An inverse Newton transform

Adi Ben-Israel (Rutgers University)

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Interdisciplinary Workshop on Fixed-Point Algorithms for
Inverse Problems in Science and Engineering
Banff International Research Station, November 2, 2009

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The **Newton transform** $\mathbf{N}f$ of a differentiable $f\colon \mathbb{R} \to \mathbb{R}$ is

$$(\mathbf{N}f)(x) := x - \frac{f(x)}{f'(x)}.$$

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Newton:=proc(f,x); x-f/diff(f,x); end:
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Example: $\left(\mathbf{N}(x^{2/3}-1)^{3/2}\right)(x) = x^{1/3}$

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simplify(Newton((x^(2/3)-1)^(3/2),x));
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$$x^{1/3}$$

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Example: $(\mathbf{N}\exp\{-x\})(x) = x + 1$

```
Newton(exp(-x),x);
```

$$x + 1$$

$u = \mathbf{N} f$

(a) If f is twice differentiable, then

$$u'(x) = \frac{f(x)f''(x)}{f'(x)^2}.$$

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ζ is a **zero** of **order** m of f if

$$f(x) = (x-\zeta)^m\, g(x), \quad m > 0,\ g(\zeta) \neq 0$$

(b) If ζ is a zero of f of order m, then

$$u'(x) = \frac{m(m-1)g(x)^2 + 2(x-\zeta)g'(x) + (x-\zeta)^2 g''(x)}{m^2 g(x)^2 + 2(x-\zeta)mg(x)g'(x) + (x-\zeta)^2 g'(x)^2} \longrightarrow \frac{m-1}{m},$$

as $x \to \zeta$, provided $\lim_{x\to\zeta}(x-\zeta)g'(x) = \lim_{x\to\zeta}(x-\zeta)^2 g''(x) = 0$.

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(c) If ζ is a zero of f of order $m < 1$, then f is not differentiable at ζ, but u may be defined and differentiable at ζ, with $u'(\zeta) = \dfrac{m-1}{m}$.

The inverse Newton transform

The **inverse Newton transform** $\mathbf{N}^{-1}u$ of a function $u : \mathbb{R} \to \mathbb{R}$, is a (differentiable) function f such that $\mathbf{N}f = u$, or,

$$x - \frac{f(x)}{f'(x)} = u(x).$$

$\mathbf{N}^{-1}u$ is defined up to a constant, since $\mathbf{N}f = \mathbf{N}(c\,f)$ for all $c \neq 0$

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Example: $\left(\mathbf{N}^{-1}x^{1/3}\right)(x) = (x^{2/3} - 1)^{3/2}$

Example: $\left(\mathbf{N}^{-1}(x+1)\right)(x) = \exp\{-x\}$

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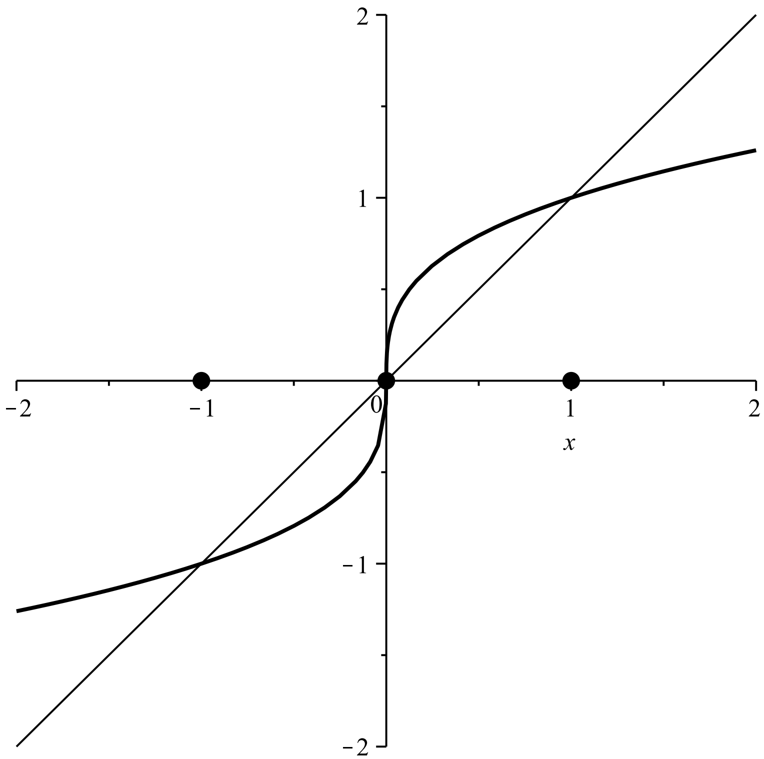
Questions:

$$\text{Fixed points}\,\{u\} \overset{?}{=} \text{Zeros}\,\{f\} \cup \text{Singularities}\,\{f'\}$$

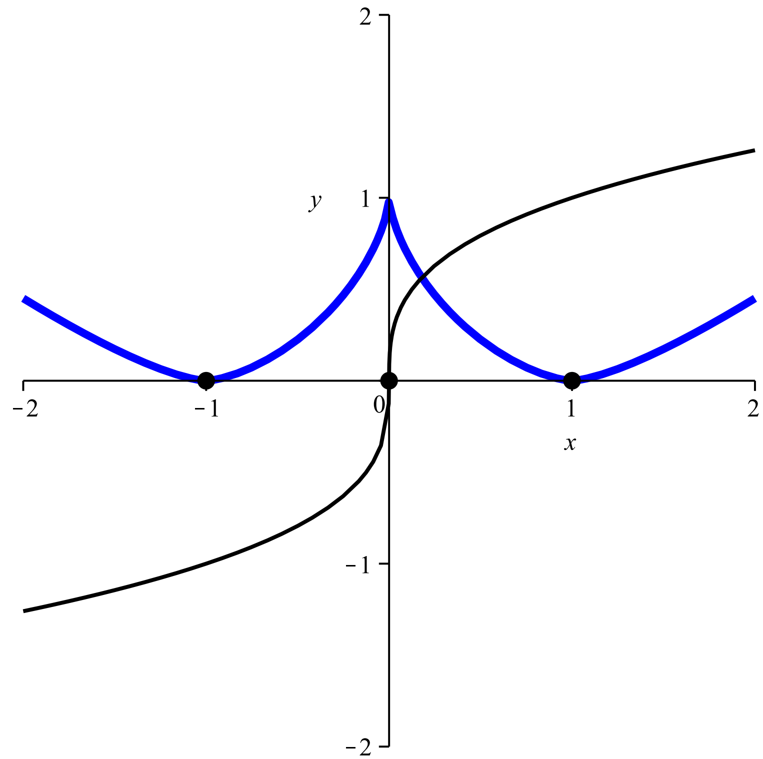
$$\text{Attracting fixed points}\,\{u\} \overset{?}{=} \text{Zeros}\,\{f\}$$

$$\text{Quadratic convergence of } u = \mathbf{N}f\,?$$

$\mathbf{u(x) = x^{1/3},\ f(x) = \left(N^{-1}u\right)(x) = \left(x^{2/3} - 1\right)^{3/2}}$



Fixed points of $u(x)$ at $0, \pm 1$



Corresponding points of $f(x)$

Fixed points $\{u\} \stackrel{?}{=}$ **Zeros** $\{f\}$

$$u(x) = x - \frac{f(x)}{f'(x)}$$

Theorem. Let f be differentiable at ζ, and in (a)–(c), $f'(\zeta) \neq 0$.

