Free pressure, free entropy and hypothesis testing

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Plan

1. Hypothesis testing: conventional framework

2. Free pressure and free entropy: microstate approach

3. Free analog of hypothesis testing – free Stein's lemma

4. The single variable case

1. Hypothesis testing: conventional framework

- $(\mathcal{H}_n)$: a sequence of finite-dimensional Hilbert spaces
- ρ_n, σ_n: states on $\mathcal{H}_n$
- Null-hypothesis (H0): the true state of the nth system is ρ_n
- Counter-hypothesis (H1): the true state of the nth system is σ_n
- Test: binary measurement $0 \le T_n \le I$ on $\mathcal{H}_n$

  T_n corresponds to outcome 0, $I - T_n$ corresponds to outcome 1

  outcome $= 0$: (H0) is accepted, outcome $= 1$: (H1) is accepted
- Error probabilities of the first/second kinds:

$$\alpha_n(T_n) := \rho_n(I_n - T_n), \quad \beta_n(T_n) := \sigma_n(T_n)$$

Bayesian error probabilities

- ρ_n and σ_n have a priori probabilities π_n and $1 - \pi_n$
- Optimal Bayesian probability of an erroneous decision:

$$P_{\min}(\rho_n : \sigma_n|\pi_n) := \min_{0 \le T_n \le I} \Big\{ \pi_n \alpha_n(T_n) + (1 - \pi_n) \beta_n(T_n) \Big\}$$

Results for i.i.d. case

- I.i.d. setting: $\mathcal{H}_n = \mathcal{H}^{\otimes n}, \quad \rho_n = \rho_1^{\otimes n}, \quad \sigma_n = \sigma_1^{\otimes n}$
- Rate function: $\psi(t) := \log \mathrm{Tr}\, \rho_1^t \sigma_1^{1-t}, \quad \varphi(a) := \max_{0 \le t \le 1} \{at - \psi(t)\}$

Stein's lemma (H-Petz, 1991; Ogawa-Nagaoka, 2000)

$$\lim_{n \to \infty} \frac{1}{n} \log \min\{\beta_n(T_n) : \alpha_n(T_n) \le \varepsilon\} = -S(\rho_1, \sigma_1) \quad \text{for any } 0 < \varepsilon < 1.$$

Chernoff bound (Audenaert-Calsamiglia-et al., Nussbaum-Szkoła, 2006)

$$\lim_{n\to\infty} \frac{1}{n} \log P_{\mathrm{min}}(\rho_n : \sigma_n | \pi) = \min_{0\le t\le 1} \psi(t) = -\varphi(0)$$

Hoeffding bound (Hayashi, Nagaoka, 2006) For any $r \in \mathbb{R}$,

$$\inf_{(T_n)} \left\{ \limsup_n \frac{1}{n} \log \beta_n(T_n) : \limsup_n \frac{1}{n} \log \alpha_n(T_n) < -r \right\} = -\max_{0\le t<1} \frac{-tr - \psi(t)}{1-t}.$$

Results for non-i.i.d. case

H-Mosonyi-Ogawa

- Large deviations and Chernoff bound for certain correlated states on the spin chain, *J. Math. Phys.*

- Error exponents in hypothesis testing for correlated states on a spin chain, *J. Math. Phys.*

2. Free pressure and free entropy: microstate approach

• For $R > 0$, $(M_N^{sa})_R := \Big\{A \in M_N(\mathbb{C}) : A = A^*,\ \|A\| \le R\Big\}$

• Λ_N: the "Lebesgue" measure on $M_N^{sa} \cong \mathbb{R}^{N^2}$

• $\mathcal{A}_R^{(n)} := C([-R,R])^{\star n}$: the n-fold universal free product C^*-algebra, i.e., the C^*-completion of $\mathbb{C}\langle X_1,\dots,X_n\rangle$ w.r.t. the norm

$$\|p\|_R := \sup\Big\{\|p(A_1,\dots,A_n)\| : A_1,\dots,A_n \in (M_N^{sa})_R,\ N \in \mathbb{N}\Big\}$$

