Minimization of Quadratic Forms in Wireless Communications

Ralf R. Müller

Department of Electronics & Telecommunications
Norwegian University of Science & Technology, Trondheim, Norway

mueller@iet.ntnu.no

Dongning Guo

Department of Electrical Engineering & Computer Science
Northwestern University, Evanston, IL, U.S.A.

dguo@northwestern.edu

Aris L. Moustakas

Physics Department
National & Capodistrian University of Athens, Greece

arislm@phys.uoa.gr

The Problem

Let

$$E := \frac{1}{K} \min_{\boldsymbol{x} \in \mathcal{X}} \boldsymbol{x}^\dagger \boldsymbol{J} \boldsymbol{x}$$

with $\boldsymbol{x} \in \mathbb{C}^K$ and $\boldsymbol{J} \in \mathbb{C}^{K \times K}$.

The Problem

Let

$$E := \frac{1}{K} \min_{\boldsymbol{x} \in \mathcal{X}} \boldsymbol{x}^\dagger \boldsymbol{J} \boldsymbol{x}$$

with $\boldsymbol{x} \in \mathbb{C}^K$ and $\boldsymbol{J} \in \mathbb{C}^{K \times K}$.

Example 1:

$$\mathcal{X} = \{\boldsymbol{x} : \boldsymbol{x}^\dagger \boldsymbol{x} = K\} \quad \Longrightarrow \quad E = \min \lambda(\boldsymbol{J})$$

The Problem

Let

$$E := \frac{1}{K} \min_{\boldsymbol{x} \in \mathcal{X}} \boldsymbol{x}^\dagger \boldsymbol{J} \boldsymbol{x}$$

with $\boldsymbol{x} \in \mathbb{C}^K$ and $\boldsymbol{J} \in \mathbb{C}^{K \times K}$.

Example 1:

$$\mathcal{X} = \{\boldsymbol{x} : \boldsymbol{x}^\dagger \boldsymbol{x} = K\} \quad \Longrightarrow \quad E = \min \lambda(\boldsymbol{J})$$

for Wishart matrix $\longrightarrow [1 - \sqrt{\alpha}]_+^2$

The Problem

Let

$$E := \frac{1}{K} \min_{\boldsymbol{x} \in \mathcal{X}} \boldsymbol{x}^\dagger \boldsymbol{J} \boldsymbol{x}$$

with $\boldsymbol{x} \in \mathbb{C}^K$ and $\boldsymbol{J} \in \mathbb{C}^{K \times K}$.

Example 1:

$$\mathcal{X} = \{\boldsymbol{x} : \boldsymbol{x}^\dagger \boldsymbol{x} = K\} \quad \Longrightarrow \quad E = \min \lambda(\boldsymbol{J})$$

for Wishart matrix $\longrightarrow [1 - \sqrt{\alpha}]_+^2$

Example 2:

$$\mathcal{X} = \{x : x^2 = 1\}^K \quad \Longrightarrow \quad ???$$

The Problem

Let

$$E := \frac{1}{K} \min_{\boldsymbol{x} \in \mathcal{X}} \boldsymbol{x}^\dagger \boldsymbol{J} \boldsymbol{x}$$

with $\boldsymbol{x} \in \mathbb{C}^K$ and $\boldsymbol{J} \in \mathbb{C}^{K \times K}$.

Example 1:

$$\mathcal{X} = \{\boldsymbol{x} : \boldsymbol{x}^\dagger \boldsymbol{x} = K\} \quad \Longrightarrow \quad E = \min \lambda(\boldsymbol{J})$$

for Wishart matrix $\longrightarrow [1 - \sqrt{\alpha}]_+^2$

Example 2:

$$\mathcal{X} = \{x : x^2 = 1\}^K \quad \Longrightarrow \quad ???$$

Example 3:

$$\mathcal{X} = \{x : |x|^2 = 1\}^K \quad \Longrightarrow \quad ???$$

The Problem

Let

$$E := \frac{1}{K} \min_{\boldsymbol{x} \in \mathcal{X}} \boldsymbol{x}^\dagger \boldsymbol{J} \boldsymbol{x}$$

with $\boldsymbol{x} \in \mathbb{C}^K$ and $\boldsymbol{J} \in \mathbb{C}^{K \times K}$.

Example 1:

$$\mathcal{X} = \{\boldsymbol{x} : \boldsymbol{x}^\dagger \boldsymbol{x} = K\} \quad \Longrightarrow \quad E = \min \lambda(\boldsymbol{J})$$

for Wishart matrix $\longrightarrow [1 - \sqrt{\alpha}]_+^2$

Example 2:

$$\mathcal{X} = \{x : x^2 = 1\}^K \quad \Longrightarrow \quad ???$$

for Wishart matrix $\longrightarrow \approx \left[1 - \frac{\alpha}{\sqrt{\pi}}\right]_+^2$

Example 3:

$$\mathcal{X} = \{x : |x|^2 = 1\}^K \quad \Longrightarrow \quad ???$$

for Wishart matrix $\longrightarrow \approx \left[1 - \frac{\sqrt{\pi}\alpha}{2}\right]_+^2$

The Problem

Let

$$E := \frac{1}{K} \min_{\boldsymbol{x} \in \mathcal{X}} \boldsymbol{x}^\dagger \boldsymbol{J} \boldsymbol{x}$$

with $\boldsymbol{x} \in \mathbb{C}^K$ and $\boldsymbol{J} \in \mathbb{C}^{K \times K}$.

Example 1:

$$\mathcal{X} = \{\boldsymbol{x} : \boldsymbol{x}^\dagger \boldsymbol{x} = K\} \quad \Longrightarrow \quad E = \min \lambda(\boldsymbol{J})$$

for Wigner matrix $\longrightarrow -2$

Example 2:

