cdot,\cdot\rfloor\big)$ and $\big(D, \lceil\cdot,\cdot\rceil\big)$ be SSD spaces and $\iota\colon\ B \to D$. ι is a *SSD–homomorphism* if ι is linear and,

$$\forall\ b, c \in B, \qquad \big\lceil \iota(b), \iota(c)\big\rceil = \lfloor b, c\rfloor.$$

- Let $\quad \widetilde{q}(d) := \frac{1}{2}\lceil d, d\rceil \quad (d \in D)$. Then $\quad \widetilde{q} \circ \iota = q$.
- Define the bilinear map $\langle\cdot,\cdot\rangle_\iota\colon B \times D \to \mathbb{R}$ by $\quad \langle b, d\rangle_\iota := \lceil \iota(b), d\rceil \quad \big((b, d) \in B \times D\big)$.

Then, $\forall\ b, c \in B$, $\big\langle b, \iota(c)\big\rangle_\iota = \big\lceil \iota(b), \iota(c)\big\rceil = \lfloor b, c\rfloor$.

- If $f \in \mathcal{PC}(B)$ and $d \in D$ let $f^*(d) := \sup_B \big[\langle\cdot, d\rangle_\iota - f\big]$. Then $\quad f^* \circ \iota = f^@$.
- Recall that, $\forall\ b \in B$, $\Phi_A(b) = q(b) - \inf q(A - b)$.
- $\iota(A)$ is a $\widetilde{q}$–positive subset of D and, moving the expression above to the "ceiling", $\forall\ d \in D$, $\Phi_{\iota(A)}(d) = \widetilde{q}(d) - \inf \widetilde{q}\big(\iota(A) - d\big)$. It follows that $\quad \Phi_{\iota(A)} \circ \iota = \Phi_A$.

$$\Phi_{\iota(A)}{}^@(d) = \sup_D \big[\lceil\cdot, d\rceil - \Phi_{\iota(A)}\big] \geq \sup_B \big[\big\lceil \iota(\cdot), d\big\rceil - \Phi_{\iota(A)} \circ \iota\big] = \sup_B \big[\langle\cdot, d\rangle_\iota - \Phi_A\big]$$

$$\Phi_A{}^*(d) = \sup_B \big[\langle\cdot, d\rangle_\iota - \Phi_A\big] \geq \sup_A \big[\langle\cdot, d\rangle_\iota - \Phi_A\big]$$

$$= \sup_A \big[\langle\cdot, d\rangle_\iota - q\big] = \sup_{\iota(A)} \big[\lceil d, \cdot\rceil - \widetilde{q}\big] = \Phi_{\iota(A)}(d).$$

Half–sandwich property

$$\Phi_{\iota(A)}{}^@ \geq \Phi_A{}^* \geq \Phi_{\iota(A)} \text{ on } D.$$

Properties of Φ_A

$\ldots$ and $\Phi_A{}^{@@} = \Phi_A$.

Half–sandwich property

$\Phi_{\iota(A)}{}^{@} \ge \Phi_A{}^{*} \ge \Phi_{\iota(A)}$ on D.

• Since $\Phi_{\iota(A)}{}^{@} \ge \Phi_A{}^{*} \ge \Phi_{\iota(A)}$ on D, we have $\Phi_{\iota(A)}{}^{@} \ge \Phi_A{}^{*@} \ge \Phi_{\iota(A)}{}^{@@}$ on D. From the "ceiling" version of the property of $\Phi.$ above, $\Phi_{\iota(A)}{}^{@@} = \Phi_{\iota(A)}$. Consequently:

Sandwich property

$\Phi_{\iota(A)}{}^{@} \ge \Phi_A{}^{*} \ge \Phi_{\iota(A)}$ on D and $\Phi_{\iota(A)}{}^{@} \ge \Phi_A{}^{*@} \ge \Phi_{\iota(A)}$ on D.

• If $f \in \mathcal{PC}(B)$, we call $f^{*@}$ the sesquiconjugate of f. So in words we have: *the conjugate and the sesquiconjugate of Φ_A are sandwiched between $\Phi_{\iota(A)}$ and its intrinsic conjugate.*

$\mathcal{P}_q\big(\Phi_A{}^{@}\big)$ theorem

Let $\Phi_A \ge q$ on B. Then Φ_A is a BC–function, and so $\mathcal{P}_q(\Phi_A{}^{@}) = \mathcal{P}_q(\Phi_A)$.

"Ceiling" $\mathcal{P}_{\tilde q}\big(\Phi_{\iota(A)}{}^{@}\big)$ theorem

Let $\Phi_{\iota(A)} \ge \widetilde{q}$ on D. Then $\Phi_{\iota(A)}$ is a BC–function, and so $\mathcal{P}_{\tilde q}(\Phi_{\iota(A)}{}^{@}) = \mathcal{P}_{\tilde q}(\Phi_{\iota(A)})$.

$\forall\ d \in D,\ \Phi_{\iota(A)}(d) = \widetilde{q}(d) - \inf \widetilde{q}\big(\iota(A) - d\big).$ It follows that $\Phi_{\iota(A)} \circ \iota = \Phi_A.$ (🐕)

Sandwich property

$${\Phi_{\iota(A)}}^{@} \ge {\Phi_A}^{*} \ge \Phi_{\iota(A)} \text{ on } D \quad \text{and} \quad {\Phi_{\iota(A)}}^{@} \ge {\Phi_A}^{*@} \ge \Phi_{\iota(A)} \text{ on } D.$$

"Ceiling" $\mathcal{P}_{\tilde{q}}\big({\Phi_{\iota(A)}}^{@}\big)$ theorem

Let $\Phi_{\iota(A)} \ge \widetilde{q}$ *on* D. *Then* $\Phi_{\iota(A)}$ *is a BC–function, and so* $\mathcal{P}_{\tilde{q}}({\Phi_{\iota(A)}}^{@}) = \mathcal{P}_{\tilde{q}}(\Phi_{\iota(A)})$.