(a) ζ is a zero of f if, and only if, it is a fixed point of u.

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(a) ζ is a zero of f if, and only if, it is a fixed point of u.

(b) If ζ is a zero of f, f and u are twice differentiable at ζ, then ζ is a superattracting fixed point of u, and convergence is (at least) quadratic.

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(c) If ζ is a zero of f of order $m > \frac{1}{2}$, and u is continuously differentiable at ζ, then ζ is an attracting fixed point of u.

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(c) If ζ is a zero of f of order $m > \frac{1}{2}$, and u is continuously differentiable at ζ, then ζ is an attracting fixed point of u.

(d) Let ζ have a neighborhood where u and f are continuously differentiable, and $f'(x) \neq 0$ except possibly at $x = \zeta$. If ζ is an attracting fixed point of u then it is a zero of f.

An integral form of $\mathbf{N}^{-1}$

Theorem. Let u be a function: $\mathbb{R} \to \mathbb{R}$, D a region where

$$\frac{1}{x - u(x)}$$

is integrable. Then in D,

$$(\mathbf{N}^{-1}u)(x) = C \cdot \exp\left\{ \int \frac{dx}{x - u(x)} \right\}, \ C \neq 0.$$

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Moreover, if $C > 0$ then $\mathbf{N}^{-1}u$ is

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Moreover, if $C > 0$ then $\mathbf{N}^{-1}u$ is

(a) increasing if $x > u(x)$,

(b) decreasing if $x < u(x)$,

(c) convex if u is differentiable and increasing, or

(d) concave if u is differentiable and decreasing.

$$(\mathbf{N^{-1}u})(\mathbf{x}) = \mathbf{C} \cdot \exp\left\{\int \frac{\mathbf{dx}}{\mathbf{x} - \mathbf{u}(\mathbf{x})}\right\}$$

Assuming $x \neq u(x)$,

$$u(x) = x - \frac{f(x)}{f'(x)} \quad \Longrightarrow \quad \frac{f'(x)}{f(x)} = \frac{1}{x - u(x)}$$

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Assuming $x \neq u(x)$,

$$u(x) = x - \frac{f(x)}{f'(x)} \quad \Longrightarrow \quad \frac{f'(x)}{f(x)} = \frac{1}{x - u(x)}$$

$$\therefore \ \ln f(x) = \int \frac{dx}{x - u(x)} + C$$

$$\therefore \ f(x) = C \exp\left\{\int \frac{dx}{x - u(x)}\right\}$$

without loss of generality, $C = 1$.

$$\mathbf{f}(\mathbf{x}) = \exp\left\{\int \frac{\mathbf{dx}}{\mathbf{x} - \mathbf{u}(\mathbf{x})}\right\}$$

$$\therefore \; f'(x) = \frac{1}{x - u(x)} \exp\left\{\int \frac{dx}{x - u(x)}\right\}$$

$$\therefore \; f''(x) = \frac{u'(x)}{(x - u(x))^2} \exp\left\{\int \frac{dx}{x - u(x)}\right\}$$

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$$x > u(x) \quad \Longrightarrow \quad f'(x) > 0$$

$$u'(x) > 0 \quad \Longrightarrow \quad f''(x) > 0$$

$$(\mathbf{N^{-1}u})(\mathbf{x}) = \exp\left\{\int \frac{\mathbf{dx}}{\mathbf{x} - \mathbf{u}(\mathbf{x})}\right\}$$

```
InverseNewton:=proc(u,x);
simplify(exp(int(1/(x-u),x)));end:
```

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```

Examples:

```
InverseNewton(Newton(f(x),x),x);
```

$$f(x)$$

```
Newton(InverseNewton(u(x),x),x);
```

$$u(x)$$

```
InverseNewton(x^2,x);
```

$$\frac{x}{x-1}$$

$$\mathbf{u(x) = x - \frac{f(x)}{f'(x) - a(x)f(x)}}, \quad \mathbf{N^{-1}u} =?$$

$$\mathbf{u}(\mathbf{x}) = \mathbf{x} - \frac{\mathbf{f}(\mathbf{x})}{\mathbf{f}'(\mathbf{x}) - \mathbf{a}(\mathbf{x})\mathbf{f}(\mathbf{x})}, \quad \mathbf{N}^{-1}\mathbf{u} =?$$

```
InverseNewton(x-f(x)/(diff(f(x),x)-a(x)*f(x)),x);
```

$$f(x) \exp\left\{-\int a(x)dx\right\}$$

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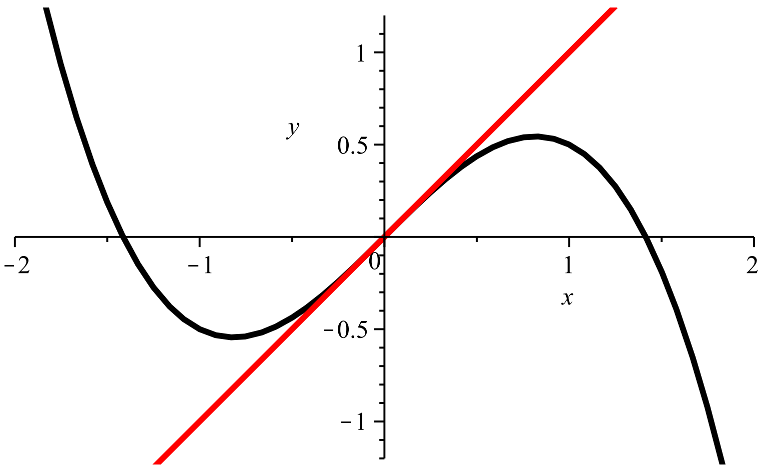
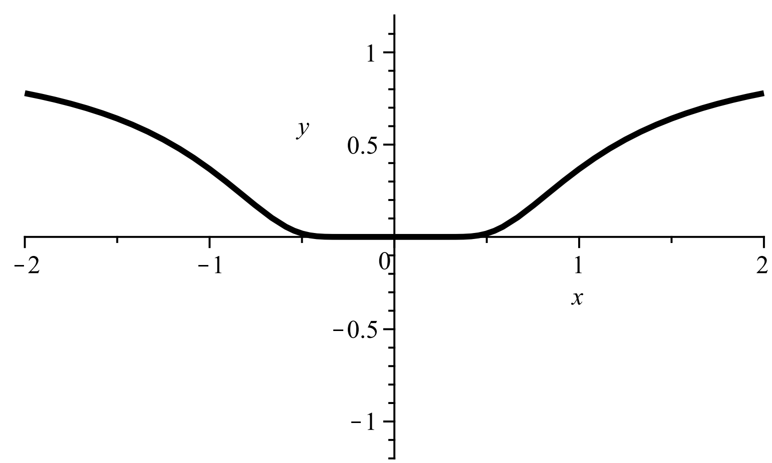
For the **Halley method**

$$H(x) := x - \frac{f(x)}{f'(x) - \dfrac{f''(x)f(x)}{2f'(x)}}$$

$$\left(\mathbf{N}^{-1}H\right)(x) = \frac{f(x)}{\sqrt{f'x}}$$

$$\mathbf{u}(\mathbf{x}) = \mathbf{x} - \frac{\mathbf{1}}{\mathbf{2}}\mathbf{x}^{\mathbf{3}}, \quad \mathbf{f}(\mathbf{x}) = (\mathbf{N}^{-\mathbf{1}}\mathbf{u})(\mathbf{x}) = \exp\left\{-\frac{\mathbf{1}}{\mathbf{x}^{\mathbf{2}}}\right\}$$





u has attracting fixed point at 0 $\qquad f^{(k)}(0) = 0, \ \forall\ k$

For $\mathbf{a} \neq \mathbf{0}$, $\quad \mathbf{N}^{-1}(\mathbf{au(x)} + \mathbf{b}) = ?$

