Polynomial families and Boolean probability

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Derivative: $(x^n)' = nx^{n-1}, \qquad x^0 = 1.$

1. Paul Appell 1880: Appell polynomials = "generalized powers"

$$A_n(x)' = nA_{n-1}(x), \qquad A_0(x) = 1$$

and

$$\int A_n(x)\, d\mu(x) = 0$$

for some probability measure μ.

Equivalently: X a random variable with distribution μ, denote A_n by A_n^X,

$$\mathbb{E}\left[A_n^X(X)\right] = 0.$$

Examples.

Hermite polynomials, $d\mu = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\, dx$,

Bernoulli polynomials, $d\mu = \mathbf{1}_{[0,1]}\, dx$.

2. Generating function:

$$\sum_{n=0}^{\infty} \frac{1}{n!} A_n(x) z^n = e^{xz-\ell(z)},$$

where

$$\ell(z) = \log \int e^{xz}\, d\mu(x).$$

3. Binomial property: if X, Y are independent random variables, then

$$A_n^{X+Y}(X+Y) = \sum_{k=0}^{n} \binom{n}{k} A_k^X(X) A_{n-k}^Y(Y)$$

(compare $(X+Y)^n$).

4. Martingale property: if $\{X_t\}$ is a Lévy process, i.e. a random process with stationary independent increments, then

$$\mathbb{E}\left[A_n^{X_t}(X_t) | \leq s\right] = A_n^{X_s}(X_s).$$

Connection with free probability.

Start with the difference quotient

$$\partial(f)(x,y) = \frac{f(x) - f(y)}{x - y}.$$

$$\partial(x^n) = \sum_{k=0}^{n-1} x^k y^{n-k-1}.$$

So define the free Appell polynomials by

$$\partial A_n(x,y) = \sum_{k=0}^{n-1} A_k(x) A_{n-k-1}(y), \qquad A_0(x) = 1$$

and

$$\int A_n(x)\, d\mu(x) = 0 \quad \text{or} \quad \mathbb{E}\left[A_n^X(X)\right] = 0.$$

Examples: Chebyshev polynomials, $d\mu = \frac{1}{2\pi}\sqrt{4-x^2}\,dx$.

2. Generating function:

$$\sum_{n=0}^{\infty} A_n(x) z^n = \frac{1}{1 - xz + zR(z)},$$

where $R(z) = R$-transform of μ.

3. Binomial property: if X, Y are freely independent random variables, then

$$\begin{aligned} A_n^{X+Y}(X+Y) = \sum A_{u(1)}^{X}(X) A_{u(2)}^{Y}(Y) A_{u(3)}^{X}(X) A_{u(4)}^{Y}(Y) \ldots \\ + \sum A_{v(1)}^{Y}(Y) A_{v(2)}^{X}(X) A_{v(3)}^{Y}(Y) A_{v(4)}^{X}(X) \ldots, \end{aligned}$$

where $u(1) + u(2) + \ldots = v(1) + v(2) + \ldots = n$.

Example.

$$
\begin{aligned}
A_3^{X+Y}(X+Y) = A_3^X(X) + A_2^X(X)A_1^Y(Y) + A_1^X(X)A_1^Y(Y)A_1^X(X) \\
+ A_1^Y(Y)A_2^X(X) + A_1^X(X)A_2^Y(Y) \\
+ A_1^Y(Y)A_1^X(X)A_1^Y(Y) + A_2^Y(Y)A_1^X(X) + A_3^Y(Y).
\end{aligned}
$$

(again compare $(X+Y)^n$).

4. Martingale property: if $\{X_t\}$ is a free Lévy process, i.e. a random process with stationary freely independent increments, then

$$
\mathbb{E}\left[A_n^{X_t}(X_t) | \leq s\right] = A_n^{X_s}(X_s).
$$

5. Polynomials with generating function

$$\sum_{n=0}^{\infty} P_n(x) z^n = \frac{1}{1 - xU(z) + U(z)R(U(z))}$$

for some $U(z)$ also martingales. Free Sheffer polynomials.

6. Free Meixner distributions = measures for which their orthogonal polynomials are free Sheffer (classical versions classified by Meixner 1934). In this case,

$$U(z) = R(z)^{\langle -1 \rangle}$$

and

$$\frac{R(z)}{z} = 1 + bR(z) + cR(z)^2.$$

Examples. Semicircular, Marchenko-Pastur, limit of Jacobi / double Wishart, arcsine, Kesten measures, Bernoulli distributions.