The Gossez extension

Let $\big(B, \lfloor\cdot,\cdot\rfloor\big)$ and $\big(D, \lceil\cdot,\cdot\rceil\big)$ be SSD spaces and $\iota\colon\ B \to D$ be an SSD–homomorphism. The Gossez extension of A is the set $A^{\mathcal{G}} = \big\{d \in D\colon\ \Phi_{\iota(A)}(d) \le \widetilde{q}(d)\big\}$.

Theorem on the Gossez extension

(a) $\iota(A) \subset A^{\mathcal{G}}$. (This justifies the term "extension".)

(b) *If* $\Phi_{\iota(A)} \ge \widetilde{q}$ *on* D *then* $A^{\mathcal{G}} = \mathcal{P}_{\tilde{q}}({\Phi_A}^{*@}) = \mathcal{P}_{\tilde{q}}({\Phi_A}^{*}) = \mathcal{P}_{\tilde{q}}({\Phi_{\iota(A)}}^{@}) = \mathcal{P}_{\tilde{q}}(\Phi_{\iota(A)})$.

Proof. (a) From (🐕), $\forall a \in A,\ \Phi_{\iota(A)}\big(\iota(a)\big) = \Phi_A(a) = q(a) = \widetilde{q}\big(\iota(a)\big)$, which gives (a). As for (b), obviously $A^{\mathcal{G}} = \mathcal{P}_{\tilde{q}}(\Phi_{\iota(A)})$, and the sandwich property and the $\mathcal{P}_{\tilde{q}}\big({\Phi_{\iota(A)}}^{@}\big)$ theorem give $\mathcal{P}_{\tilde{q}}({\Phi_{\iota(A)}}^{@}) \subset \mathcal{P}_{\tilde{q}}({\Phi_A}^{*}) \subset \mathcal{P}_{\tilde{q}}(\Phi_{\iota(A)}) = \mathcal{P}_{\tilde{q}}({\Phi_{\iota(A)}}^{@})$ and $\mathcal{P}_{\tilde{q}}({\Phi_{\iota(A)}}^{@}) \subset \mathcal{P}_{\tilde{q}}({\Phi_A}^{*@}) \subset \mathcal{P}_{\tilde{q}}(\Phi_{\iota(A)}) = \mathcal{P}_{\tilde{q}}({\Phi_{\iota(A)}}^{@})$. $\square$

— Banach SSD spaces and classes of monotone sets —

- Let $\big(B, \lfloor\cdot,\cdot\rfloor, \|\cdot\|\big)$ be a Banach SSD space.
- From (🗝), $\forall\ b \in B$, $|q(b)| = \frac{1}{2}|\lfloor b, b\rfloor| \le \frac{1}{2}\|b\|\|b\|$. Now define the function p on B by $p := \frac{1}{2}\|\cdot\|^2 + q$. Then $\inf_B p = 0$.

VZ functions

Let $f \in \mathcal{PC}(B)$. We say that f is a *VZ function* if (writing ∇ for inf–convolution)

$$(f - q)\,\nabla\, p = 0 \text{ on } B.$$

Theorem on lower semicontinuous VZ functions

Let $f \in \mathcal{PCLSC}(B)$ be a VZ function. Then $\mathcal{P}_q(f)$ is a maximally q–positive subset of B and, $\forall\ c \in B$,

$$\operatorname{dist}\big(c, \mathcal{P}_q(f)\big) \le \sqrt{2}\sqrt{(f-q)(c)}.$$

Note that $\sqrt{2}$ is the best constant possible: take $\big(\mathbb{R}\times\mathbb{R}, \lfloor\cdot,\cdot\rfloor, \|\cdot\|\big)$ and $f := g$.

- We say that a subset A of B is *p–dense in B* if, $\forall\ c \in B$, $\inf p(A - c) = 0$.

p-density criterion for a VZ function

Let $f \in \mathcal{PCLSC}(B)$. Then f is a VZ function $\iff$ $f \ge q$ on B and $\mathcal{P}_q(f)$ is p–dense in B.

- *If $f \in \mathcal{PC}(B)$ is a VZ function then $f^{@}$ is a VZ function.*
- These results depend heavily on the completeness of B. For full details, see the last of the items available at <www.math.ucsb.edu/~simons/Banff.html>.

The map ι

From (🔑) and standard algebraic arguments, $\exists$ a linear map $\iota\colon\ B \to B^*$ such that $\|\iota\| \le 1$ and

$$\forall\ b, c \in B, \quad \big\langle b, \iota(c)\big\rangle = \lfloor b, c\rfloor. \qquad (🍦)$$

Banach SSD duals

Let $\big(B, \lfloor\cdot,\cdot\rfloor, \|\cdot\|\big)$ be a Banach SSD space, $(B^*, \|\cdot\|)$ be the Banach space dual of B and the linear map $\iota\colon\ B \to B^*$ be defined as in (🍦). Let $\big(B^*, \lceil\cdot,\cdot\rceil, \|\cdot\|\big)$ also be a Banach SSD space. We say that $\big(B^*, \lceil\cdot,\cdot\rceil, \|\cdot\|\big)$ is a *Banach SSD dual* of $\big(B, \lfloor\cdot,\cdot\rfloor, \|\cdot\|\big)$ if $\langle\cdot,\cdot\rangle_\iota = \langle\cdot,\cdot\rangle$ on $B \times B^*$, that is to say

$$\forall\ b \in B \text{ and } c^* \in B^*, \qquad \big\lceil \iota(b), c^*\big\rceil = \langle b, c^*\rangle. \qquad (🐦)$$

Let $\big(B^*, \lceil\cdot,\cdot\rceil, \|\cdot\|\big)$ be a Banach SSD dual of $\big(B, \lfloor\cdot,\cdot\rfloor, \|\cdot\|\big)$.