Corollary. If $a \neq 0$ and b are reals, and

$$f := \mathbf{N}^{-1}(u(a\,x + b)),$$

$$\text{then} \quad (\mathbf{N}^{-1}(a\,u + b))(x) = f\left(\frac{x-b}{a}\right).$$

Proof.

$$\int \frac{dx}{x - (a\,u(x) + b)} = \int \frac{dx}{a\left(\left(\frac{x-b}{a}\right) - u(x)\right)}, \text{ etc.}$$

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Equivalently, if

$$\phi(x) = ax + b, \ a \neq 0$$

then

$$\mathbf{N}^{-1}(\phi u) = \phi^{-1}\mathbf{N}^{-1}(u\phi),$$

Reverse iteration

If u is monotone then $x_+ := u(x)$ is reversed by $x := u^{-1}(x_+)$

Corollary. Let u be monotone and differentiable, and let,

$$f(x) := \exp\left\{\int \frac{u'(x)\,dx}{u(x) - x}\right\}$$

Then

$$(\mathbf{N}^{-1}(u^{-1}))(x) = f(u^{-1}(x)).$$

Proof. The inverse Newton transform of u^{-1} is

$$(\mathbf{N}^{-1}(u^{-1}))(x_+) = \exp\left\{\int \frac{dx_+}{x_+ - u^{-1}(x_+)}\right\}$$

changing variables to $x = u^{-1}(x_+)$ we get

$$(\mathbf{N}^{-1}(u^{-1}))(u(x)) = \exp\left\{\int \frac{u'(x)\,dx}{u(x) - x}\right\}$$

proving the corollary. □

$$(\mathbf{N^{-1}}(\mathbf{u^{-1}}))(\mathbf{x}) = \mathbf{f}(\mathbf{u^{-1}}(\mathbf{x}))$$

$$\mathbf{f}(\mathbf{x}) := \exp\left\{\int \frac{\mathbf{u}'(\mathbf{x})\,\mathbf{dx}}{\mathbf{u}(\mathbf{x}) - \mathbf{x}}\right\}$$

```
ReverseNewton:=proc(u,x);
simplify(exp(int(diff(u,x)/(u-x),x)));end:
```

Example. $u(x) = x^3$, $u^{-1}(x) = x^{1/3}$.

```
subs(x=x^(1/3),ReverseNewton(x^3,x));
```

$$(x^{1/3} - 1)^{3/2}\,(x^{1/3} + 1)^{3/2}$$

again

$$\left(\mathbf{N}^{-1}(x^{1/3})\right)(x) = (x^{2/3} - 1)^{3/2}$$

The logistic iteration

$$\mathbf{u}(\mathbf{x}) = \mu\,\mathbf{x}\,(\mathbf{1}-\mathbf{x}),\ \mathbf{0} \le \mathbf{x} \le \mathbf{1},\ \mathbf{1} \le \mu \le \mathbf{4}$$

```
expand(InverseNewton(μ*x*(1-x),x));
```

$$\frac{(1-\mu+\mu\,x)^{(-1+\mu)^{-1}}}{x^{(-1+\mu)^{-1}}}$$

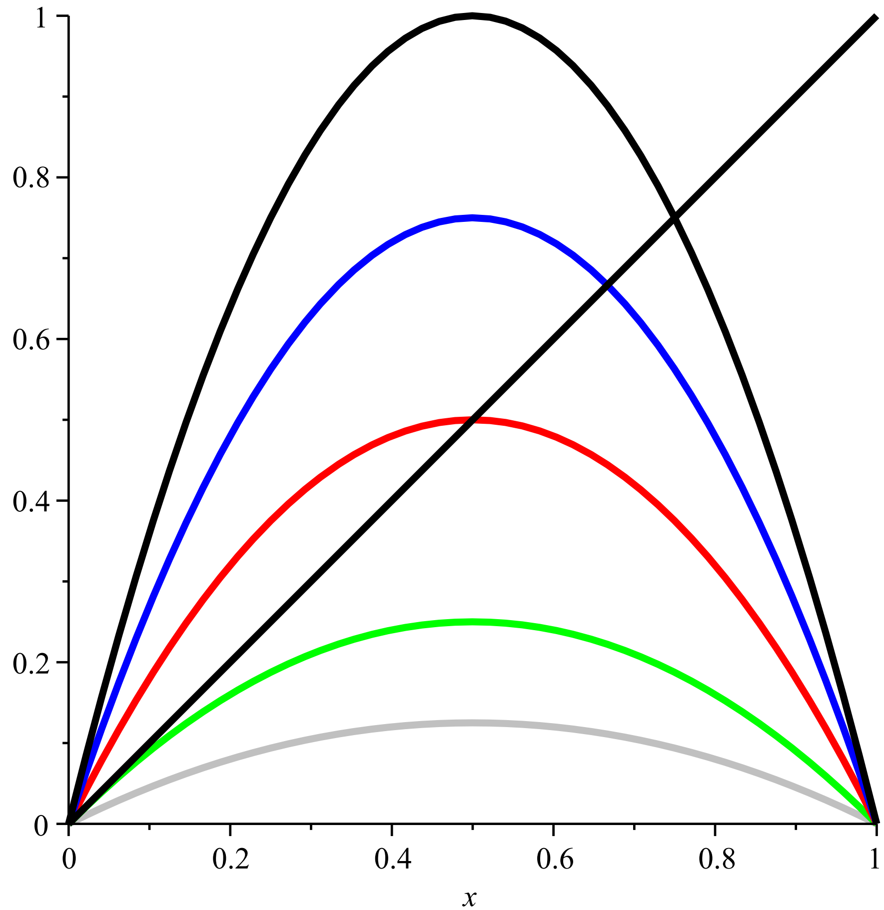
(a) $\therefore\ f(x) = \left(\mathbf{N}^{-1}u\right)(x) = \left(\dfrac{x-\frac{\mu-1}{\mu}}{x}\right)^{\frac{1}{\mu-1}}$