In today's talk: start with a very simple derivative operator

$$Df(x) = \frac{f(x) - f(0)}{x}.$$

The $q = 0$ version of the q-derivative operator

$$D_q f(x) = \frac{f(x) - f(qx)}{(1-q)x}.$$

$$D(x^n) = x^{n-1}, \qquad x^0 = 1.$$

So define the (Boolean) Appell polynomials by

$$DA_n(x) = A_{n-1}(x), \qquad A_0(x) = 1$$

and

$$\int A_n(x)\, d\mu(x) = 0 \quad \text{or} \quad \mathbb{E}\left[A_n^X(X)\right] = 0.$$

2. Generating function:

$$\sum_{n=0}^{\infty} A_n(x) z^n = \frac{1-\eta_\mu(z)}{1-xz}.$$

What is $\eta_\mu(z)$?

$$1 = \int \frac{1-\eta(1/z)}{1-x/z}\, d\mu = z \int \frac{1-\eta(1/z)}{z-x}\, d\mu = z(1-\eta(1/z))G_\mu(z).$$

So

$$\eta(1/z) = 1 - \frac{1}{zG(z)}.$$

This function appears in Boolean non-commutative probability theory.

$\mathcal{A}$ an algebra, φ a state on it. Non-unital subalgebras $\mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_k$ are Boolean independent if for $b_i \in \mathcal{B}_{u(i)}$, $u(1) \neq u(2) \neq \ldots \neq u(n)$,

$$\varphi[b_1 b_2 \ldots b_n] = \varphi[b_1]\varphi[b_2]\ldots\varphi[b_n].$$

Example. In $\mathbb{C}\langle x_1, x_2, \ldots, x_d \rangle$ with the state

$$\varphi\left[x_{u(1)} x_{u(2)} \ldots x_{u(n)}\right] = 0, \qquad \varphi[1] = 1,$$

$x_1, \ldots, x_d$ are freely independent.

In $\mathbb{C}\langle x_1, x_2, \ldots, x_d \rangle$ with the state

$$\varphi\left[x_{u(1)} x_{u(2)} \ldots x_{u(n)}\right] = e^{-n},$$

$x_1, \ldots, x_d$ are Boolean independent.

Combinatorics governed by the lattice of interval partitions, isomorphic to the Boolean lattice of subsets.

3. Binomial property: if X, Y are Boolean independent random variables, then

$$A_n^{X+Y}(X+Y) = A_n^X(X) + \sum_{k=1}^{n-1} (X+Y)^{k-1} Y A_{n-k}^X(X) + A_n^Y(Y) + \sum_{k=1}^{n-1} (X+Y)^{k-1} X A_{n-k}^Y(Y).$$

Example.

$$(X+Y)^3 = X^3 + YX^2 + (X+Y)YX + (X+Y)^2 Y + Y^3 + XY^2 + (X+Y)XY + (X+Y)^2 X.$$

4. Martingale property: if $\{X_t\}$ is a Boolean Lévy process, i.e. a random process with stationary Boolean independent increments, then

$$\mathbb{E}\left[A_n(X_t)| \leq s\right] = A_n(X_s).$$

Boolean states typically not tracial, so this does not immediately imply the Markov property; known due to Franz 2003.

5. Boolean Sheffer polynomials

$$\sum_{n=0}^{\infty} P_n(x) z^n = \frac{1 - \eta(V(z))}{1 - xV(z)}.$$

Proposition. These are the same as free:

$$\frac{1}{1 - xU(z) + U(z)R(U(z))} = \frac{1 - \eta(V(z))}{1 - xV(z)},$$

where

$$V(z) = \Big(1 + U(z)R(U(z))\Big)^{-1} U(z).$$

Remark. Everything works in the multivariate situation. Start with "left" partial derivatives $D_1, D_2, \ldots, D_d$,

$$D_i(x_j x_{u(1)} x_{u(2)} \ldots x_{u(k)}) = \delta_{ij} x_{u(1)} x_{u(2)} \ldots x_{u(k)}$$

6. Corollary. Boolean Meixner distributions = free Meixner distributions.

Moreover,

$$V(z) = (D\eta(z))^{\langle -1 \rangle}$$

and

$$D^2\eta(z) = 1 + bD\eta(z) + (1+c)(D\eta(z))^2.$$

Recall

$$D^2(zR(z)) = 1 + bD(zR(z)) + c(D(zR(z)))^2$$

and

$$\ell(z)'' = 1 + \beta\ell(z)' + \gamma(\ell(z)')^2.$$

Bercovici, Pata: there are bijections between infinitely divisible, freely infinitely divisible, Boolean infinitely divisible distributions.

$$\ell_\mu(z) \quad = \quad z R_\nu(z) \quad = \quad \eta_\zeta(z),$$

$$\mu \quad \leftrightarrow \quad \nu \quad \leftrightarrow \quad \zeta,$$

Gaussian $\leftrightarrow$ Semicircular $\leftrightarrow$ Symmetric Bernoulli,

Poisson $\leftrightarrow$ Marchenko-Pastur $\leftrightarrow$ Asymmetric Bernoulli.