$$\mathcal{X} = \{x : x^2 = 1\}^K \quad \Longrightarrow \quad ???$$

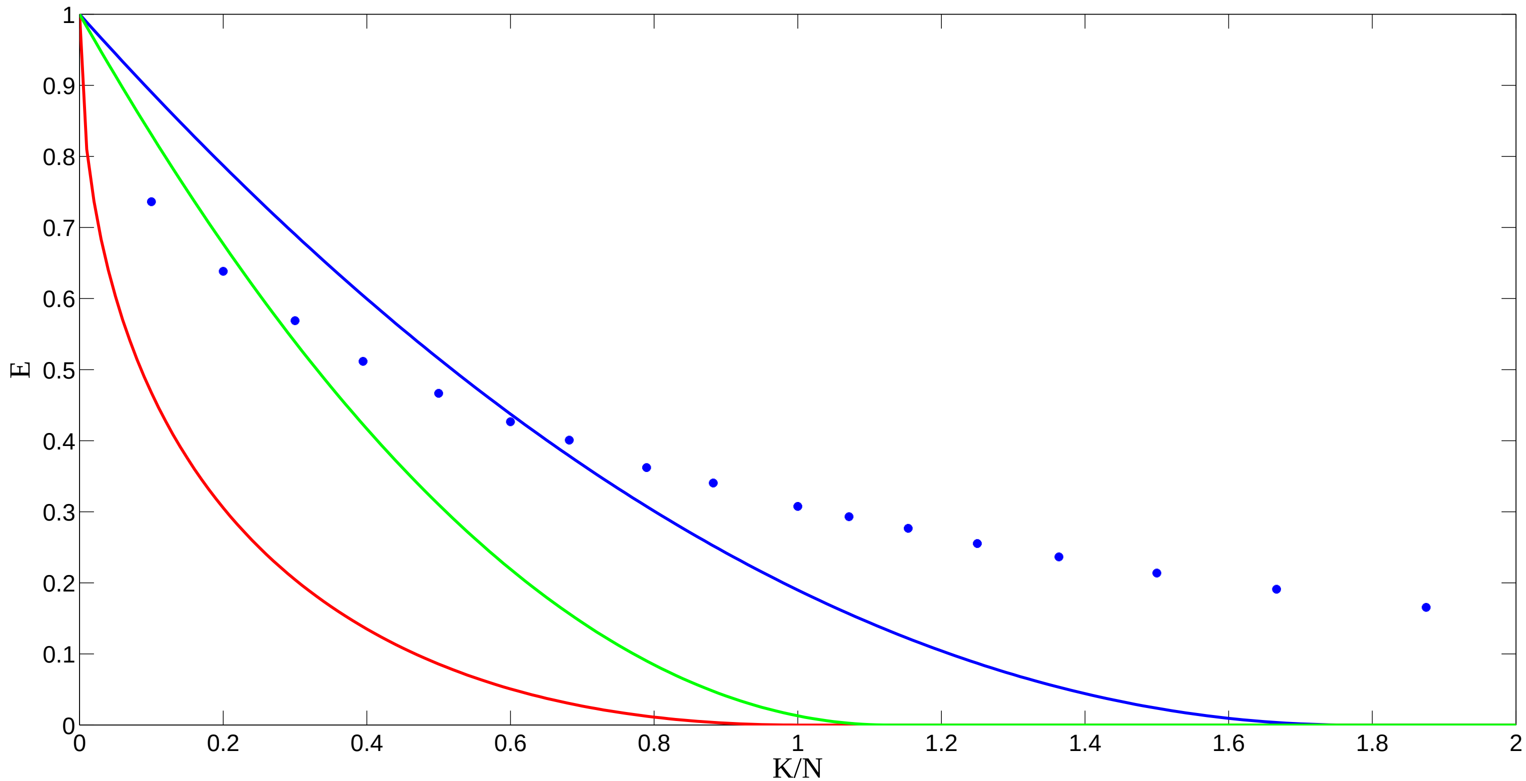
for Wigner matrix $\longrightarrow \approx -\frac{2}{\sqrt{\pi}}$

Example 3:

$$\mathcal{X} = \{x : |x|^2 = 1\}^K \quad \Longrightarrow \quad ???$$

for Wigner matrix $\longrightarrow \approx -\sqrt{\pi}$

Wishart Matrix



$\boldsymbol{x}^\dagger \boldsymbol{x} = K$ $\quad |x|^2 = 1$ $\quad x^2 = 1 : K = 15, \infty$

The Gaussian Vector Channel

Let the received vector be given by

$$\boldsymbol{r} = \boldsymbol{H}\boldsymbol{t} + \boldsymbol{n}$$

where

- $\boldsymbol{t}$ is the transmitted vector
- $\boldsymbol{n}$ is uncorrelated (white) Gaussian noise
- $\boldsymbol{H}$ is a coupling matrix accounting for crosstalk

In many applications, e.g. antenna arrays, code-division multiple-access, the coupling matrix is modelled as a random matrix with independent identically distributed entries (i.i.d. model).

Crosstalk can be processed either at receiver or transmitter

Processing at Transmitter

If the transmitter is a base-station and the receiver is a hand-held device one would prefer to have the complexity at the transmitter.

E.g. let the transmitted vector be

$$\boldsymbol{t} = \boldsymbol{H}^\dagger (\boldsymbol{H}\boldsymbol{H}^\dagger)^{-1} \boldsymbol{x}$$

where $\boldsymbol{x}$ is the data to be sent.

Then,

$$\boldsymbol{r} = \boldsymbol{x} + \boldsymbol{n}.$$

No crosstalk anymore due to channel inversion.

Problems of Simple Channel Inversion

Channel inversion implies a significant power amplification, i.e.

$$\boldsymbol{x}^\dagger \left(\boldsymbol{H}\boldsymbol{H}^\dagger\right)^{-1} \boldsymbol{x} > \boldsymbol{x}^\dagger \boldsymbol{x}.$$

In particular, let

- $\alpha = \frac{K}{N} \leq 1$;
- the entries of $\boldsymbol{H}$ are i.i.d. with variance $1/N$.

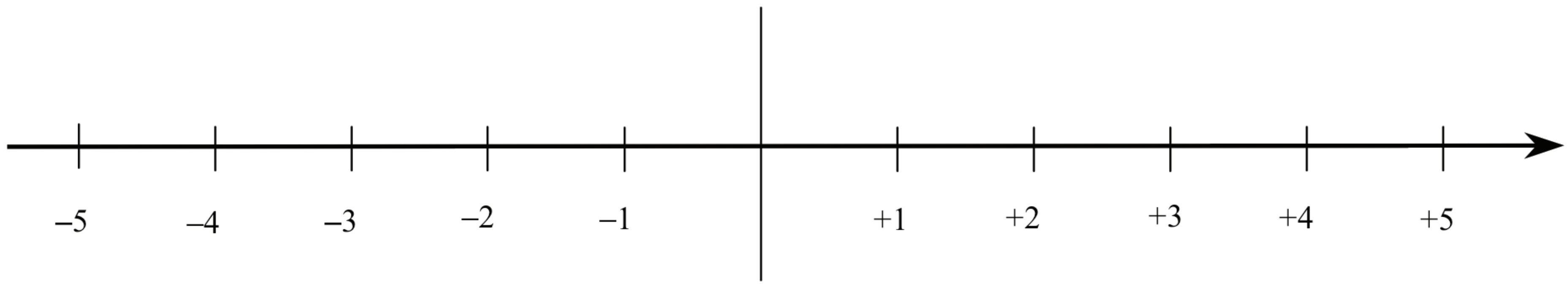
Then, for fixed aspect ratio α

$$\lim_{K\to\infty} \frac{\boldsymbol{x}^\dagger \left(\boldsymbol{H}\boldsymbol{H}^\dagger\right)^{-1} \boldsymbol{x}}{\boldsymbol{x}^\dagger \boldsymbol{x}} = \frac{1}{1-\alpha}$$

with probability 1.