(b) Fixed points $\{u\} = \left\{0,\ \frac{\mu-1}{\mu}\right\}$

(c) The fixed point $\frac{\mu-1}{\mu}$ is attracting for $1 \le \mu < 3$

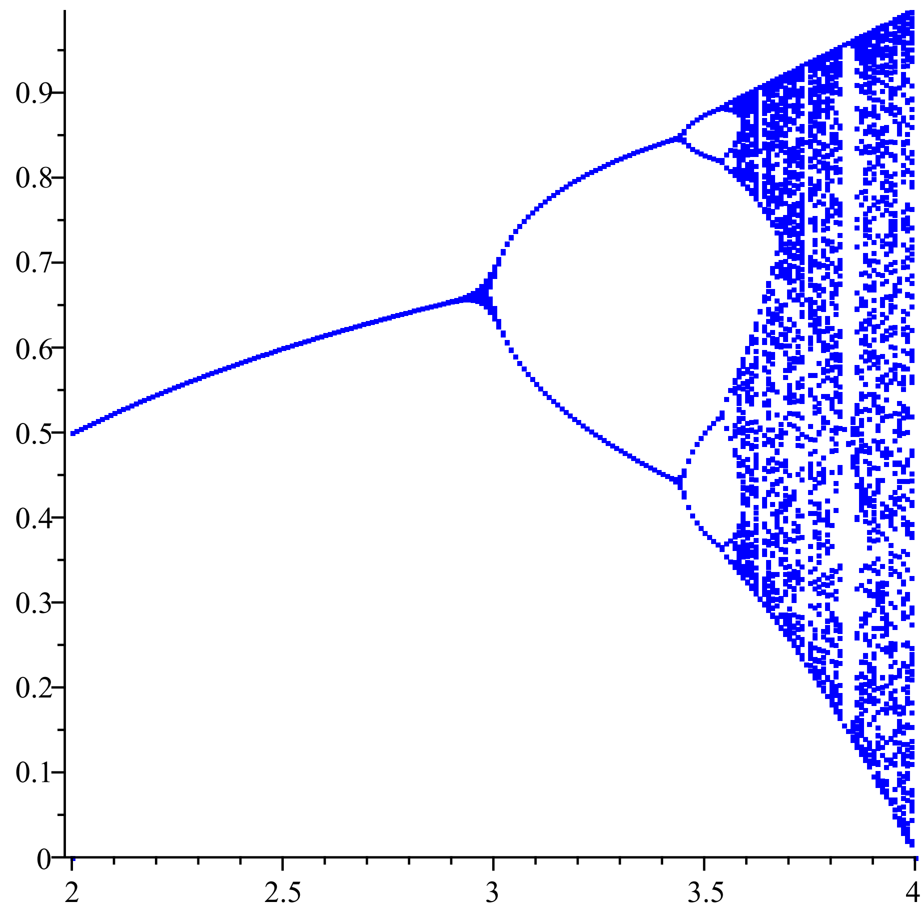
(d) $f(x)$ is convex [concave] for $x < \frac{1}{2}$ $[x > \frac{1}{2}]$

$$\mathbf{u}(\mathbf{x}) = \mu\,\mathbf{x}\,(\mathbf{1} - \mathbf{x})$$



The logistic function with $\mu = 0.5, 1, 2, 3, 4$

Chaos

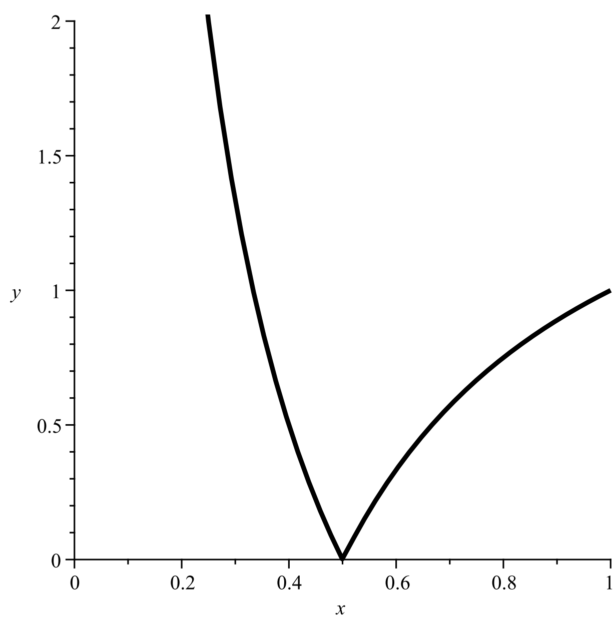


100 iterates of the logistic function for selected values of $2 \leq \mu \leq 4$

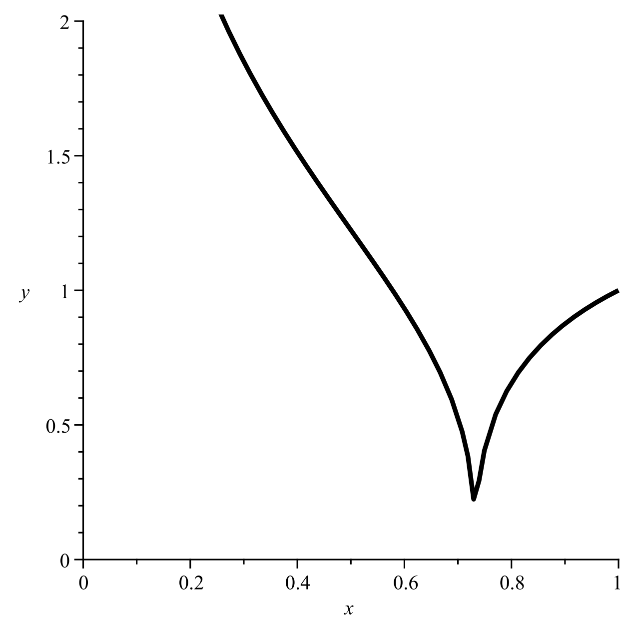
Chaos explained

The inverse Newton transform of $u(x) = \mu\, x\,(1-x)$

$$\left(\mathbf{N}^{-1}u\right)(x) = \left(\frac{x - \frac{\mu-1}{\mu}}{x}\right)^{\frac{1}{\mu-1}}$$