Does not take classical Meixner to free Meixner.

Takes free Meixner to Boolean Meixner: $\mu_{b,c} \mapsto \mu_{b,1+c}$.

More general results on the behavior under the Belinschi-Nica transformation.

Again, this is all true in the multi-variable case.

If μ is a Meixner distribution, the orthogonal polynomials for μ^{*t} satisfy recursion relations

$$xP_n(x) = P_{n+1}(x) + (t\beta_0 + nb)P_n(x) + n(t\gamma_1 + (n-1)c)P_{n-1}.$$

If μ is a free / Boolean Meixner distribution, the orthogonal polynomials for $\mu^{\boxplus t}$ satisfy recursion relations

$$\begin{aligned} xP_0(x) &= P_1(x) + t\beta_0 P_0(x), \\ xP_1(x) &= P_2(x) + (t\beta_0 + b)P_1(x) + t\gamma_1 P_0, \\ xP_n(x) &= P_{n+1}(x) + (t\beta_0 + b)P_n(x) + (t\gamma_1 + c)P_{n-1}. \end{aligned}$$

In contrast, if μ is any distribution, the orthogonal polynomials for $\mu^{\uplus t}$ satisfy recursion relations

$$\begin{aligned} xP_0(x) &= P_1(x) + t\beta_0 P_0(x), \\ xP_1(x) &= P_2(x) + \beta_1 P_1(x) + t\gamma_1 P_0, \\ xP_n(x) &= P_{n+1}(x) + \beta_n P_n(x) + \gamma_n P_{n-1}. \end{aligned}$$

Proof using (multivariate) continued fractions.

$\varphi =$ state (with monic orthogonal polynomials).

$$1 + M(\mathbf{z}) = 1 + \sum_i \varphi[x_i] z_i + \sum_{i,j} \varphi\left[x_i x_j\right] z_i z_j + \ldots$$

its moment generating function.

Stieltjes continued fraction: one-variable case.

$$\begin{aligned} 1 + M(z) &= 1 + \varphi[x] z + \varphi\left[x^2\right] z^2 + \ldots \\ &= \cfrac{1}{1 - \alpha_0 z - \cfrac{\omega_1 z^2}{1 - \alpha_1 z - \cfrac{\omega_2 z^2}{1 - \alpha_2 z - \cfrac{\omega_3 z^2}{1 - \ldots}}}}. \end{aligned}$$

Proposition. For $k = 1, 2, \ldots$, there are diagonal non-negative $d^k \times d^k$ matrices $\mathcal{C}^{(k)}$ and Hermitian matrices $\mathcal{T}_i^{(k)}$, such that

$$1 + M(\mathbf{z}) =$$

$$\frac{1}{1 - \sum_{i_0} z_{i_0} \mathcal{T}_{i_0}^{(0)} - \frac{\sum_{j_1} z_{j_1} E_{j_1} \mathcal{C}^{(1)} | \sum_{k_1} E_{k_1} z_{k_1}}{1 - \sum_{i_1} z_{i_1} \mathcal{T}_{i_1}^{(1)} - \frac{\sum_{j_2} z_{j_2} E_{j_2} \mathcal{C}^{(2)} | \sum_{k_2} E_{k_2} z_{k_2}}{1 - \sum_{i_2} z_{i_2} \mathcal{T}_{i_2}^{(2)} - \frac{\sum_{j_3} z_{j_3} E_{j_3} \mathcal{C}^{(3)} | \sum_{k_3} E_{k_3}}{1 - \ldots}}}}$$

$$M_{d^2 \times d^2} = M_{d \times d} \otimes M_{d \times d}.$$

LAHA-LUKACS PROPERTY.

Proposition. Suppose X, Y are (appropriately) independent, self-adjoint, non-degenerate and there are numbers $\alpha, \alpha_0, C, a, b \in \mathbb{R}$ such that

$$\varphi[X|X+Y] = \alpha(X+Y) + \alpha_0$$

and

$$\mathrm{Var}[X|X+Y] = C\Big(1 + a(X+Y) + b(X+Y)^2\Big).$$

X, Y independent $\Rightarrow$ Meixner (Laha, Lukacs).

X, Y freely independent $\Rightarrow$ free Meixner (Bożejko, Bryc).

X, Y Boolean independent $\Rightarrow$ Bernoulli.