- From (🐦) and (🍦), $\forall\ b, c \in B$, $\big\lceil \iota(b), \iota(c)\big\rceil = \big\langle b, \iota(c)\big\rangle = \lfloor b, c\rfloor$. So ι is an SSD-homomorphism from $\big(B, \lfloor\cdot,\cdot\rfloor\big)$ into $\big(B^*, \lceil\cdot,\cdot\rceil\big)$.

The map $\widetilde{\iota}$

By analogy with (🍦), we define the linear map $\widetilde{\iota}\colon B^* \to B^{**}$ such that $\|\widetilde{\iota}\| \le 1$ and

$$\forall\ c^*, b^* \in B^*, \qquad \big\langle c^*, \widetilde{\iota}(b^*)\big\rangle = \lceil c^*, b^*\rceil. \qquad (🌳)$$

- We had $f \in \mathcal{PC}(B) \Longrightarrow f^* \circ i = f^@$. “Ceiling version”: $h \in \mathcal{PC}(B^*) \Longrightarrow h^* \circ \widetilde{\iota} = h^@$.

Let $\big(B^*, \lceil\cdot,\cdot\rceil, \|\cdot\|\big)$ be a Banach SSD dual of $\big(B, \lfloor\cdot,\cdot\rfloor, \|\cdot\|\big)$.

So far . . .

$$\forall\ b \in B \text{ and } c^* \in B^*, \qquad \big\lceil \iota(b), c^* \big\rceil = \langle b, c^* \rangle. \tag{𓅃}$$

$$\forall\ c^*, b^* \in B^*, \qquad \lceil c^*, b^* \rceil = \big\langle c^*, \widetilde{\iota}(b^*) \big\rangle. \tag{🌳}$$

The automatic factorization of the canonical map $\widehat{\ }\colon\ B \to B^{**}$

$$\forall\ b \in B.\quad \widehat{b} = \widetilde{\iota} \circ \iota(b).$$

Proof. Let $b \in B$ and $c^* \in B^*$. Then, from the definition of $\widehat{b}$, (𓅃) and (🌳),

$$\big\langle c^*, \widehat{b} \big\rangle = \langle b, c^* \rangle = \big\lceil \iota(b), c^* \big\rceil = \big\lceil c^*, \iota(b) \big\rceil = \big\langle c^*, \widetilde{\iota} \circ \iota(b) \big\rangle. \qquad \square$$

If $f \in \mathcal{PCLSC}(B)$ then, from the Fenchel–Moreau theorem, $\forall b \in B,\ f(b) = f^{**}\big(\widehat{b}\big)$. Thus $f = f^{**} \circ \widetilde{\iota} \circ \iota = (f^*)^* \circ \widetilde{\iota} \circ \iota = (f^*)^{@} \circ \iota = f^{*@} \circ \iota$. So we get the following

- Fenchel–Moreau theorem for sesquiconjugates: $\quad f \in \mathcal{PCLSC}(B) \Longrightarrow f = f^{*@} \circ \iota$.
- Define the function $\widetilde{p}\colon\ B^* \to \mathbb{R}$ by $\widetilde{p} := \frac{1}{2}\|\cdot\|^2 + \widetilde{q}$. Then $\quad \widetilde{p} \ge 0$ on B^*.

The "–" equality

If $f \in \mathcal{PC}(B)$ then $\quad -\big((f - q)\,\nabla\, p\big) = \big((f^* - \widetilde{q})\,\nabla\, \widetilde{p}\big) \circ \iota$ *on B.*

Proof. This follows from Rockafellar's version of the Fenchel duality theorem and the fact that the conjugate of the function $b \mapsto \frac{1}{2}\|b\|^2$ is the function $b^* \mapsto \frac{1}{2}\|b^*\|^2$. $\square$

- We say that $\iota(B)$ is $\widetilde{p}$*–dense in* B^* if, $\quad \forall\ b^* \in B^*, \quad \inf \widetilde{p}\big(\iota(B) - b^*\big) = 0$.

Let $\big(B^*, \lceil\cdot,\cdot\rceil, \|\cdot\|\big)$ be a Banach SSD dual of $\big(B, \lfloor\cdot,\cdot\rfloor, \|\cdot\|\big)$.

VZ functions

Let $f \in \mathcal{PC}(B)$. f is a *VZ function* if $\;(f - q) \,\nabla\, p = 0$ on B.

The "–" equality

If $f \in \mathcal{PC}(B)$ then $\;-\big((f - q) \,\nabla\, p\big) = \big((f^* - \widetilde{q}) \,\nabla\, \widetilde{p}\big) \circ \iota$ *on B.*

MAS functions

Let $f \in \mathcal{PC}(B)$. f is an *MAS function* if $\;f \ge q$ on $B\;$ and $\;f^* \ge \widetilde{q}$ on B^*.