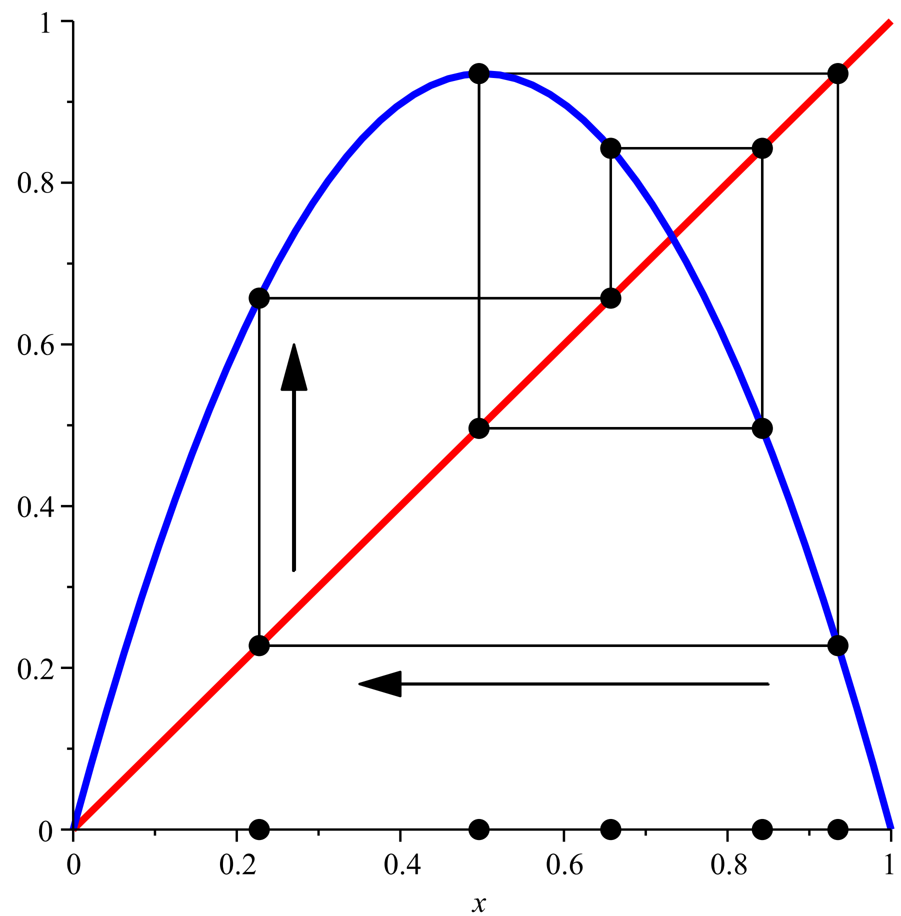


$\mu = 2.0$



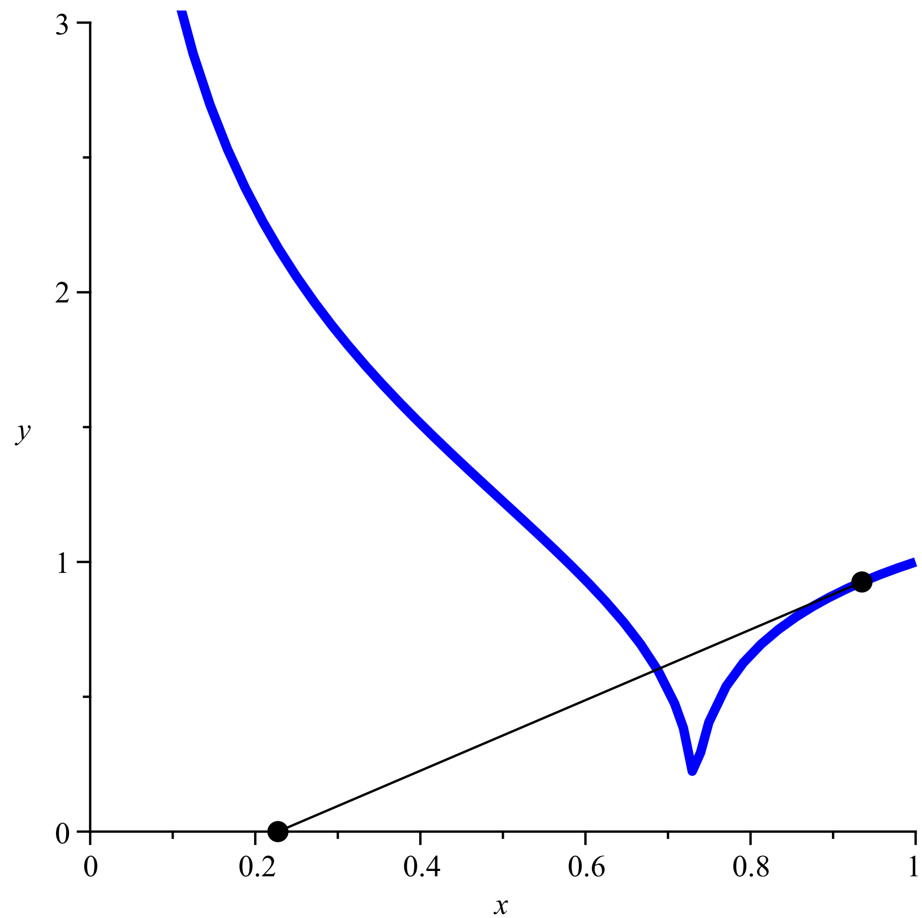
$\mu = 3.74$

5–cycle for $\mu = 3.74$

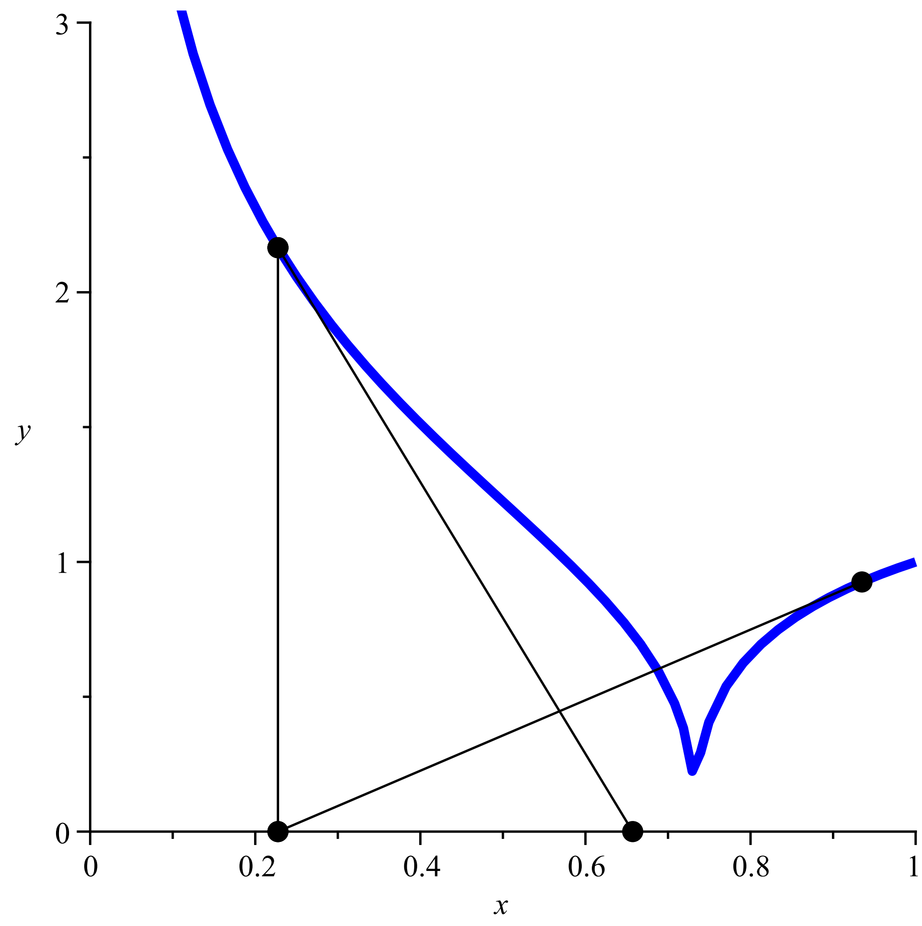


Starting at and returning to .9349453234

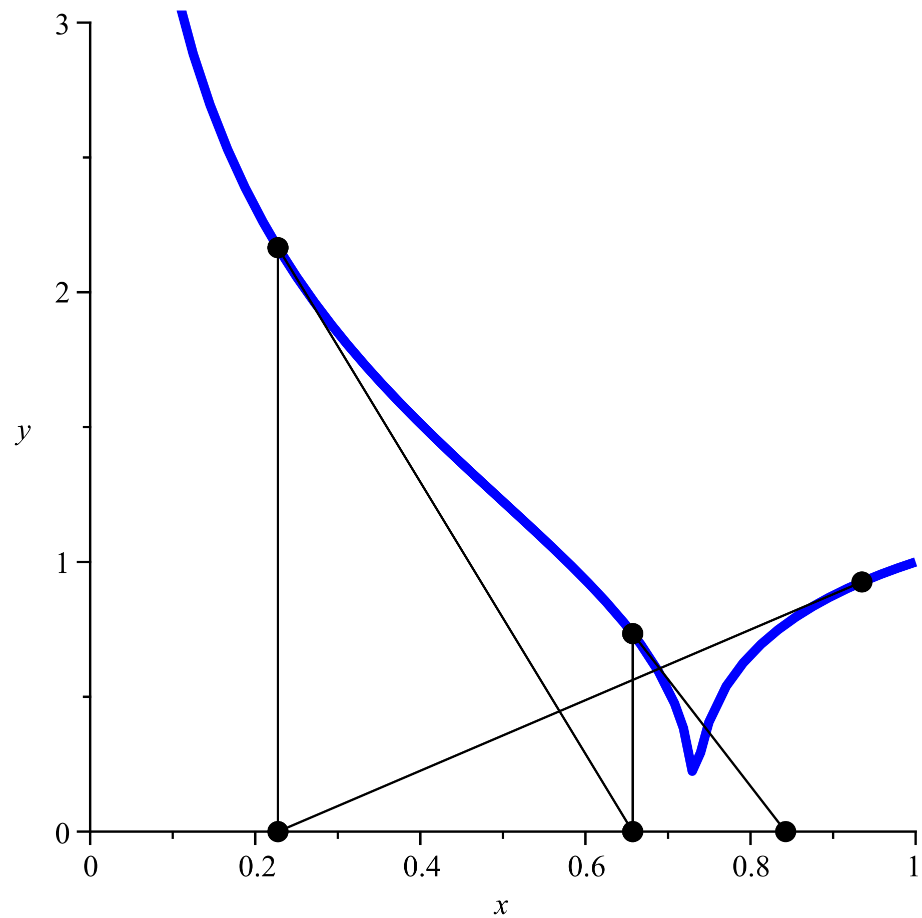
Ping pong – 1