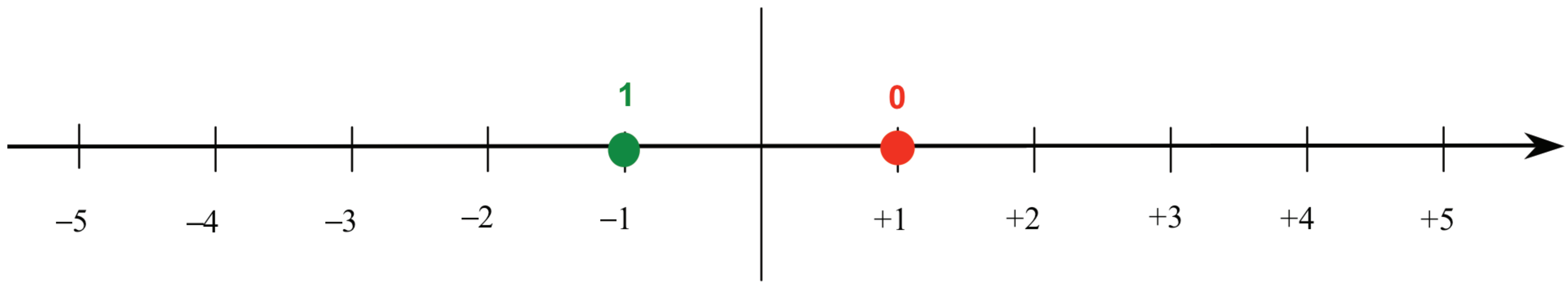
Tomlinson-Harashima Precoding

Tomlinson '71, Harashima & Miyakawa '72



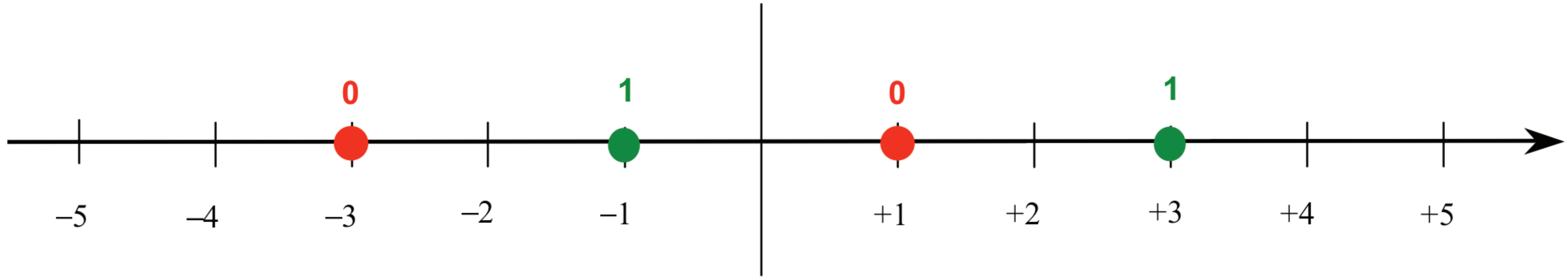
Tomlinson-Harashima Precoding

Tomlinson '71, Harashima & Miyakawa '72



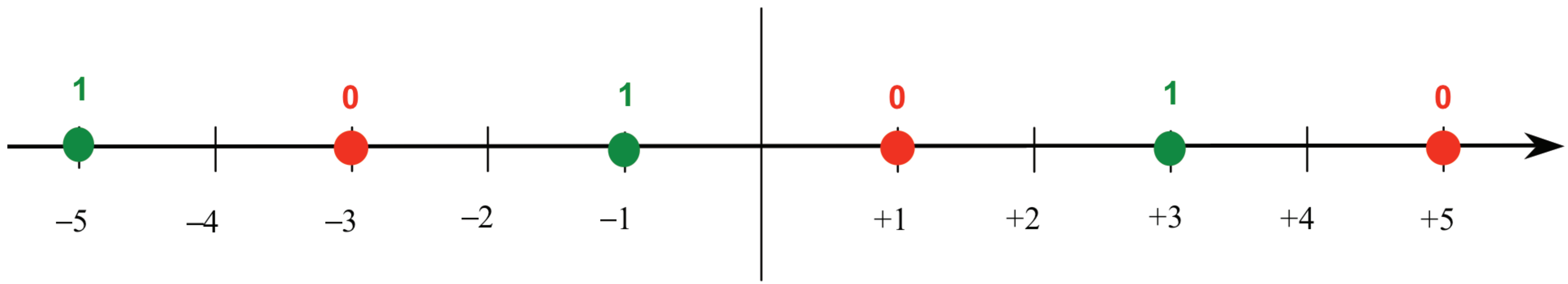
Tomlinson-Harashima Precoding

Tomlinson '71; Harashima & Miyakawa '72



Tomlinson-Harashima Precoding

Tomlinson '71, Harashima & Miyakawa '72



Instead of representing the logical "0" by $+1$, we present it by any element of the set $\{\ldots, -7, -3, +1, +5, \ldots\} = 4\mathbb{Z} + 1$. Correspondingly, the logical "1" is represented by any element of the set $4\mathbb{Z} - 1$.

Choose that representation that gives the smallest transmit power.

Generalized TH Precoding

Let $\mathcal{B}_0$ and $\mathcal{B}_1$ denote the sets presenting 0 and 1, resp.
Let $(s_1, s_2, s_3, \ldots, s_K)$ denote the data to be transmitted.

Then, the transmitted energy per data symbol is given by

$$E = \frac{1}{K} \min_{\boldsymbol{x} \in \mathcal{X}} \boldsymbol{x}^\dagger \boldsymbol{J} \boldsymbol{x}$$

with

$$\mathcal{X} = \mathcal{B}_{s_1} \times \mathcal{B}_{s_2} \times \cdots \times \mathcal{B}_{s_K}$$

and

$$\boldsymbol{J} = (\boldsymbol{H}\boldsymbol{H}^\dagger)^{-1}.$$

Zero Temperature Formulation

Quadratic programming is the problem of finding the zero temperature limit (ground state energy) of a quadratic Hamiltonian.

The transmitted power is written as a zero temperature limit

$$E = -\lim_{\beta\to\infty} \frac{1}{\beta K} \log \sum_{\boldsymbol{x}\in\mathcal{X}} \mathrm{e}^{-\beta K \,\mathrm{Tr}(\boldsymbol{J}\boldsymbol{x}\boldsymbol{x}^\dagger)}$$

with $\frac{1}{\beta}$ denoting temperature.

Zero Temperature Formulation

Quadratic programming is the problem of finding the zero temperature limit (ground state energy) of a quadratic Hamiltonian.

The transmitted power is written as a zero temperature limit

$$\begin{aligned} E &= -\lim_{\beta\to\infty} \frac{1}{\beta K} \log \sum_{\boldsymbol{x}\in\mathcal{X}} \mathrm{e}^{-\beta K \,\mathrm{Tr}(\boldsymbol{J}\boldsymbol{x}\boldsymbol{x}^\dagger)} \\ &\longrightarrow -\lim_{\beta\to\infty} \lim_{K\to\infty} \mathop{\mathrm{E}}_{\boldsymbol{J}} \frac{1}{\beta K} \log \sum_{\boldsymbol{x}\in\mathcal{X}} \mathrm{e}^{-\beta K \,\mathrm{Tr}(\boldsymbol{J}\boldsymbol{x}\boldsymbol{x}^\dagger)} \end{aligned}$$

with $\frac{1}{\beta}$ denoting temperature.

Free Fourier Transform

We want

$$\lim_{K\to\infty} \frac{1}{K} \mathop{\mathrm{E}}_{\boldsymbol{J}} \log \sum_{\boldsymbol{x}\in\mathcal{X}} \mathrm{e}^{-\beta K\,\mathrm{Tr}(\boldsymbol{J}\boldsymbol{x}\boldsymbol{x}^\dagger)}.$$

Free Fourier Transform

We want

$$\lim_{K\to\infty} \frac{1}{K} \mathop{\mathrm{E}}_{\boldsymbol{J}} \log \sum_{\boldsymbol{x}\in\mathcal{X}} \mathrm{e}^{-\beta K \operatorname{Tr}(\boldsymbol{J}\boldsymbol{x}\boldsymbol{x}^\dagger)}.$$

We know

$$\lim_{K\to\infty} \frac{1}{K} \log \mathop{\mathrm{E}}_{\boldsymbol{J}} \mathrm{e}^{-K \operatorname{Tr} \boldsymbol{J}\boldsymbol{P}} = -\sum_{a=1}^{n} \int\limits_{0}^{\lambda_a(\boldsymbol{P})} R_{\boldsymbol{J}}(-w)\mathrm{d}w.$$

Free Fourier Transform

We want

$$\lim_{K\to\infty} \frac{1}{K} \mathop{\mathrm{E}}_{\boldsymbol{J}} \log \sum_{\boldsymbol{x}\in\mathcal{X}} \mathrm{e}^{-\beta K \operatorname{Tr}(\boldsymbol{J}\boldsymbol{x}\boldsymbol{x}^\dagger)}.$$

We know

$$\lim_{K\to\infty} \frac{1}{K} \log \mathop{\mathrm{E}}_{\boldsymbol{J}} \mathrm{e}^{-K \operatorname{Tr} \boldsymbol{J}\boldsymbol{P}} = -\sum_{a=1}^{n} \int_0^{\lambda_a(\boldsymbol{P})} R_{\boldsymbol{J}}(-w)\mathrm{d}w.$$