MASVZ theorem

Let $\iota(B)$ be $\widetilde{p}$–dense in B^* and $f \in \mathcal{PC}(B)$. Then

$$f \text{ is an MAS function} \iff f \text{ is a VZ function}.$$

Proof. $(\Longrightarrow)$ We have $\inf_B \big[f - q\big] \ge 0$, $\inf_B p \ge 0$, $\inf_{B^*} \big[f^* - \widetilde{q}\big] \ge 0$ and $\inf_{B^*} \widetilde{p} \ge 0$. Consequently, $\inf_B \big[(f - q) \,\nabla\, p\big] \ge 0$ and $\inf_B \big[\big((f^* - \widetilde{q}) \,\nabla\, \widetilde{p}\big) \circ \iota\big] \ge 0$, and $(\Longrightarrow)$ follows from the "–" equality.

$(\Longleftarrow)$ It is easily seen that $\;f \ge q$ on B. Now let $b^* \in B^*$ and $c \in B$. Then

$$(f^* - \widetilde{q})(b^*) + \widetilde{p}\big(\iota(c) - b^*\big) \ge \big((f^* - \widetilde{q}) \,\nabla\, \widetilde{p}\big)\big(\iota(c)\big) = -\big((f - q) \,\nabla\, p\big)(c) = 0.$$

Taking the infimum over $c \in B$ and using the $\widetilde{p}$–density, $\;(f^* - \widetilde{q})(b^*) \ge 0$ on B^*. Since this holds for all $b^* \in B^*$, f is an MAS function, giving $(\Longleftarrow)$. □

Let $\big(B^*, \lceil\cdot,\cdot\rceil, \|\cdot\|\big)$ be a Banach SSD dual of $\big(B, \lfloor\cdot,\cdot\rfloor, \|\cdot\|\big)$.

Compatible topologies on B^*

We say that $\mathcal{T}$ is a compatible topology on B^* if (a)–(c) below are satisfied:

(a) $\mathcal{T} \supset w(B^*, B^*)$.

(b) If $f \in \mathcal{PCLSC}(B)$ and $b^* \in B^*$ then $\exists$ a net $\{b_\gamma\}$ of elements of B such that $\iota(b_\gamma) \to b^*$ in $\mathcal{T}$ and $f(b_\gamma) \to f^{*@}(b^*)$.

(c) If $\{b_\gamma\}$ and $\{a_\gamma\}$ are nets of elements of B, $b^* \in B^*$, $\iota(b_\gamma) \to b^*$ in $\mathcal{T}$ and $\|a_\gamma - b_\gamma\| \to 0$ then $\iota(a_\gamma) \to b^*$ in $\mathcal{T}$.

• Fenchel–Moreau theorem for sesquiconjugates: $f \in \mathcal{PCLSC}(B) \Longrightarrow f = f^{*@} \circ \iota$.
Consequently

$$f(b_\gamma) \to f^{*@}(b^*)$$
$$\Updownarrow$$
$$f^{*@}\big(\iota(b_\gamma)\big) \to f^{*@}(b^*).$$

• $\mathcal{CLB}(B)$ is defined as the set of all convex functions $h\colon B \to \mathbb{R}$ that are bounded above on the bounded subsets of B.

• $\mathcal{T}_{\mathcal{D}}(B^*)$ is defined as the coarsest topology on B^* making all the sesquiconjugates $h^{*@}\colon B^* \to \mathbb{R}$ $\big(h \in \mathcal{CLB}(B)\big)$ continuous.

Theorem on $\mathcal{T}_{\mathcal{D}}(B^*)$

$\mathcal{T}_{\mathcal{D}}(B^*)$ is a compatible topology on B^* and $\widetilde{q}$ is $\mathcal{T}_{\mathcal{D}}(B^*)$–continuous.

$\forall\ b^* \in B^*,\ \Phi_{\iota(A)}(b^*) = \widetilde{q}(b^*) - \inf \widetilde{q}\big(\iota(A) - b^*\big)$. It follows that $\Phi_{\iota(A)} \circ \iota = \Phi_A$. (🐕)

- The Gossez extension of A is the set $A^{\mathcal{G}} = \big\{b^* \in B^*\colon\ \Phi_{\iota(A)}(b^*) \le \widetilde{q}(b^*)\big\}$.

Property (a) of compatible topologies

(a) *If $\mathcal{T}$ is a compatible topology on B^* then $\mathcal{T} \supset w(B^*, B^*)$.*

Main theorem

Let $\mathcal{T}$ be a compatible topology on B^*, $\widetilde{q}$ be $\mathcal{T}$–continuous and A be a maximally q–positive subset of B. Then the conditions (a)–(c) *below are equivalent.*

(a) $\forall\ b^* \in A^{\mathcal{G}}$, $\exists$ *a net* $\{a_\gamma\}$ *of elements of A such that* $\iota(a_\gamma) \to b^*$ *in* $\mathcal{T}$*.*

(b) $\forall\ b^* \in A^{\mathcal{G}}$, $\inf \widetilde{q}\big(\iota(A) - b^*\big) \le 0$.