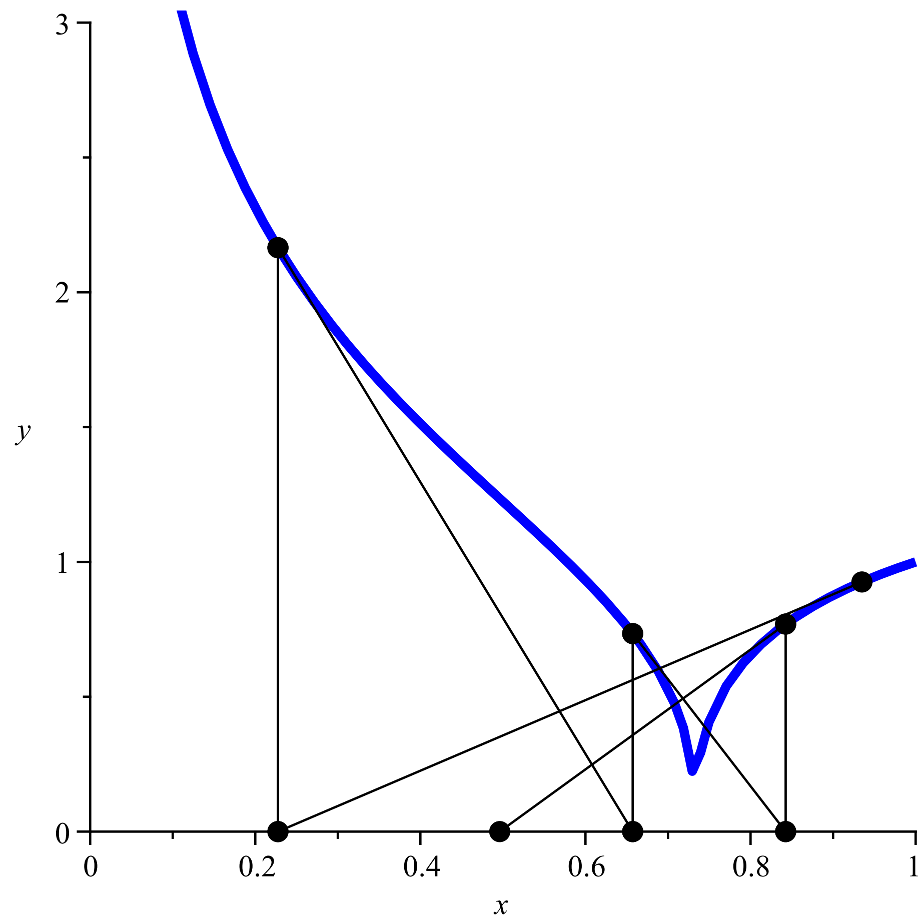
Ping pong – 2



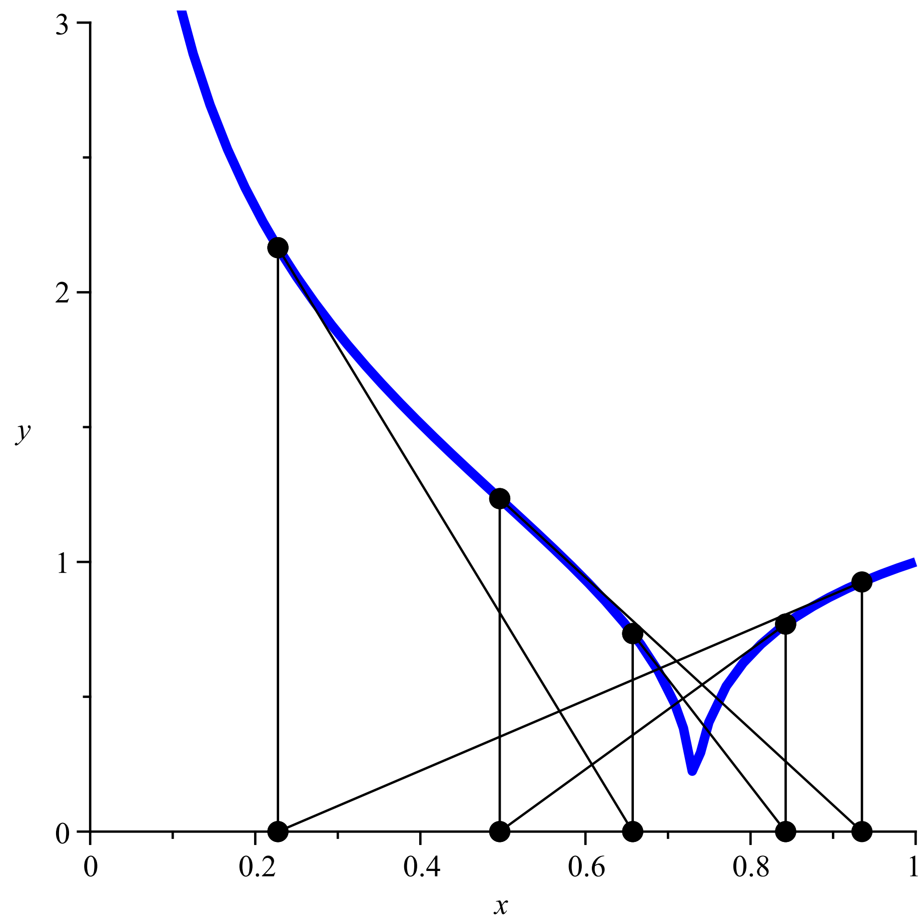
Ping pong – 3



Ping pong – 4



Ping pong – 5



Complex Newton iteration: Geometry

$$\mathbf{z}_+ := \mathbf{z} - \frac{\mathbf{f}(\mathbf{z})}{\mathbf{f}'(\mathbf{z})},\ \ \mathbf{f}'(\mathbf{z}) \neq \mathbf{0}$$