• $TS(\mathcal{A}_R^{(n)})$: the set of tracial states on $\mathcal{A}_R^{(n)}$

- Free entropy: for $\mu \in TS(\mathcal{A}_R^{(n)})$,

$$\chi_R(\mu) := \lim_{m\to\infty,\delta\searrow 0} \limsup_{N\to\infty} \left[\frac{1}{N^2} \log \Lambda_N^{\otimes n}(\Gamma_R(\mu; N, m, \delta)) + \frac{n}{2} \log N\right]$$

- Free pressure (free energy): for $h \in (\mathcal{A}_R^{(n)})^{sa}$ (considered as a free probabilistic potential),

$$\pi_R(h) := \limsup_{N\to\infty} \left[\frac{1}{N^2} \log \int_{(M_N^{sa})_R^n} \exp\Bigl(-N^2 \mathrm{tr}_N(h(A_1, \dots, A_n))\Bigr) \, d\Lambda_N^{\otimes n}(A_1, \dots, A_n) + \frac{n}{2} \log N\right]$$

- η-version of free entropy: for $\mu \in TS(\mathcal{A}_R^{(n)})$,

$$\eta_R(\mu) := \inf\Bigl\{\mu(h) + \pi_R(h) : h \in (\mathcal{A}_R^{(n)})^{sa}\Bigr\},$$

the (minus) Legendre transform of π_R.

• For every $h_0 \in (\mathcal{A}_R^{(n)})^{sa}$,

$$\pi_R(h_0) = \max\left\{-\mu(h_0) + \eta_R(\mu) : \mu \in TS(\mathcal{A}_R^{(n)})\right\}.$$

$\mu_0 \in TS(\mathcal{A}_R^{(n)})$ is called an equilibrium tracial state associated with h_0 if

$$\pi_R(h_0) = -\mu_0(h_0) + \eta_R(\mu_0).$$

Note An equilibrium tracial state exists for every $h \in (\mathcal{A}_R^{(n)})^{sa}$, and is unique for almost all $h \in (\mathcal{A}_R^{(n)})^{sa}$ (i.e., in a dense G_δ subset) by the Baire category theorem. But it is not easy to prove the uniqueness for a given h.

Fact

$$\eta_R(\mu) \geq \chi_R(\mu)$$

and equality holds if $X_1, \ldots, X_n$ are free w.r.t. μ.

3. Free analog of hypothesis testing – free Stein's lemma

- Micro Gibbs measure: for $h \in (\mathcal{A}_R^{(n)})^{sa}$ and $N \in \mathbb{N}$,

$$d\lambda_{R,N}^h(A_1,\dots,A_n) := \frac{1}{Z_{R,N}^h} \exp\Bigl(-N^2 \mathrm{tr}_N(h(A_1,\dots,A_n))\Bigr) \times \chi_{(M_N^{sa})_R^n}(A_1,\dots,A_n)\, d\Lambda_N^{\otimes n}(A_1,\dots,A_n)$$

with normalization constant $Z_{R,N}^h$.

- Micro pressure: for $h \in (\mathcal{A}_R^{(n)})^{sa}$,

$$P_{R,N}(h) := \log Z_{R,N}^h = \log \int_{(M_N^{sa})_R^n} \exp\Bigl(-N^2 \mathrm{tr}_N(h(A_1,\dots,A_n))\Bigr)\, d\Lambda_N^{\otimes n}(A_1,\dots,A_n)$$

- N-level tracial state: for each $h_0 \in (\mathcal{A}_R^{(n)})^{sa}$, $\mu_{R,N}^{h_0} \in TS(\mathcal{A}_R^{(n)})$ is defined by

$$\mu_{R,N}^{h_0}(h) := \int_{(M_N^{sa})_R^n} \mathrm{tr}_N(h(A_1,\dots,A_n))\, d\lambda_{R,N}^{h_0}(A_1,\dots,A_n) \quad \text{for } h \in \mathcal{A}_R^{(n)}.$$

Fact If limit exists in the definition of $\pi_R(h_0)$, i.e.,

$$\pi_R(h_0) = \lim_{N\to\infty}\left(\frac{1}{N^2}P_{R,N}(h_0) + \frac{n}{2}\log N\right),$$

then any limit point of $(\mu_{R,N}^{h_0})_{N\in\mathbb{N}}$ is an equilibrium tracial state associated with h_0.