We would like to exchange expectation and logarithm:

$$\mathop{\mathrm{E}}_{X} \log X = \lim_{n\to 0} \frac{1}{n} \log \mathop{\mathrm{E}}_{X} X^n.$$

Replica Continuity

We want

$$\lim_{K\to\infty}\frac{1}{K}\mathop{\mathrm{E}}_{\boldsymbol{J}}\log\sum_{\boldsymbol{x}\in\mathcal{X}}\mathrm{e}^{-\beta K\,\mathrm{Tr}(\boldsymbol{J}\boldsymbol{x}\boldsymbol{x}^\dagger)} = \lim_{K\to\infty}\lim_{n\to 0}\frac{1}{nK}\log\mathop{\mathrm{E}}_{\boldsymbol{J}}\left(\sum_{\boldsymbol{x}\in\mathcal{X}}\mathrm{e}^{-\beta K\,\mathrm{Tr}(\boldsymbol{J}\boldsymbol{x}\boldsymbol{x}^\dagger)}\right)^n$$

Replica Continuity

We want

$$\lim_{K\to\infty} \frac{1}{K} \mathop{\mathrm{E}}_{\boldsymbol{J}} \log \sum_{\boldsymbol{x}\in\mathcal{X}} \mathrm{e}^{-\beta K \,\mathrm{Tr}(\boldsymbol{J}\boldsymbol{x}\boldsymbol{x}^\dagger)} = \lim_{K\to\infty} \lim_{n\to 0} \frac{1}{nK} \log \mathop{\mathrm{E}}_{\boldsymbol{J}} \left(\sum_{\boldsymbol{x}\in\mathcal{X}} \mathrm{e}^{-\beta K \,\mathrm{Tr}(\boldsymbol{J}\boldsymbol{x}\boldsymbol{x}^\dagger)} \right)^n$$

$$= \lim_{K\to\infty} \lim_{n\to 0} \frac{1}{nK} \log \mathop{\mathrm{E}}_{\boldsymbol{J}} \prod_{a=1}^{n} \sum_{\boldsymbol{x}_a\in\mathcal{X}} \mathrm{e}^{-\beta K \,\mathrm{Tr}\left(\boldsymbol{J}\boldsymbol{x}_a\boldsymbol{x}_a^\dagger\right)}$$

Replica Continuity

We want

$$\lim_{K\to\infty}\frac{1}{K}\mathop{\mathrm{E}}_{\boldsymbol{J}}\log\sum_{\boldsymbol{x}\in\mathcal{X}}\mathrm{e}^{-\beta K\,\mathrm{Tr}(\boldsymbol{J}\boldsymbol{x}\boldsymbol{x}^\dagger)} = \lim_{K\to\infty}\lim_{n\to 0}\frac{1}{nK}\log\mathop{\mathrm{E}}_{\boldsymbol{J}}\left(\sum_{\boldsymbol{x}\in\mathcal{X}}\mathrm{e}^{-\beta K\,\mathrm{Tr}(\boldsymbol{J}\boldsymbol{x}\boldsymbol{x}^\dagger)}\right)^n$$

$$= \lim_{K\to\infty}\lim_{n\to 0}\frac{1}{nK}\log\mathop{\mathrm{E}}_{\boldsymbol{J}}\prod_{a=1}^{n}\sum_{\boldsymbol{x}_a\in\mathcal{X}}\mathrm{e}^{-\beta K\,\mathrm{Tr}\left(\boldsymbol{J}\boldsymbol{x}_a\boldsymbol{x}_a^\dagger\right)}$$

$$= \lim_{K\to\infty}\lim_{n\to 0}\frac{1}{nK}\log\mathop{\mathrm{E}}_{\boldsymbol{J}}\sum_{\boldsymbol{x}_1\in\mathcal{X}}\cdots\sum_{\boldsymbol{x}_n\in\mathcal{X}}\mathrm{e}^{-K\,\mathrm{Tr}\left(\boldsymbol{J}\beta\sum_{a=1}^{n}\boldsymbol{x}_a\boldsymbol{x}_a^\dagger\right)}$$

Replica Continuity

We want

$$\lim_{K\to\infty} \frac{1}{K} \mathop{\mathrm{E}}_{\boldsymbol{J}} \log \sum_{\boldsymbol{x}\in\mathcal{X}} \mathrm{e}^{-\beta K \,\mathrm{Tr}(\boldsymbol{J}\boldsymbol{x}\boldsymbol{x}^\dagger)} = \lim_{K\to\infty} \lim_{n\to 0} \frac{1}{nK} \log \mathop{\mathrm{E}}_{\boldsymbol{J}} \left(\sum_{\boldsymbol{x}\in\mathcal{X}} \mathrm{e}^{-\beta K \,\mathrm{Tr}(\boldsymbol{J}\boldsymbol{x}\boldsymbol{x}^\dagger)} \right)^n$$

$$= \lim_{K\to\infty} \lim_{n\to 0} \frac{1}{nK} \log \mathop{\mathrm{E}}_{\boldsymbol{J}} \prod_{a=1}^{n} \sum_{\boldsymbol{x}_a\in\mathcal{X}} \mathrm{e}^{-\beta K \,\mathrm{Tr}\left(\boldsymbol{J}\boldsymbol{x}_a\boldsymbol{x}_a^\dagger\right)}$$

$$= \lim_{K\to\infty} \lim_{n\to 0} \frac{1}{nK} \log \mathop{\mathrm{E}}_{\boldsymbol{J}} \sum_{\boldsymbol{x}_1\in\mathcal{X}} \cdots \sum_{\boldsymbol{x}_n\in\mathcal{X}} \mathrm{e}^{-K \,\mathrm{Tr}\left(\boldsymbol{J}\beta \sum_{a=1}^{n} \boldsymbol{x}_a\boldsymbol{x}_a^\dagger\right)}$$

$$= -\lim_{n\to 0} \frac{1}{n} \sum_{a=1}^{n} \mathop{\mathrm{E}}_{\boldsymbol{Q}} \int_0^{\beta\lambda_a(\boldsymbol{Q})} R_{\boldsymbol{J}}(-w)\,\mathrm{d}w$$

with

$$Q_{ab} := \frac{1}{K} \boldsymbol{x}_a^\dagger \boldsymbol{x}_b.$$

Replica Symmetry

$$\boldsymbol{Q} := \begin{bmatrix} q+\frac{\chi}{\beta} & q & \cdots & q & q \\ q & q+\frac{\chi}{\beta} & \ddots & q & q \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ q & q & \ddots & q+\frac{\chi}{\beta} & q \\ q & q & \cdots & q & q+\frac{\chi}{\beta} \end{bmatrix}$$

with some macroscopic parameters q and χ.

This is the most critical step. In general, the structure of $\boldsymbol{Q}$ is more complicated. Generalizations are called replica symmetry breaking (RSB).

Main Result

Let $\mathrm{P}(s)$ denote the limit of the empirical distribution of the information symbols $s_1, s_2, \ldots, s_K$ as $K \to \infty$. Let q and χ be the simultaneous solutions to

$$q = \iint \operatorname*{argmin}_{x \in \mathcal{B}_s}{}^2 \left| z\sqrt{2qR'(-\chi)} - 2xR(-\chi) \right| \mathrm{D}z\, \mathrm{dP}(s)$$

$$\chi = \frac{1}{\sqrt{2qR'(-\chi)}} \iint \operatorname*{argmin}_{x \in \mathcal{B}_s} \left| z\sqrt{2qR'(-\chi)} - 2xR(-\chi) \right| z\mathrm{D}z\, \mathrm{dP}(s)$$

where $\mathrm{D}z = \exp(-z^2/2)\mathrm{d}z/\sqrt{2\pi}$, $R(\cdot)$ is the R-transform of the limiting eigenvalue spectrum of $\boldsymbol{J}$, and $0 < \chi < \infty$.