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(A) Let

$$z = x + i\,y \longleftrightarrow (x, y)$$

be the natural correspondence between $\mathbb{C}$ and $\mathbb{R}^2$, and let

$$F(x, y) := f(z) \text{ for } z \longleftrightarrow (x, y).$$

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(B) Let $T \subset \mathbb{R}^3$ be the plane tangent to the graph of $|F|$ at the point $(x, y, |F(x, y)|)$, and let L be the line of intersection of T and the (x, y)–plane (L is nonempty by the assumption that $f'(z) \neq 0$.)

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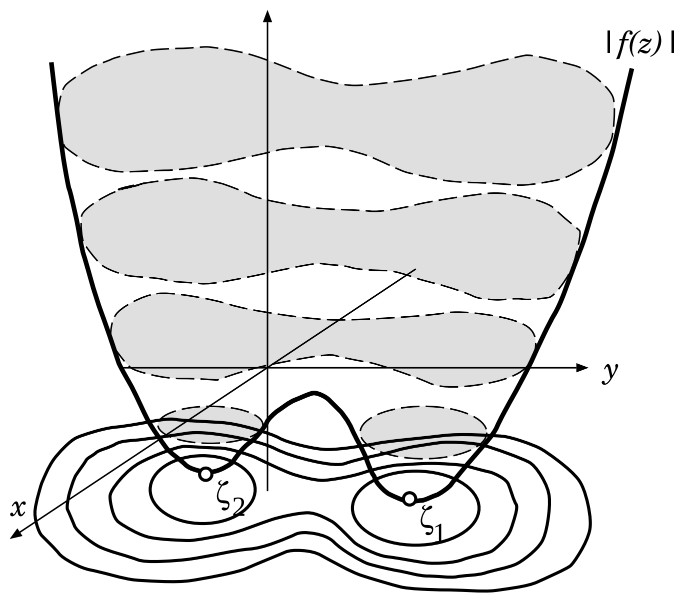
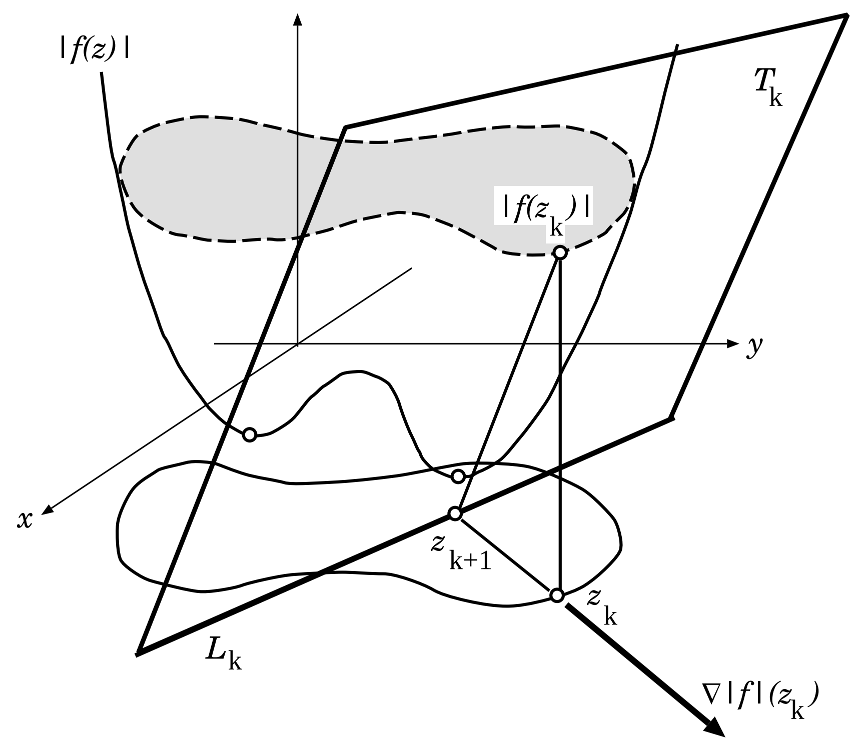
(C) Then

$$z_+ \longleftrightarrow (x_+, y_+),$$

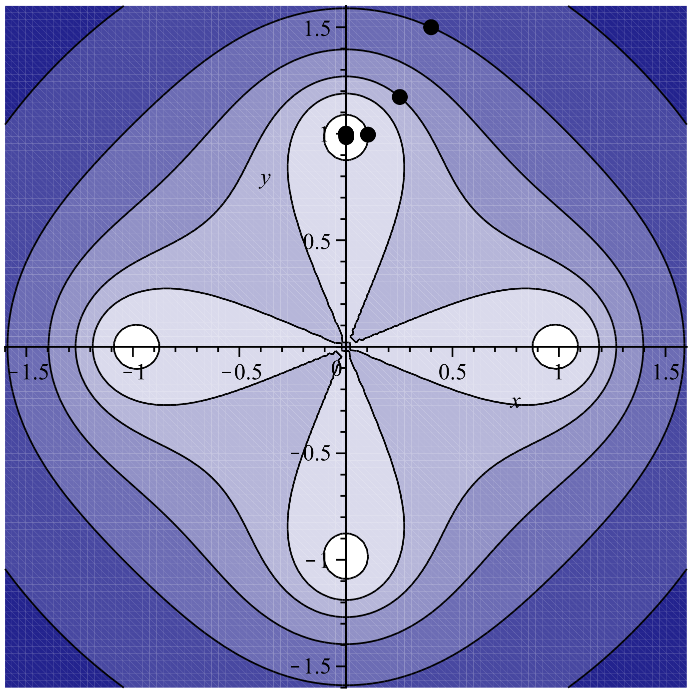
the perpendicular projection of (x, y) on L.