(c) $\Phi_{\iota(A)} \ge \widetilde{q}$ *on* B^**.*

Proof that (a)$\Longrightarrow$(b)$\Longrightarrow$(c). Let $\{a_\gamma\}$ be a net of elements of A such that $\iota(a_\gamma) \to b^*$ in $\mathcal{T}$. From property (a) above, $\big\lceil \iota(a_\gamma), b^* \big\rceil \to \big\lceil b^*, b^* \big\rceil = 2\widetilde{q}(b^*)$. From the $\mathcal{T}$–continuity of $\widetilde{q}$, $\widetilde{q}\big(\iota(a_\gamma)\big) \to \widetilde{q}(b^*)$. Thus

$$\widetilde{q}\big(\iota(a_\gamma) - b^*\big) = \widetilde{q}\big(\iota(a_\gamma)\big) - \big\lceil \iota(a_\gamma), b^* \big\rceil + \widetilde{q}(b^*) \to \widetilde{q}(b^*) - 2\widetilde{q}(b^*) + \widetilde{q}(b^*) = 0,$$

and so (a) $\Longrightarrow$ (b). If (b) is true then, from (🐕), $b^* \in A^{\mathcal{G}} \Longrightarrow \Phi_{\iota(A)}(b^*) \ge \widetilde{q}(b^*)$. On the other hand, $b^* \in B^* \setminus A^{\mathcal{G}} \Longrightarrow \Phi_{\iota(A)}(b^*) > \widetilde{q}(b^*)$. Thus (b) $\Longrightarrow$ (c).

Theorem on the Gossez extension

(b) *If* $\Phi_{\iota(A)} \ge \widetilde{q}$ *on* D *then* $A^{\mathcal{G}} = \mathcal{P}_{\tilde{q}}({\Phi_A}^{*@}) = \mathcal{P}_{\tilde{q}}({\Phi_A}^{*}) = \mathcal{P}_{\tilde{q}}({\Phi_{\iota(A)}}^{@}) = \mathcal{P}_{\tilde{q}}(\Phi_{\iota(A)})$.

Property (b) of compatible topologies

(b) *If* $f \in \mathcal{PCLSC}(B)$ *and* $b^* \in B^*$ *then* $\exists$ *a net* $\{b_\gamma\}$ *of elements of* B *such that* $\iota(b_\gamma) \to b^*$ *in* $\mathcal{T}$ *and* $f(b_\gamma) \to f^{*@}(b^*)$.

Main theorem $\big($**(c)**$\Longrightarrow$**(a)**$\big)$

Let $\mathcal{T}$ *be a compatible topology on* B^*, $\widetilde{q}$ *be* $\mathcal{T}$*–continuous,* A *be a maximally* q*–positive subset of* B, $\Phi_{\iota(A)} \ge \widetilde{q}$ *on* B^* *and* $b^* \in A^{\mathcal{G}}$. *Then* $\exists$ *a net* $\{a_\gamma\}$ *of elements of* A *such that* $\iota(a_\gamma) \to b^*$ *in* $\mathcal{T}$.

First part of proof. We know that $\Phi_A \in \mathcal{PCLSC}(B)$. From (b) of the theorem on the Gossez extension,

$$ {\Phi_A}^{*}(b^*) = {\Phi_A}^{*@}(b^*) = \widetilde{q}(b^*). $$

From property (b) of compatible topologies, $\exists$ a net $\{b_\gamma\}$ of elements of B such that

$$ \iota(b_\gamma) \to b^* \text{ in } \mathcal{T} \quad \text{and} \quad \Phi_A(b_\gamma) \to {\Phi_A}^{*@}(b^*) = \widetilde{q}(b^*). $$

Since $\widetilde{q}$ is $\mathcal{T}$–continuous, $q(b_\gamma) = \widetilde{q} \circ \iota(b_\gamma) \to \widetilde{q}(b^*)$ and so

$$ (\Phi_A - q)(b_\gamma) = \Phi_A(b_\gamma) - q(b_\gamma) \to \widetilde{q}(b^*) - \widetilde{q}(b^*) = 0. $$

To be continued...

The convex function given by a maximally q–positive set

Let A be a maximally q–positive subset of B. Then… $\Phi_A \ge q$ *on* B …

Half–sandwich property

$$\Phi_{\iota(A)}{}^{@} \ge \Phi_A{}^{*} \ge \Phi_{\iota(A)} \text{ on } D.$$

MAS functions

Let $f \in \mathcal{PC}(B)$. f is an *MAS function* if $f \ge q$ on B and $f^* \ge \widetilde{q}$ on B^*.

MASVZ theorem

Let $\iota(B)$ be $\widetilde{p}$–dense in B^* and $f \in \mathcal{PC}(B)$. Then

f *is an MAS function* $\iff$ f *is a VZ function.*

Main theorem $\big(\textbf{(c)}\Longrightarrow\textbf{(a)}\big)$

Let $\mathcal{T}$ be a compatible topology on B^*, $\widetilde{q}$ be $\mathcal{T}$–continuous, A be a maximally q–positive subset of B, $\Phi_{\iota(A)} \ge \widetilde{q}$ on B^* and $b^* \in A^{\mathcal{G}}$. Then $\exists$ a net $\{a_\gamma\}$ of elements of A such that $\iota(a_\gamma) \to b^*$ in $\mathcal{T}$.

Second part of proof. Now $\Phi_A \ge q$ on B and $\Phi_A{}^* \ge \Phi_{\iota(A)} \ge \widetilde{q}$ on B^*. Thus Φ_A is an MAS function. From the MASVZ theorem, Φ_A is a VZ function.

To be continued…

Theorem on lower semicontinuous VZ functions
Let $f \in \mathcal{PCLSC}(B)$ be a VZ function. Then, $\forall\ c \in B$, $\operatorname{dist}\big(c, \mathcal{P}_q(f)\big) \le \sqrt{2}\sqrt{(f-q)(c)}$.