Let $h_0, h_1 \in (\mathcal{A}_R^{(n)})^{sa}$ and consider the hypothesis testing for

$(\lambda_{R,N}^{h_0})_{N\in\mathbb{N}}$ (null-hypothesis) vs. $(\lambda_{R,N}^{h_1})_{N\in\mathbb{N}}$ (counter-hypothesis).

For a Borel subset (test) $T \subset (M_N^{sa})_R^n$,

$$\alpha_N(T) := \lambda_{R,N}^{h_0}(T^c), \quad \beta_N(T) := \lambda_{R,N}^{h_1}(T).$$

For the free Stein's lemma, define for $0 < \varepsilon < 1$

$$\beta_\varepsilon(\lambda_{R,N}^{h_1}\|\lambda_{R,N}^{h_0}) := \min\Big\{\lambda_{R,N}^{h_0}(T) : T \subset (M_N^{sa})_R^n,\ \lambda_{R,N}^{h_1}(T^c) \le \varepsilon\Big\}, \qquad N \in \mathbb{N},$$

$$\underline{B}((\lambda_{R,N}^{h_1})\|(\lambda_{R,N}^{h_0})) := \inf_{(T_N)}\left\{\liminf_{N\to\infty} \frac{1}{N^2}\log\lambda_{R,N}^{h_0}(T_N) : \lim_{N\to\infty}\lambda_{R,N}^{h_1}(T_N^c) = 0\right\},$$

$$\overline{B}((\lambda_{R,N}^{h_1})\|(\lambda_{R,N}^{h_0})) := \inf_{(T_N)}\left\{\limsup_{N\to\infty} \frac{1}{N^2}\log\lambda_{R,N}^{h_0}(T_N) : \lim_{N\to\infty}\lambda_{R,N}^{h_1}(T_N^c) = 0\right\},$$

$$B((\lambda_{R,N}^{h_1})\|(\lambda_{R,N}^{h_0})) := \inf_{(T_N)}\left\{\lim_{N\to\infty} \frac{1}{N^2}\log\lambda_{R,N}^{h_0}(T_N) : \lim_{N\to\infty}\lambda_{R,N}^{h_1}(T_N^c) = 0\right\}.$$

$$\begin{aligned}\sup_{\varepsilon>0}\left(\liminf_{N\to\infty}\frac{1}{N^2}\log\beta_\varepsilon(\lambda_{R,N}^{h_1}\|\lambda_{R,N}^{h_0})\right) &= \underline{B}((\lambda_{R,N}^{h_1})\|(\lambda_{R,N}^{h_0})) \\ \le \overline{B}((\lambda_{R,N}^{h_1})\|(\lambda_{R,N}^{h_0})) &= \sup_{\varepsilon>0}\left(\limsup_{N\to\infty}\frac{1}{N^2}\log\beta_\varepsilon(\lambda_{R,N}^{h_1}\|\lambda_{R,N}^{h_0})\right)\end{aligned}$$

Theorem Assume that there is a unique equilibrium tracial state μ_{h_1} associated with h_1. Then for every $0 < \varepsilon < 1$,

$$\limsup_{N\to\infty} \frac{1}{N^2} \log \beta_\varepsilon(\lambda_{R,N}^{h_1} \| \lambda_{R,N}^{h_0}) \geq \eta_R(\mu_{h_1}) - \mu_{h_1}(h_0) - \pi_R(h_0)$$
$$\geq \chi_R(\mu_{h_1}) - \mu_{h_1}(h_0) - \pi_R(h_0).$$

If, moreover, limit exists in the definition of $\pi_R(h_1)$, then for every $0 < \varepsilon < 1$,

$$\liminf_{N\to\infty} \frac{1}{N^2} \log \beta_\varepsilon(\lambda_{R,N}^{h_1} \| \lambda_{R,N}^{h_0}) \geq \eta_R(\mu_{h_1}) - \mu_{h_1}(h_0) - \pi_R(h_0).$$