Then, replica symmetry implies

$$\frac{1}{K} \min_{\boldsymbol{x} \in \mathcal{X}} \boldsymbol{x}^\dagger \boldsymbol{J} \boldsymbol{x} \to q \frac{\partial}{\partial \chi} \chi R(-\chi)$$

as $K \to \infty$.

Some R-Transforms

$$\mathbf{I}: \quad R(w) = 1$$

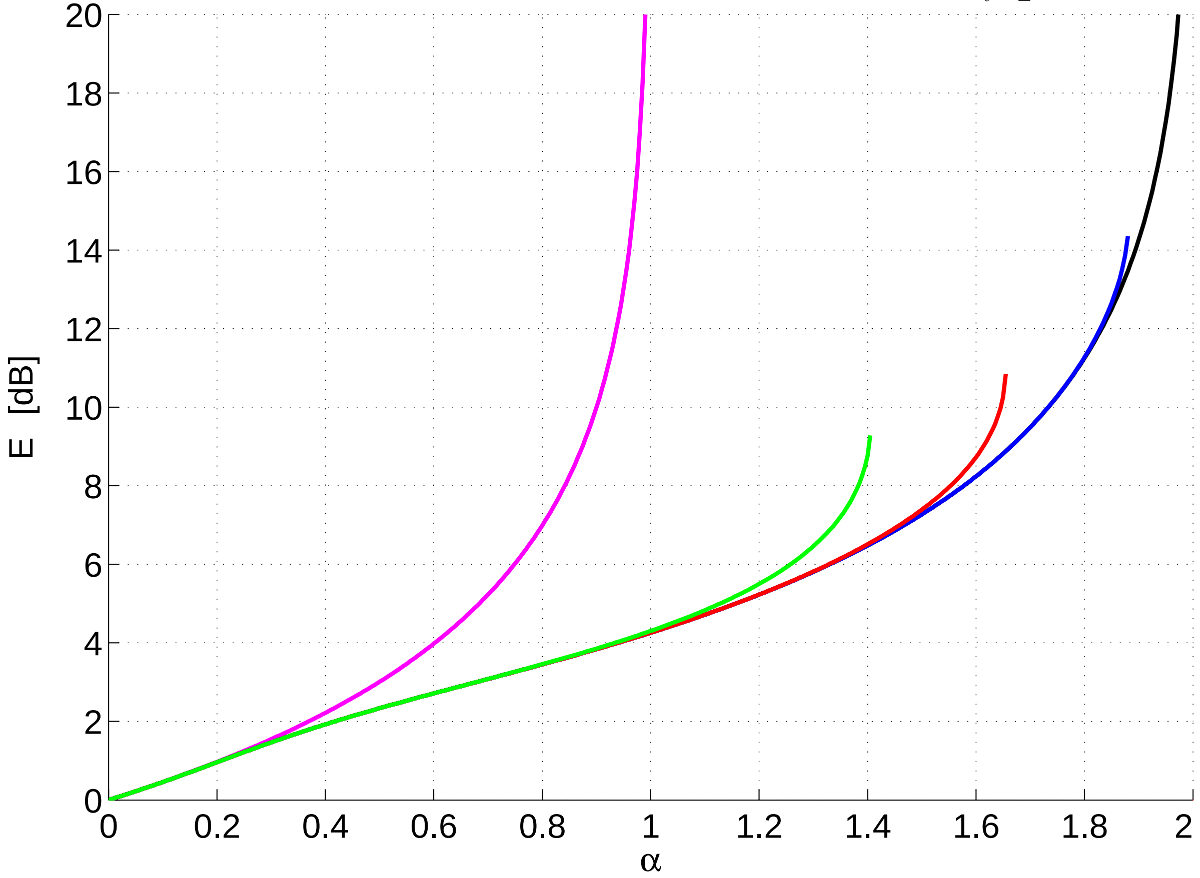
$$\boldsymbol{H}\boldsymbol{H}^\dagger: \quad R(w) = \frac{1}{1-\alpha w}$$ Marchenko-Pastur (MP) law

$$(\boldsymbol{H}\boldsymbol{H}^\dagger)^{-1}: \quad R(w) = \frac{1-\alpha-\sqrt{(1-\alpha)^2-4\alpha w}}{2\alpha w}$$ inv. MP

$$\boldsymbol{U}+\boldsymbol{U}^\dagger: \quad R(w) = \frac{-1+\sqrt{1+4w^2}}{w}$$

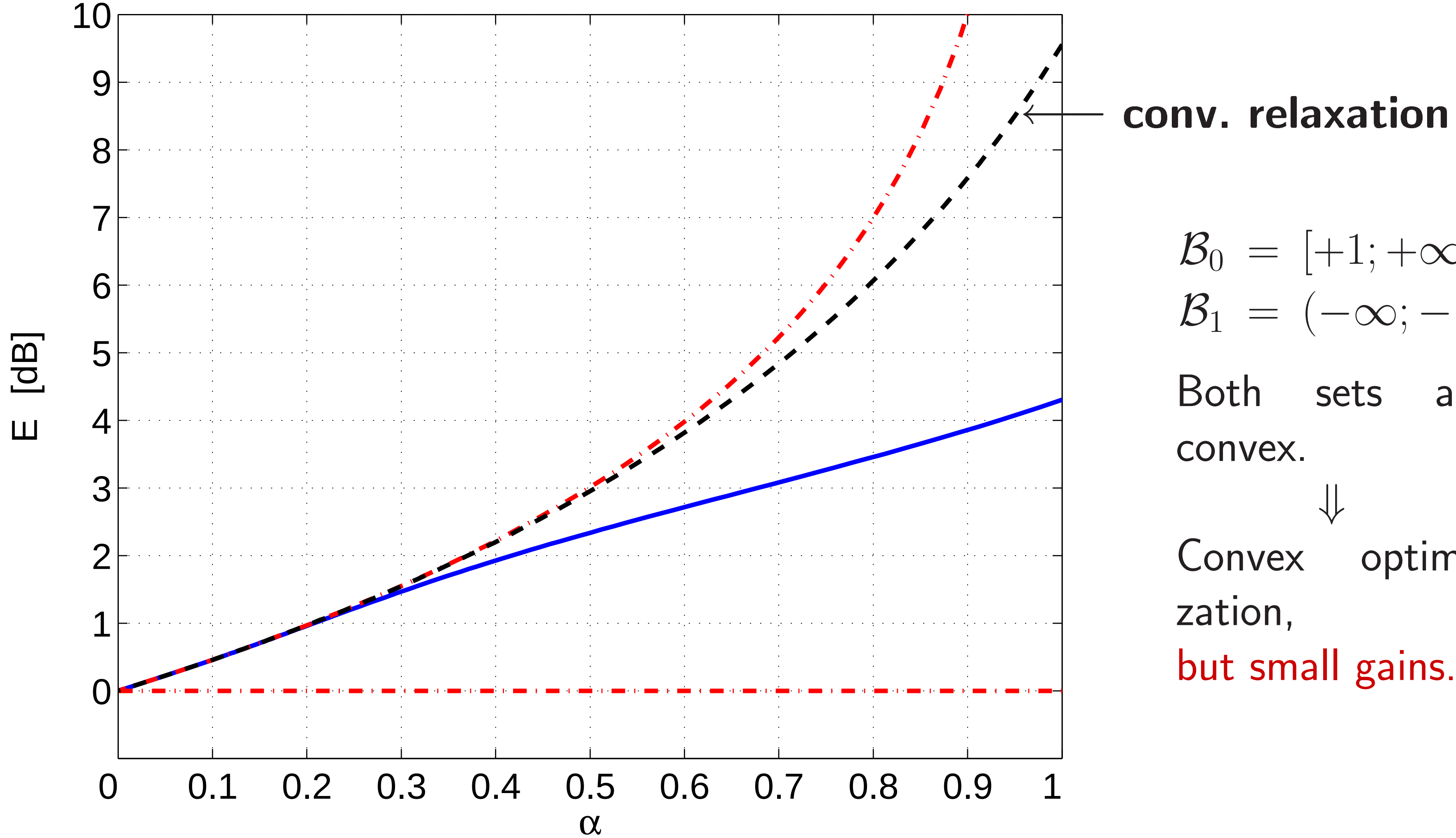
Inv. MP with Odd Integer Lattice (TH Precoding)