The convex function given by a maximally q–positive set
Let A be a maximally q–positive subset of B. Then… $\quad \mathcal{P}_q(\Phi_A) = A \quad \ldots$

Property (c) of compatible topologies
(c) *Let $\mathcal{T}$ be a compatible topology on B^*, $\{b_\gamma\}$ and $\{a_\gamma\}$ be nets of elements of B, $b^* \in B^*$,* $\quad \iota(b_\gamma) \to b^*$ *in $\mathcal{T}$ and* $\quad \|a_\gamma - b_\gamma\| \to 0$. *Then* $\quad \iota(a_\gamma) \to b^*$ *in $\mathcal{T}$.*

Main theorem $\big((\mathbf{c}) \Longrightarrow (\mathbf{a})\big)$
Let $\mathcal{T}$ be a compatible topology on B^*, $\widetilde{q}$ be $\mathcal{T}$–continuous, A be a maximally q–positive subset of B, $\quad \Phi_{\iota(A)} \ge \widetilde{q}$ *on B^* and $b^* \in A^{\mathcal{G}}$. Then $\exists$ a net $\{a_\gamma\}$ of elements of A such that* $\quad \iota(a_\gamma) \to b^*$ *in $\mathcal{T}$.*

End of proof. So far, we know that Φ_A is a VZ function,

$$\iota(b_\gamma) \to b^* \text{ in } \mathcal{T} \quad \text{and} \quad (\Phi_A - q)(b_\gamma) \to 0.$$

Since $\Phi_A \in \mathcal{PCLSC}(B)$, for all γ, $\operatorname{dist}\big(b_\gamma, \mathcal{P}_q(\Phi_A)\big) \le \sqrt{2}\sqrt{(\Phi_A - q)(b_\gamma)}$, and so

$$\operatorname{dist}\big(b_\gamma, \mathcal{P}_q(\Phi_A)\big) \to 0.$$

Since $\mathcal{P}_q(\Phi_A) = A$, $\exists\ a_\gamma \in A$ such that $\|a_\gamma - b_\gamma\| \to 0$. From property (c) of $\mathcal{T}$,

$$\iota(a_\gamma) \to b^* \text{ in } \mathcal{T}. \qquad \square$$

p-density criterion for a VZ function

Let $f \in \mathcal{PCLSC}(B)$. Then f is a VZ function $\iff$ $f \ge q$ on B and $\mathcal{P}_q(f)$ is p–dense in B.

The function Ψ_A

Define $\Psi_A\colon B \to]-\infty, \infty]$ by $\Psi_A := \sup_{b^* \in B^*} \big[\langle \cdot, b^* \rangle - \Phi_{\iota(A)}(b^*)\big]$.
(a) $\Psi_A \le q$ on A. So $\Psi_A \in \mathcal{PCLSC}(B)$. Further, $\Phi_{\iota(A)} \ge {\Psi_A}^*$ on B^*.
(b) Let $f \in \mathcal{PCLSC}(B)$, $f \ge q$ on B and $A = \mathcal{P}_q(f)$. Then $\Psi_A \ge f$ on B^* and $\mathcal{P}_q(\Psi_A) = \mathcal{P}_q(f)$.
(c) Let $f \in \mathcal{PCLSC}(B)$, f be a VZ function and $A = \mathcal{P}_q(f)$. Then Ψ_A is a VZ function and $\Phi_{\iota(A)} \ge {\Psi_A}^*$ on B^*.

Proof. (a) Let $b^* \in B^*$. Note that $\Phi_{\iota(A)}(b^*) = \sup_{\iota(A)} \big[\lceil b^*, \cdot \rceil - \widetilde{q}\big] = \sup_A \big[\langle \cdot, b^* \rangle - q\big]$. Consequently $\langle \cdot, b^* \rangle - \Phi_{\iota(A)}(b^*) \le q$ on A. Take the supremum over b^*.
Now let $b^* \in B^*$ and $b \in B$. Then $\Psi_A(b) \ge \langle b, b^* \rangle - \Phi_{\iota(A)}(b^*)$. Consequently $\Phi_{\iota(A)}(b^*) \ge \langle b, b^* \rangle - \Psi_A(b)$. Take the supremum over b.
(b) Let $b^* \in B^*$. From (a), $\Phi_{\iota(A)}(b^*) = \sup_{\iota(A)} \big[\lceil b^*, \cdot \rceil - \widetilde{q}\big] = \sup_A \big[\langle \cdot, b^* \rangle - q\big] = \sup_A \big[\langle \cdot, b^* \rangle - f\big] \le \sup_B \big[\langle \cdot, b^* \rangle - f\big] = f^*(b^*)$. Thus the Fenchel–Moreau theorem gives $\Psi_A = \sup_{b^* \in B^*} \big[\langle \cdot, b^* \rangle - \Phi_{\iota(A)}(b^*)\big] \ge \sup_{b^* \in B^*} \big[\langle \cdot, b^* \rangle - f^*(b^*)\big] = f$. So $\Psi_A \ge f \ge q$ on B^*, from which $\mathcal{P}_q(\Psi_A) \subset \mathcal{P}_q(f) = A$. Combining with (a), $\Psi_A = q$ on A, consequently $A \subset \mathcal{P}_q(\Psi_A)$.
(c) This is immediate from the p–density criterion, (b) and (a). □

Dictionary for Example (e)

- $B = E \times E^*$,
$\lfloor (x, x^*), (y, y^*) \rfloor := \langle x, y^* \rangle + \langle y, x^* \rangle$ and $q(x, x^*) = \langle x, x^* \rangle$.
$\|(x, x^*)\| := \sqrt{\|x\|^2 + \|x^*\|^2}$.