Theorem Assume that limit exists in the definition of $\pi_R(h_1)$. Then for any limit point μ of $(\mu_{R,N}^{h_1})_{N\in\mathbb{N}}$,

$$\overline{B}((\lambda_{R,N}^{h_1}) \| (\lambda_{R,N}^{h_0})) \geq \eta_R(\mu) - \mu(h_0) - \pi_R(h_0).$$

Moreover, there exists a limit point μ_1 of $(\mu_{R,N}^{h_1})_{N\in\mathbb{N}}$ such that

$$\underline{B}((\lambda_{R,N}^{h_1}) \| (\lambda_{R,N}^{h_0})) \geq \eta_R(\mu_1) - \mu_1(h_0) - \pi_R(h_0).$$

In particular when $h_0 = 0$, the theorems give

Cor. Let $h \in (\mathcal{A}_R^{(n)})^{sa}$ and assume that there is a unique equilibrium tracial state μ_h associated with h. Then

$$\chi_R(\mu_h) \le \eta_R(\mu_h)$$
$$\le \limsup_{N\to\infty}\left[\frac{1}{N^2}\log\Bigl(\min\Bigl\{\Lambda_N^{\otimes n}(T) : T \subset (M_N^{sa})_R^n,\ \lambda_{R,N}^h(T^c) \le \varepsilon\Bigr\}\Bigr) + \frac{n}{2}\log N\right]$$

for every $0 < \varepsilon < 1$. If, moreover, limit exists in the definition of $\pi_R(h)$, then for every $0 < \varepsilon < 1$,

$$\eta_R(\mu_h)$$
$$\le \liminf_{N\to\infty}\left[\frac{1}{N^2}\log\Bigl(\min\Bigl\{\Lambda_N^{\otimes n}(T) : T \subset (M_N^{sa})_R^n,\ \lambda_{R,N}^h(T^c) \le \varepsilon\Bigr\}\Bigr) + \frac{n}{2}\log N\right].$$

Cor. Let $h \in (\mathcal{A}_R^{(n)})^{sa}$ and assume that limit exists in the definition of $\pi_R(h)$. Then for any limit point μ of $(\mu_{R,N}^h)_{N\in\mathbb{N}}$,

$$\eta_R(\mu) \\ \le \inf_{(T_N)}\left\{\limsup_{N\to\infty}\left(\frac{1}{N^2}\log\Lambda_N^{\otimes n}(T_N)+\frac{n}{2}\log N\right) : \lim_{N\to\infty}\lambda_{R,N}^h(T_N^c)=0\right\}.$$

Moreover, for some limit point μ_1 of $(\mu_{R,N}^h)_{N\in\mathbb{N}}$,

$$\eta_R(\mu_1) \\ \le \inf_{(T_N)}\left\{\liminf_{N\to\infty}\left(\frac{1}{N^2}\log\Lambda_N^{\otimes n}(T_N)+\frac{n}{2}\log N\right) : \lim_{N\to\infty}\lambda_{R,N}^h(T_N^c)=0\right\}.$$

Let $h_0 \in (\mathcal{A}_R^{(n)})^{sa}$. For each $(A_,\dots,A_n) \in (M_N^{sa})_R^n$ define

$$\mu_{N,(A_1,\dots,A_n)}(h) := \mathrm{tr}_N(h(A_1,\dots,A_n)), \qquad h \in \mathcal{A}_R^{(n)},$$

which is a random tracial state when $(A_1,\dots,A_n)$ is distributed under $\lambda_{R,N}^{h_0}$.

Fact

(1) If the random tracial state $\mu_{N,(A_1,\dots,A_n)}$ satisfies LDP in the scale N^{-2} with a good rate function having a unique minimizer μ_0, then $\mu_{N,(A_1,\dots,A_n)}$ weakly* converges to μ_0 almost surely and so $\lambda_{R,N}^{h_0}(\Gamma_R(\mu_0; N, m, \delta)) \to 1$ as $N \to \infty$ for every $m \in \mathbb{N}$ and $\delta > 0$.

(2) If $\lambda_{R,N}^{h_0}(\Gamma_R(\mu_0; N, m, \delta)) \to 1$ as $N \to \infty$ for every $m \in \mathbb{N}$ and $\delta > 0$, then $\mu_{R,N}^{h_0} \to \mu_0$ weakly* as $N \to \infty$.