Let $\boldsymbol{J} = (\boldsymbol{H}\boldsymbol{H}^\dagger)^{-1}$ and $\chi < \infty$: $E = \dfrac{c_1^2 + \sum_{i=2}^{L} \left(c_i^2 - c_{i-1}^2\right) \mathrm{Q}\left(\frac{c_i + c_{i-1}}{\sqrt{2\alpha E}}\right)}{1 - \alpha + \sqrt{\frac{\alpha}{\pi E}} \sum_{i=2}^{L} (c_i - c_{i-1}) \exp\left(-\frac{(c_i + c_{i-1})^2}{4\alpha E}\right)}$



$L = 1, 2, 3, 6, 100$

Convex Relaxation



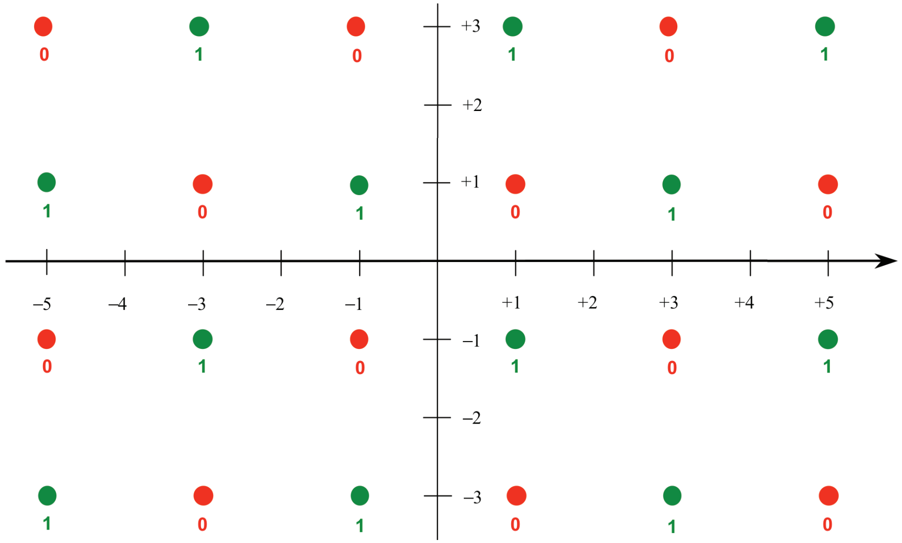
$$\mathcal{B}_0 = [+1; +\infty)$$
$$\mathcal{B}_1 = (-\infty; -1]$$

Both sets are convex.

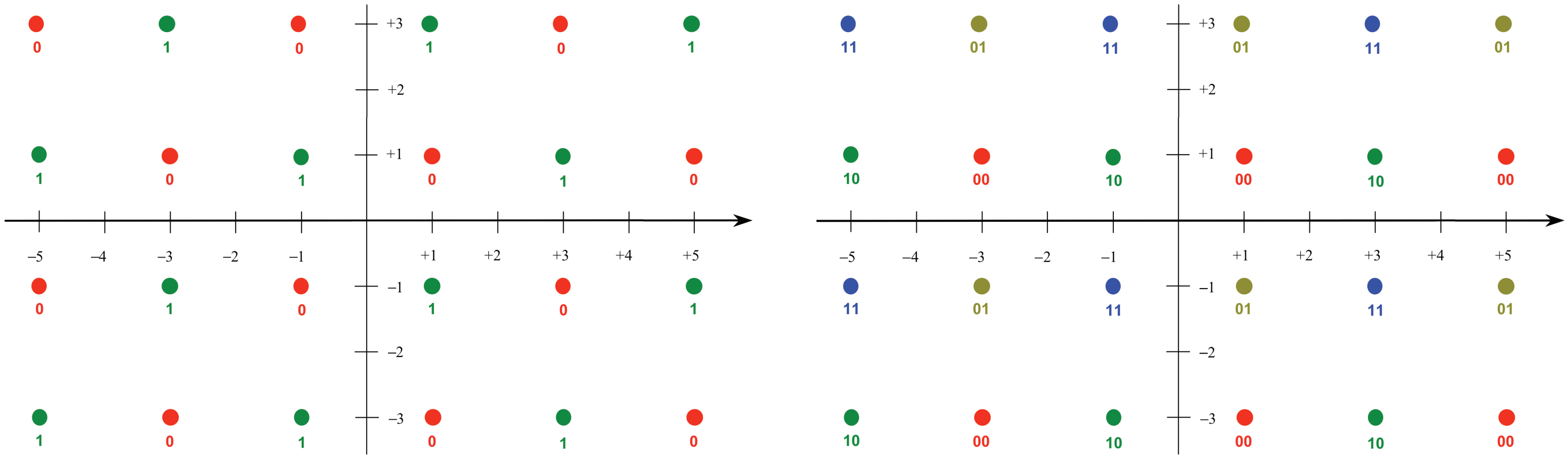
$\Downarrow$

Convex optimization, but small gains.