- $B^* = E^* \times E^{**}$ under the pairing $\big\langle (x, x^*), (y^*, y^{**}) \big\rangle := \langle x, y^* \rangle + \langle x^*, y^{**} \rangle$
$(B^*)^* = E^{**} \times E^{***}$ under the pairing $\big\langle (y^*, y^{**}), (w^{**}, w^{***}) \big\rangle := \langle y^*, w^{**} \rangle + \langle y^{**}, w^{***} \rangle$.
$\big(E^* \times E^{**}, \lceil \cdot, \cdot \rceil, \| \cdot \|\big)$ is a Banach SSD dual of $\big(E \times E^*, \lfloor \cdot, \cdot \rfloor, \| \cdot \|\big)$.

- $D = B^*$,
$\lceil (x^*, x^{**}), (y^*, y^{**}) \rceil := \langle y^*, x^{**} \rangle + \langle x^*, y^{**} \rangle$ and $\widetilde{q}(y^*, y^{**}) = \langle y^*, y^{**} \rangle$.
$\|(y^*, y^{**})\| := \sqrt{\|y^*\|^2 + \|y^{**}\|^2}$.

- $\iota(x, x^*) = (x^*, \widehat{x})$ and $\widetilde{\iota}(y^*, y^{**}) = (y^{**}, \widehat{y^*})$.

- $\iota(E \times E^*)$ is $\widetilde{p}$–dense in $E^* \times E^{**}$.

- If $(a, a^*) \in B$ and $(y^*, y^{**}) \in B^*$ then
$\widetilde{q}\big(\iota(a, a^*) - (y^*, y^{**})\big) = \widetilde{q}\big((a^*, \widehat{a}) - (y^*, y^{**})\big) = \widetilde{q}(a^* - y^*, \widehat{a} - y^{**}) = \langle a^* - y^*, \widehat{a} - y^{**} \rangle$.

- Consequently, if A is a nonempty q–positive subset of $B = E \times E^*$ then

$$\inf \widetilde{q}\big(\iota(A) - (y^*, y^{**})\big) = \inf_{(a,a^*) \in A} \langle a^* - y^*, \widehat{a} - y^{**} \rangle.$$

— Banach SSD spaces and classes of monotone sets —

- Consequently, if A is a nonempty q–positive subset of $B = E \times E^*$ then
$$\inf \widetilde{q}\big(\iota(A) - (y^*, y^{**})\big) = \inf_{(a,a^*)\in A} \langle a^* - y^*, \widehat{a} - y^{**}\rangle.$$

- $\iota(A)$ is a $\widetilde{q}$–positive subset of $B^* = E^* \times E^{**}$ and,
$$\forall\ (y^*, y^{**}) \in B^*,\ \ \Phi_{\iota(A)}(y^*, y^{**}) = \widetilde{q}(y^*, y^{**}) - \inf \widetilde{q}\big(\iota(A) - (y^*, y^{**})\big)\ldots.$$

- Let $A \subset B = E \times E^*$. A is *maximally monotone of type (NI)* if A is maximally monotone and, $\ \forall\ (y^*, y^{**}) \in B^*,\quad \inf_{(a,a^*)\in A} \langle a^* - y^*, \widehat{a} - y^{**}\rangle \le 0.$

SSD characterization of type (NI)

Let $A \subset B = E \times E^*$. A is maximally monotone of type (NI) $\iff$ A is maximally monotone and $\Phi_{\iota(A)} \ge \widetilde{q}$ on B^*.

- We had: A maximally q–positive $\Longrightarrow \Phi_A \ge q$ on B and $\mathcal{P}_q(\Phi_A) = A$.

Half–sandwich property

$$\Phi_{\iota(A)}{}^{@} \ge \Phi_A{}^* \ge \Phi_{\iota(A)} \text{ on } B^*.$$

SSD characterization of type (NI)

Let $A \subset B = E \times E^*$. A is maximally monotone of type (NI) $\iff$ A is maximally monotone and

$$\Phi_{\iota(A)} \ge \widetilde{q} \text{ on } B^*.$$

- Let $A \subset B = E \times E^*$. A is *strongly representable* if $\exists\ f \in \mathcal{PCLSC}(B)$ such that f is an MAS function and $A = \mathcal{P}_q(f)$.

A result of Marques Alves and Svaiter

Let $A \subset B = E \times E^*$. Then

A is maximally monotone of type (NI) $\iff$ A is strongly representable.

Proof. ($\Longrightarrow$) We have $\Phi_A \ge q$ on B and $\Phi_A{}^* \ge \Phi_{\iota(A)} \ge \widetilde{q}$ on B^* and so Φ_A is an MAS function. Since $\mathcal{P}_q(\Phi_A) = A$, A is strongly representable.

• Let $A \subset B = E \times E^*$. A is *strongly representable* if $\exists\, f \in \mathcal{PCLSC}(B)$ such that f is an MAS function and $A = \mathcal{P}_q(f)$.

MASVZ theorem

Let $f \in \mathcal{PC}(E \times E^*)$. Then f is an MAS function $\iff$ f is a VZ function.

SSD characterization of type (NI)

Let $A \subset B = E \times E^*$. A is maximally monotone of type (NI) $\iff$ A is maximally monotone and $\Phi_{\iota(A)} \ge \widetilde{q}$ on B^*.

The function Ψ_A

(c) Let $f \in \mathcal{PCLSC}(B)$, f be a VZ function and $A = \mathcal{P}_q(f)$. Then Ψ_A is a VZ function and $\Phi_{\iota(A)} \ge {\Psi_A}^*$ on B^*.