Cor In addition to the assumption of (2), assume

(i) μ_0 is a unique equilibrium tracial state associated with h_0, or

(ii) limit exists in the definition of $\pi_R(h_0)$.

Then $\eta_R(\mu_0) = \chi_R(\mu_0)$. Moreover, in the case (ii), μ_0 is regular.

Guionnet – Maurel-Segala, 2006

Consider a potential $V_{\mathbf{t}} = \sum_{i=1}^{k} t_i q_i \in \mathbb{C}\langle X_1, \ldots, X_n\rangle^{sa}$ with monomials q_i. There exists an $\varepsilon > 0$ such that when $|\mathbf{t}| < \varepsilon$ the following hold:

(i) The Schwinger-Dyson equation $\mu \otimes \mu(\partial_i h) = \mu((D_i V_{\mathbf{t}} + X_i)h)$ has a unique solution $\mu_{\mathbf{t}}$ $(\in TS(\mathcal{A}_R^{(n)}))$.

(ii) Limit exists in the definition of $\pi_R(V_{\mathbf{t}})$.

(iii) $\mu_{N,(A_1,\ldots,A_N)}$, under $\lambda_{R,N}^{V_{\mathbf{t}}}$, weakly* converges to $\mu_{\mathbf{t}}$ almost surely.

(iv) $\mu_{\mathbf{t}}$ is regular.

Consequently, the above $\mu_{\mathbf{t}}$ is an equilibrium (unique?) tracial state associated with $V_{\mathbf{t}}$ and $\eta_R(\mu_{\mathbf{t}}) = \chi_R(\mu_{\mathbf{t}})$.

4. The single variable case

$\mathcal{A}_R^{(1)} = C([-R,R])$, $TS(\mathcal{A}_R^{(1)}) = \mathrm{Prob}([-R,R])$.

Fact For every $h \in C_{\mathbb{R}}([-R,R])$,

- $$\pi_R(h) = \lim_{N\to\infty}\left[\frac{1}{N^2}\int_{(M_N^{sa})_R} \exp\Big(-N^2 \mathrm{tr}_N(h(A))\Big)\, d\Lambda_N(A) + \frac{1}{2}\log N\right],$$

- $$\eta_R(\mu) = \chi_R(\mu) = \iint \log|x-y|\, d\mu(x)\, d\mu(y) + \frac{1}{2}\log 2\pi + \frac{3}{4},$$

- there exists a unique equilibrium measure $\mu_h \in \mathrm{Prob}([-R,R])$:

$$\pi_R(\mu_h) = -\mu_h(h) + \chi_R(\mu_h),$$

- π_R is Gâteaux differentiable and

$$\lim_{t\to 0}\frac{\pi_R(h+th') - \pi_R(h)}{t} = -\mu_h(h'), \quad h' \in C_{\mathbb{R}}([-R,R]).$$

Let $h_0, h_1 \in C_{\mathbb{R}}([-R, R])$.

Theorem (Chernoff bound) Define

$$\psi(s) := \pi_R(sh_1 + (1-s)h_0) - s\pi_R(h_1) - (1-s)\pi_R(h_0), \quad 0 \le s \le 1,$$
$$\varphi(a) := \sup_{0 \le s \le 1} \{as - \psi(s)\}, \quad a \in \mathbb{R}.$$

Then for every $a \in \mathbb{R}$ with $a \neq \psi'(0), \psi'(1)$,

$$\lim_{N\to\infty} \frac{1}{N^2} \log \min_{T \subset (M_N^{sa})_R} \left\{ e^{-N^2 a} \lambda_{R,N}^{h_1}(T^c) + \lambda_{R,N}^{h_0}(T) \right\} = -\varphi(a).$$

Fact

$$\psi'(0) = \chi_R(\mu_{h_0}) - \mu_{h_0}(h_1) - \pi_R(h_1),$$

$$\psi'(1) = -\chi_R(\mu_{h_1}) + \mu_{h_1}(h_0) + \pi_R(h_0),$$

$\psi'(1)$ is the relative free entropy of μ_{h_1} with respect to h_0 (or μ_{h_0}) introduced by **Biane-Speicher, 2001**.