Odd Integer Quadrature Lattice

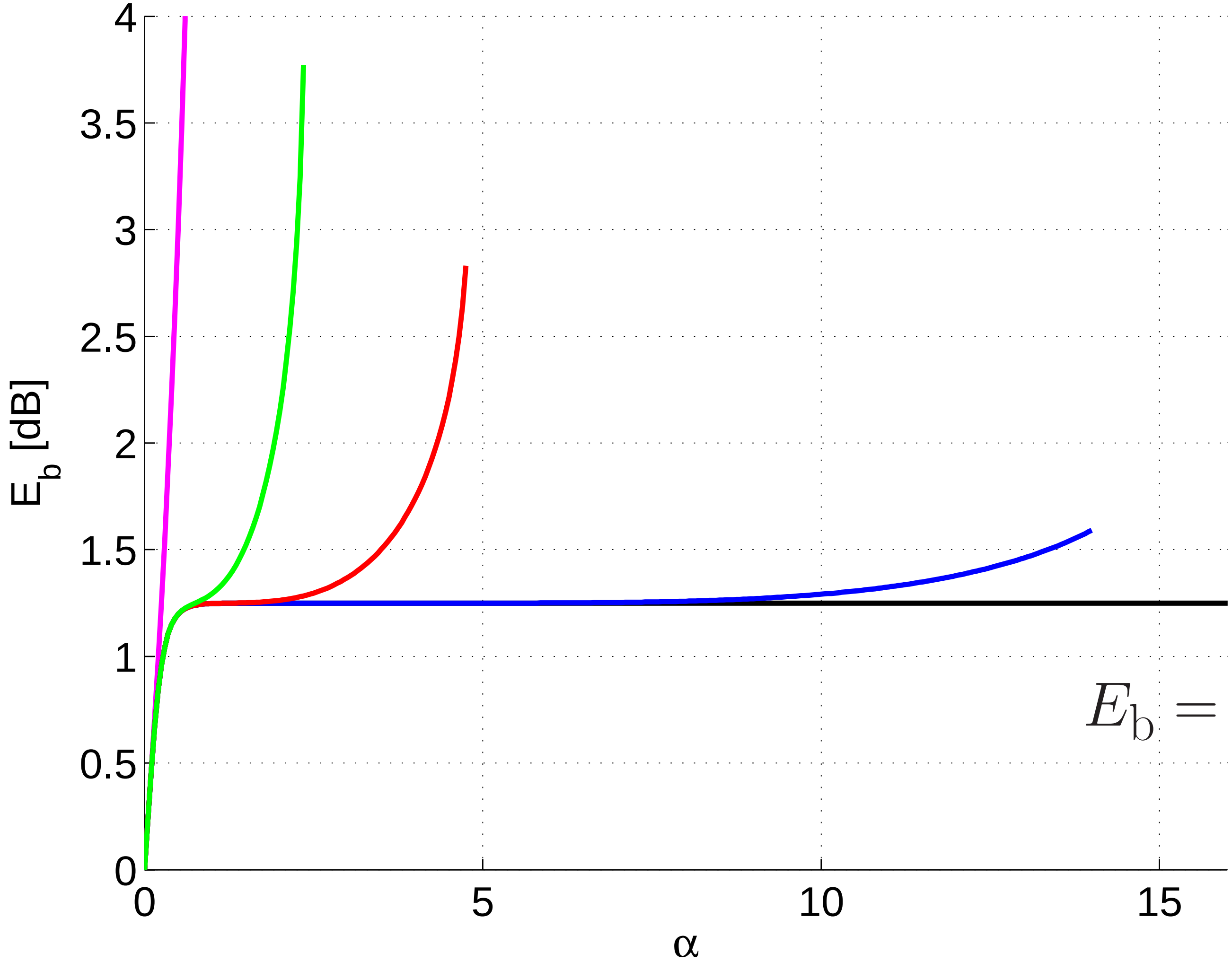


Odd Integer Quadrature Lattice



Same energy per bit $E_\mathrm{b} = \dfrac{E}{\log_2 |\mathcal{S}|}$ in both cases.

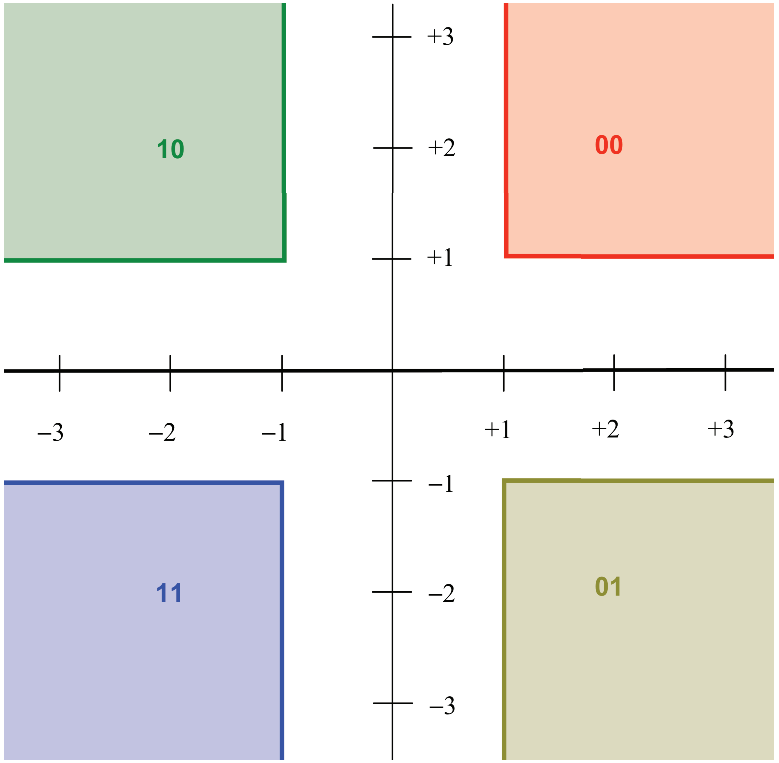
Complex TH Precoding



$L = 1, 2, 3, 6, 100$

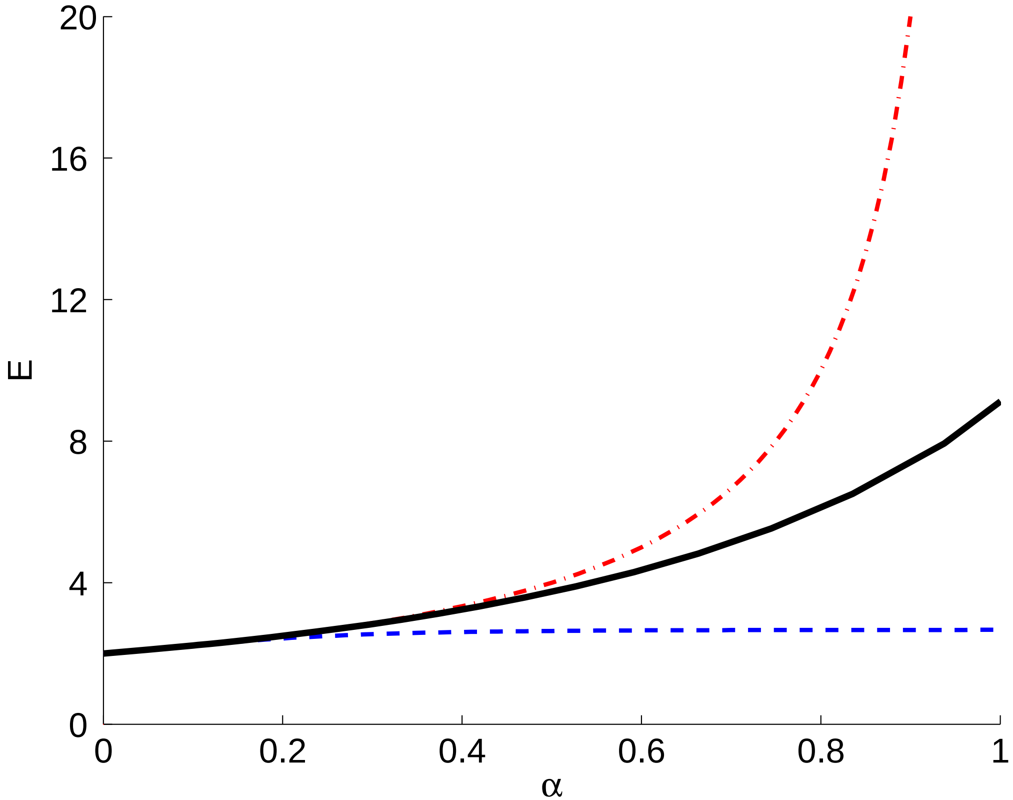
$E_\mathrm{b} = \frac{E}{2} = \frac{4}{3}$ for $L \to \infty$.

Complex Convex Relaxation



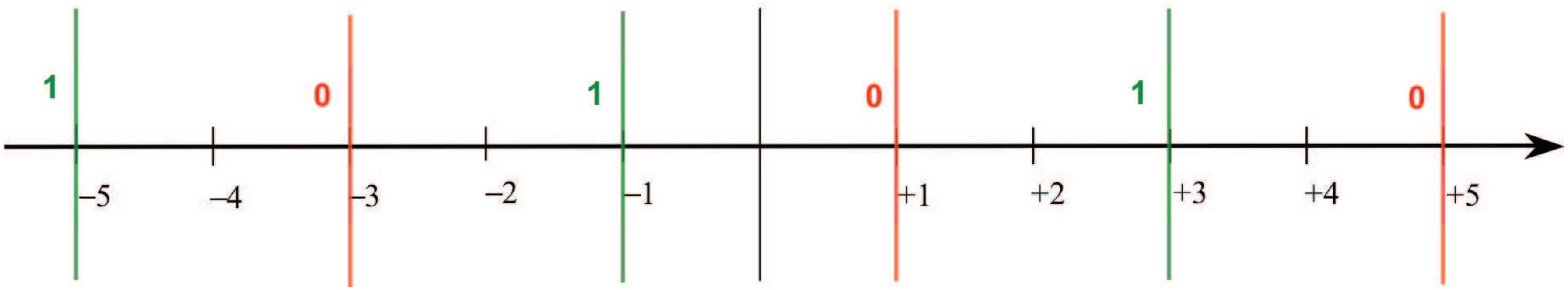
. . . allows for convex programming.