A result of Marques Alves and Svaiter

Let $A \subset B = E \times E^*$. Then

A is maximally monotone of type (NI) $\iff$ A is strongly representable.

Proof. ($\Longleftarrow$) Suppose that $f \in \mathcal{PCLSC}(B)$, f is an MAS function and $A = \mathcal{P}_q(f)$. The MASVZ theorem implies that f is a VZ function. From the theorem on lower semicontinuous VZ functions and (c) above, A is maximally monotone, Ψ_A is a VZ function and $\Phi_{\iota(A)} \ge {\Psi_A}^*$ on B^*. The MASVZ theorem now implies that Ψ_A is an MAS function, and so ${\Psi_A}^* \ge \widetilde{q}$ on B^*. Thus $\Phi_{\iota(A)} \ge \widetilde{q}$ on B^*, from which A is of type (NI). □

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- The Gossez extension of A is the set $A^{\mathcal{G}} = \big\{b^* \in B^*\colon\ \Phi_{\iota(A)}(b^*) \le \widetilde{q}(b^*)\big\}$.
- $\iota(A)$ is a $\widetilde{q}$–positive subset of $B^* = E^* \times E^{**}$ and,

$$\forall\ (y^*, y^{**}) \in B^*,\ \ \Phi_{\iota(A)}(y^*, y^{**}) = \widetilde{q}(y^*, y^{**}) - \inf \widetilde{q}\big(\iota(A) - (y^*, y^{**})\big)\ldots.$$

- Consequently, if A is a nonempty q–positive subset of $B = E \times E^*$ then

$$\inf \widetilde{q}\big(\iota(A) - (y^*, y^{**})\big) = \inf_{(a,a^*)\in A}\langle a^* - y^*, \widehat{a} - y^{**}\rangle.$$

The Gossez extension in Example (e)

$$(y^*, y^{**}) \in A^{\mathcal{G}} \iff \inf_{(a,a^*)\in A}\langle a^* - y^*, \widehat{a} - y^{**}\rangle \ge 0.$$

- In the situation of Example (e), $\overline{A}$ is normally written instead of $A^{\mathcal{G}}$.

SSD characterization of type (NI)

Let $A \subset B = E \times E^*$. A is maximally monotone of type (NI) $\iff$ A is maximally monotone and

$$\Phi_{\iota(A)} \geq \widetilde{q} \text{ on } B^*.$$

Main theorem in Example (e)

Let $\mathcal{T}$ be a compatible topology on $B^* = E^* \times E^{**}$, $\widetilde{q}$ be $\mathcal{T}$–continuous and A be a maximally monotone subset of $B = E \times E^*$. Then the conditions (a)–(c) below are equivalent.

(a) $\forall\ b^* \in A^{\mathcal{G}}$, $\exists$ *a net* $\{a_\gamma\}$ *of elements of* A *such that* $\iota(a_\gamma) \to b^*$ *in* $\mathcal{T}$.

(b) $\forall\ b^* \in A^{\mathcal{G}}$, $\inf \widetilde{q}\big(\iota(A) - b^*\big) \leq 0$.

(c) $\Phi_{\iota(A)} \geq \widetilde{q}$ *on* B^*.

Theorem on $\mathcal{T}_{\mathcal{D}}(B^*)$

$\mathcal{T}_{\mathcal{D}}(B^*)$ is a compatible topology on B^* and $\widetilde{q}$ is $\mathcal{T}_{\mathcal{D}}(B^*)$–continuous.

Corollary

Let A be a maximally monotone subset of $B = E \times E^*$ of type (NI). Then, $\forall\ b^* \in A^{\mathcal{G}}$, $\exists$ a net $\{a_\gamma\}$ of elements of A such that $\iota(a_\gamma) \to b^*$ in $\mathcal{T}_{\mathcal{D}}(B^*)$.

The Gossez extension in Example (e)

$$(y^*, y^{**}) \in A^{\mathcal{G}} \iff \inf_{(a,a^*)\in A}\langle a^* - y^*, \widehat{a} - y^{**}\rangle \geq 0.$$

Corollary

Let A be a maximally monotone subset of $B = E \times E^*$ of type (NI). Then, $\forall\ b^* \in A^{\mathcal{G}}$, $\exists$ a net $\{a_\gamma\}$ of elements of A such that $\iota(a_\gamma) \to b^*$ in $\mathcal{T}_{\mathcal{D}}(B^*)$.

- Various classes of maximally monotone sets have been discussed since Gossez introduced *type (D)* and *dense type*.
- In chronological order, we mention here *type (NI)*, *type (WD)*, and *type (ED)*.
- The easy implications are that, for maximally monotone sets,

$$\text{type (ED)} \Longrightarrow \text{dense type} \Longrightarrow \text{type (D)} \Longrightarrow \text{type (WD)} \Longrightarrow \text{type (NI)}.$$

- Marques Alves and Svaiter proved recently that type (NI) $\Longrightarrow$ type (D).
- The Corollary above gives the stronger result that type (NI) $\Longrightarrow$ type (ED), so all of the above five classes are identical.
- It is already known that maximally monotone sets of type (ED) are of *type (FP)* (= *locally maximally monotone*), *type (FPV)* (= *maximally monotone locally*) and *strongly maximally monotone*, and that they possess strong Brøndsted–Rockafellar properties and properties related to the surjectivity of approximate resolvents.
- As we have already observed, a set is maximally monotone of type (NI) $\iff$ it is strongly representable. Thus we are led to new results about strongly representable sets, as well as some new proofs of known results.