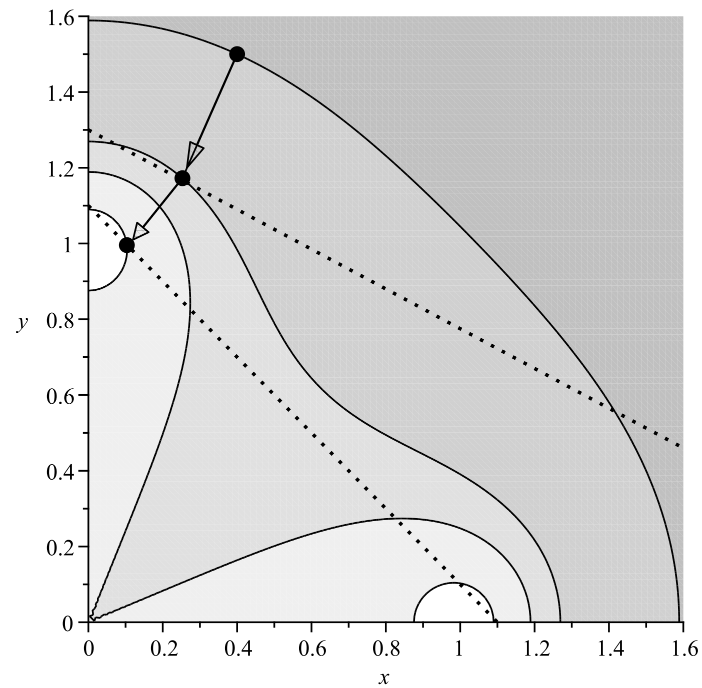
Illustration





$\mathbf{z^4 = 1}$





Level sets of $|z^4 - 1|$ and iterates converging to i

The Mandelbrot set

$$\mathcal{M} := \{\mathbf{c} : \{\mathbf{z_k} : \mathbf{z_{k+1}} := \mathbf{z_k^2} + \mathbf{c},\ \mathbf{z_0} = \mathbf{0}\} \text{ is bounded}\}$$

```
InverseNewton(z^2+c,z);
```

$$\exp\left\{-\frac{2}{\sqrt{4\,c-1}}\arctan\left(\frac{2\,z-1}{\sqrt{4\,c-1}}\right)\right\}$$

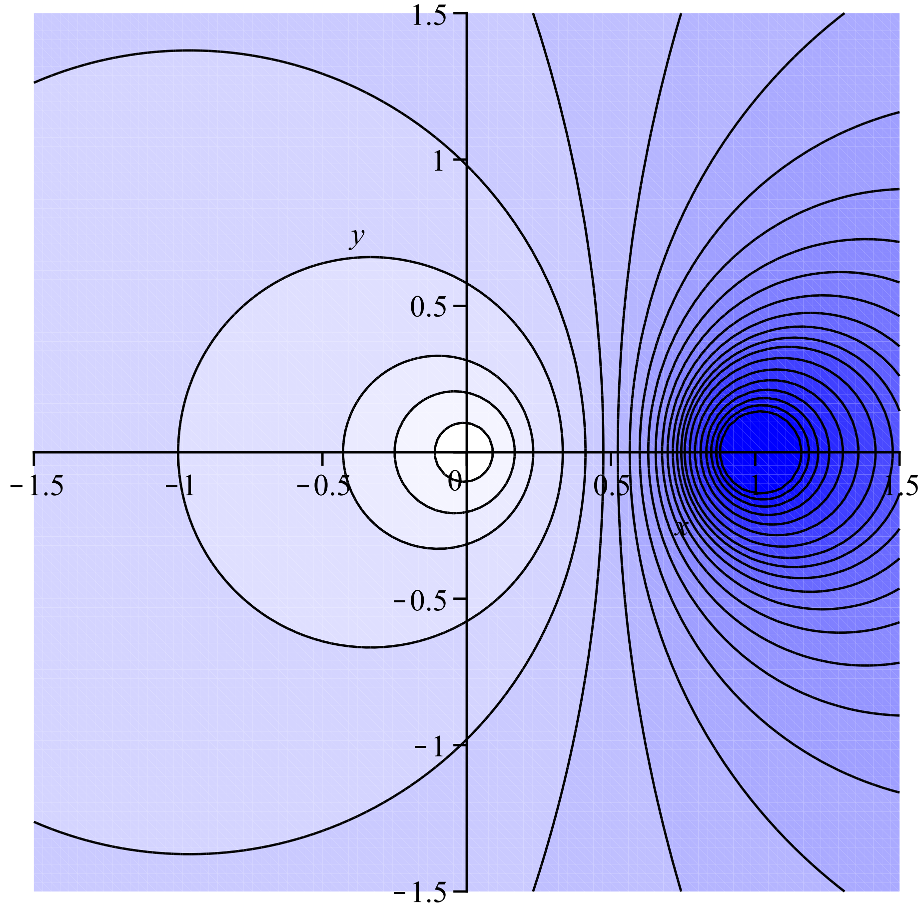
```
InverseNewton(z^2+(1/4),z);
```

$$\exp\left\{\frac{2}{2z-1}\right\}$$

```
InverseNewton(z^2,z);
```

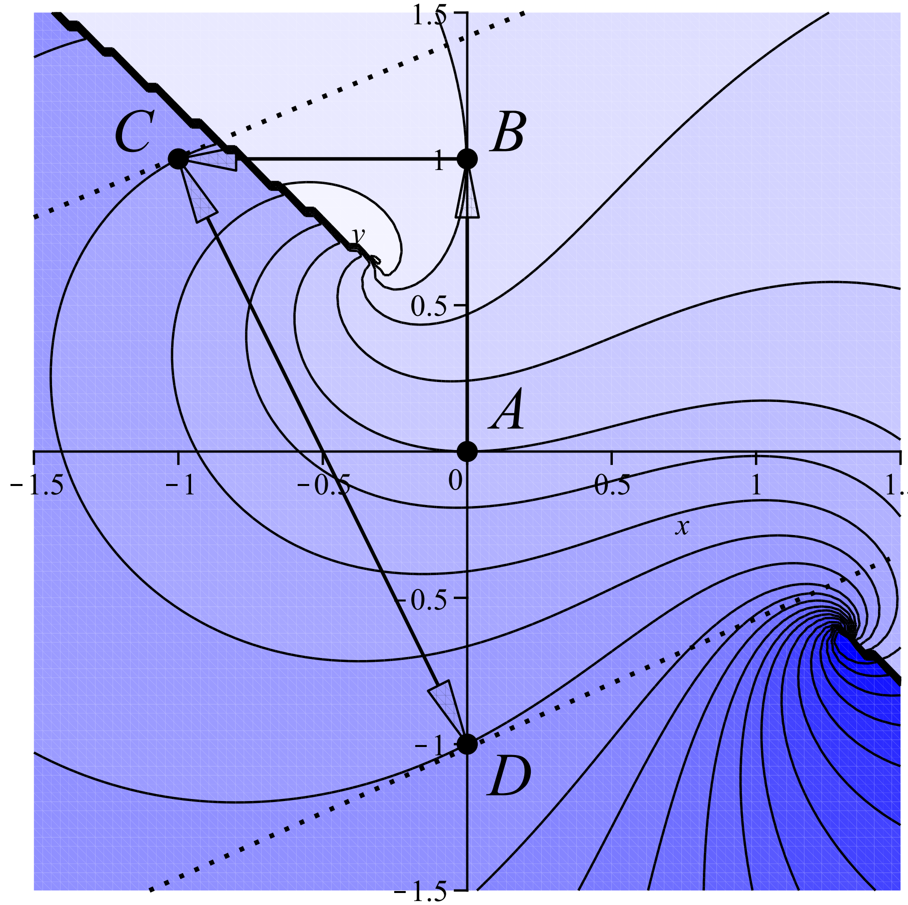
$$\frac{z}{z-1}$$

$\mathbf{0} \in \mathcal{M}$



Level sets of $|\mathbf{N}^{-1}(z^2)|$

$\mathbf{i} \in \mathcal{M}$



Level sets of $|\mathbf{N}^{-1}(z^2 + i)|$

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