Complex Convex Relaxation *(cont'd)*



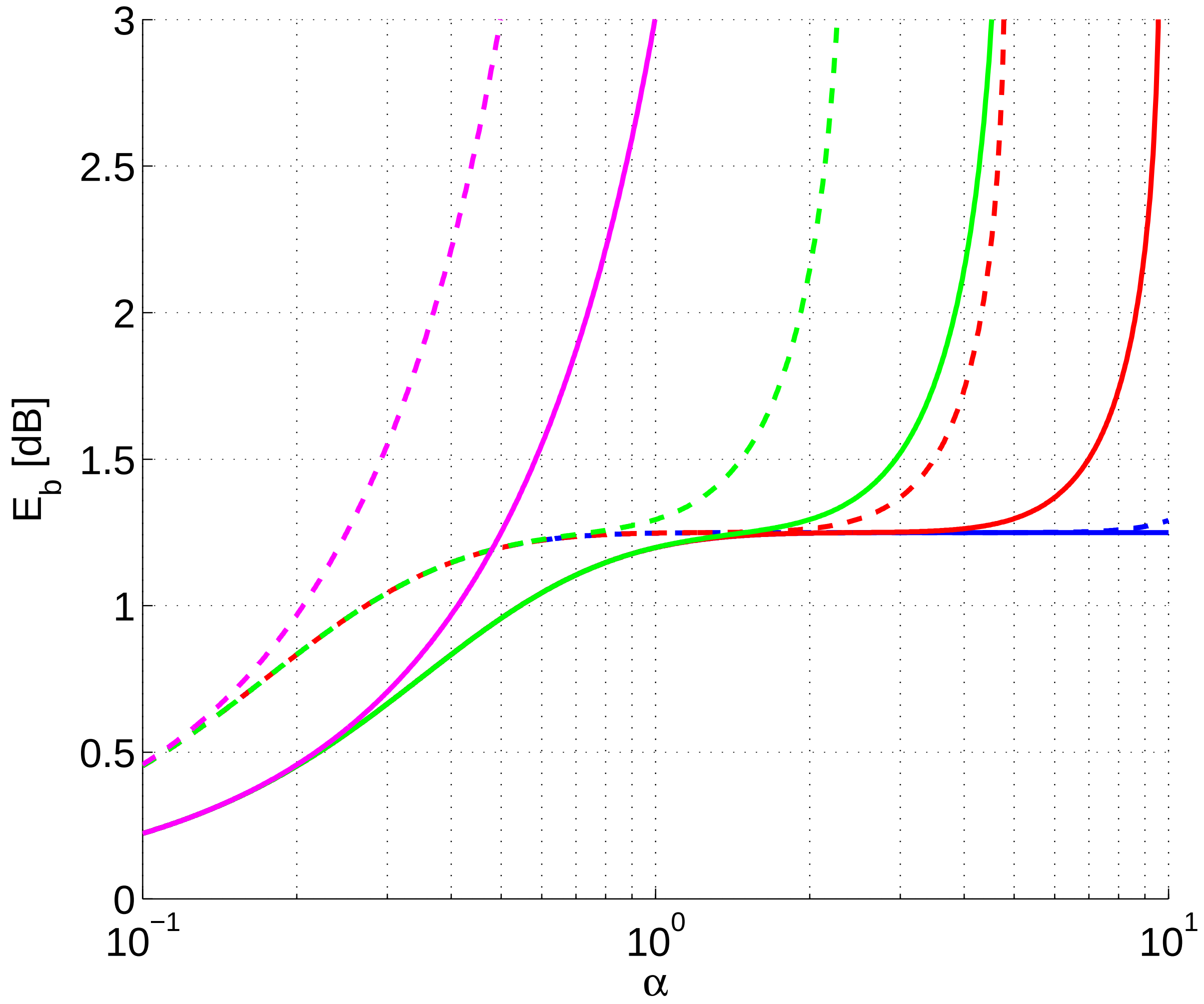
... achieves part of the gain of TH precoding.

Complex Semi-Discrete Set



The imaginary part is purely used to reduce transmit energy.

Complex Semi-Discrete Set *(solid lines)*



$L = 1, 2, 3, 6$

$E_\mathrm{b} = \frac{4}{3}$ for $L \to \infty$.

Convex opt. for $L = 1$.

A Fake Gain

The inverse MP kernel has the following property:

Let $\boldsymbol{H}$ and $\boldsymbol{H'}$ be random matrices of size $K \times N$ and $K' \times N$ respectively, with $K' > K$ and with i.i.d. entries of zero mean and variance $1/N$. Then,

$$\min_{\boldsymbol{x}\in\mathcal{X}} \frac{\boldsymbol{x}^\dagger(\boldsymbol{H}\boldsymbol{H}^\dagger)^{-1}\boldsymbol{x}}{K} \quad - \min_{\boldsymbol{x}\in\mathcal{X}\times\mathbb{C}^{K'-K}} \frac{\boldsymbol{x}^\dagger(\boldsymbol{H'}\boldsymbol{H'}^\dagger)^{-1}\boldsymbol{x}}{K} \longrightarrow 0.$$

The redundant symbols serve no purpose.

Wanted

$$\lim_{K\to\infty} \frac{1}{K} \log \mathop{\mathrm{E}}_{\boldsymbol{A},\boldsymbol{B}} \mathrm{e}^{-K \operatorname{Tr} \boldsymbol{APBP}} = f\left\{R_{\boldsymbol{A}}(\cdots), R_{\boldsymbol{B}}(\cdot), \ldots, \right\}.$$

Rigorous or Hand-Waving

Discovering Antimatter

What happens if the MP-law has a mass point at zero $(K > N)$?

Can we precode without interference?

Discovering Antimatter

What happens if the MP-law has a mass point at zero $(K > N)$?

Can we precode without interference?

The precoder produces

$$\lim_{\epsilon \longrightarrow 0} \operatorname*{argmin}_{\boldsymbol{x} \in \mathcal{X}} \frac{\boldsymbol{x}^\dagger (\boldsymbol{H}\boldsymbol{H}^\dagger + \epsilon \mathbf{I})^{-1} \boldsymbol{x}}{K}$$

The received signal becomes

$$\boldsymbol{r} = \lim_{\epsilon \longrightarrow 0} \boldsymbol{H}\boldsymbol{H}^\dagger (\boldsymbol{H}\boldsymbol{H}^\dagger + \epsilon \mathbf{I})^{-1} \boldsymbol{x} + \boldsymbol{n}.$$

Discovering Antimatter

What happens if the MP-law has a mass point at zero $(K > N)$?

Can we precode without interference?

The precoder produces

$$\lim_{\epsilon \longrightarrow 0} \operatorname*{argmin}_{\boldsymbol{x} \in \mathcal{X}} \frac{\boldsymbol{x}^\dagger (\boldsymbol{H}\boldsymbol{H}^\dagger + \epsilon \mathbf{I})^{-1} \boldsymbol{x}}{K}$$

The received signal becomes

$$\boldsymbol{r} = \lim_{\epsilon \longrightarrow 0} \boldsymbol{H}\boldsymbol{H}^\dagger (\boldsymbol{H}\boldsymbol{H}^\dagger + \epsilon \mathbf{I})^{-1} \boldsymbol{x} + \boldsymbol{n}.$$

If the energy is finite, there is no interference.