For $r \in \mathbb{R}$ introduce

$$\begin{aligned}
&\underline{B}(r|\mu_{h_1}\|\mu_{h_0}) \\
&\quad := \inf_{(T_N)} \left\{ \liminf_{N\to\infty} \frac{1}{N^2} \log \lambda^{h_0}_{R,N}(T_N) : \limsup_{N\to\infty} \frac{1}{N^2} \log \lambda^{h_1}_{R,N}(T_N^c) < -r \right\}, \\
&\overline{B}(r|\mu_{h_1}\|\mu_{h_0}) \\
&\quad := \inf_{(T_N)} \left\{ \limsup_{N\to\infty} \frac{1}{N^2} \log \lambda^{h_0}_{R,N}(T_N) : \limsup_{N\to\infty} \frac{1}{N^2} \log \lambda^{h_1}_{R,N}(T_N^c) < -r \right\}, \\
&B(r|\mu_{h_1}\|\mu_{h_0}) \\
&\quad := \inf_{(T_N)} \left\{ \lim_{N\to\infty} \frac{1}{N^2} \log \lambda^{h_0}_{R,N}(T_N) : \limsup_{N\to\infty} \frac{1}{N^2} \log \lambda^{h_1}_{R,N}(T_N^c) < -r \right\}.
\end{aligned}$$

Theorem (Hoeffding bound) For any $r \in \mathbb{R}$,

$$\underline{B}(r|\mu_{h_1}\|\mu_{h_0}) = \overline{B}(r|\mu_{h_1}\|\mu_{h_0}) = B(r|\mu_{h_1}\|\mu_{h_0}) = -\sup_{0\le s\le 1} \frac{-sr - \psi(s)}{1-s}.$$

Theorem (free Stein's lemma) For every $0 < \varepsilon < 1$,

$$\begin{aligned}
&\lim_{N\to\infty} \frac{1}{N^2} \log \beta_\varepsilon(\lambda_{R,N}^{h_1} \| \lambda_{R,N}^{h_0}) \\
&\quad = \underline{B}(\mu_{h_1} \| \mu_{h_0}) = \overline{B}(\mu_{h_1} \| \mu_{h_0}) = B(\mu_{h_1} \| \mu_{h_0}) \\
&\quad = \underline{B}(0 | \mu_{h_1} \| \mu_{h_0}) = \overline{B}(0 | \mu_{h_1} \| \mu_{h_0}) = B(0 | \mu_{h_1} \| \mu_{h_0}) \\
&\quad = -\psi'(1) \\
&\quad = \chi_R(\mu_{h_1}) - \mu_{h_1}(h_0) - \pi_R(h_0).
\end{aligned}$$

In particular when $h_0 = 0$, the above theorem gives

Cor. For every $h \in C_{\mathbb{R}}([-R, R])$ and $0 < \varepsilon < 1$,

$$
\begin{aligned}
&\chi_R(\mu_h) \\
&\quad = \lim_{N\to\infty} \left[\frac{1}{N^2} \log\Bigl(\min\Bigl\{ \Lambda_N(T) : T \subset (M_N^{sa})_R,\ \lambda_{R,N}^h(T^c) \le \varepsilon \Bigr\} \Bigr) + \frac{1}{2} \log N \right] \\
&\quad = \inf_{(T_N)} \left\{ \liminf_{N\to\infty} \left(\frac{1}{N^2} \log \Lambda_N(T_N) + \frac{1}{2} \log N \right) : \lim_{N\to\infty} \lambda_{R,N}^h(T_N^c) = 0 \right\} \\
&\quad = \inf_{(T_N)} \left\{ \limsup_{N\to\infty} \left(\frac{1}{N^2} \log \Lambda_N(T_N) + \frac{1}{2} \log N \right) : \lim_{N\to\infty} \lambda_{R,N}^h(T_N^c) = 0 \right\} \\
&\quad = \inf_{(T_N)} \left\{ \lim_{N\to\infty} \left(\frac{1}{N^2} \log \Lambda_N(T_N) + \frac{1}{2} \log N \right) : \lim_{N\to\infty} \lambda_{R,N}^h(T_N^c) = 0 \right\}.
\end{aligned}
$$

Thank you for attention!