“Special Structures in Riemannian Geometry”

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1 Overview of the Field

Special structures often arise naturally in Riemannian geometry. They are usually given by the existence of globally defined tensors satisfying some (usually elliptic, almost always non-linear) partial differential equations.

One important type of structure is a special curvature condition on the Riemannian metric itself. The most important class of such structures is that of constant Ricci-curvature, or Einstein metrics. Such metrics are often the best candidates for the role of canonical metrics on manifolds, and thus play a very important role in topology. They have also long been a subject of intense interest in theoretical physics, initially in general relativity but more recently in superstring theories as well. Other typical examples in this setting are constant scalar curvature metrics, conformally flat metrics, and others. (The two best survey volumes on Einstein manifolds are [**?**] and [**?**].)

Sometimes a Riemannian manifold admits a globally defined tensor which is parallel with respect to the induced Levi-Civita connection. In this case the Riemannian holonomy reduces. For instance, a manifold admits a parallel almost complex structure if and only if it is a Kahler manifold. Manifolds with special holonomy give examples of Einstein metrics. Calabi-Yau, hyperKahler, G_2, and $Spin(7)$ manifolds are always Ricci-flat, while quaternionic-Kahler manifolds are positive Einstein. The special holonomy condition is a first-order reduction of the second-order Einstein condition. (See [**?**] for background on Riemannian holonomy.)

Another example of special structures are those isometrically immersed submanifolds which themselves satisfy some non-linear condition on their second fundamental forms, such as minimal submanifolds, CMC (constant mean curvature) submanifolds, or more complicated fully non-linear relations among the eigenvalues of the second fundamental form. The notion of a calibration on a Riemannian manifold fits into this category. The associated calibrated submanifolds are always minimizing, although this is a reduction to a first order equation, similar to the special holonomy condition. ([**?**] is a good source on calibrated manifolds and their relation to special holonomy.) The interesting calibrations seem to be in one-to-one correspondence with the special holonomies, a relationship which is still not fully understood.

Finally, vector bundles over Riemannian manifolds can admit connections with special properties. An important example is that of Yang-Mills connections, of which anti-self dual connections are a special case. Many of the most spectacular results in low-dimensional topology and geometry from the last twenty years has have arisen via the study of these gauge theories. Recently, analogues of anti-self dual instantons in the context of G_2 and $Spin(7)$ geometry have been discovered, the Donaldson-Thomas connections. Much work remains to be done in this area.

It is always of interest to find explicit examples of special structures, and usually easier to do so on non-compact manifolds. Often, we can even specify the asymptotic behaviour of such a structure. For the cases of exceptional holonomy, for example, there are several explicit examples of such metrics of cohomogeneity one on non-compact manifolds, due to Bryant, Salamon, and the physicists Gibbons-Page-Pope, and many others([**?**]). It would be interesting to find explicit examples of higher cohomogeneity. In the case of general Einstein metrics, the work of Graham-Lee and Anderson(see the survey article [**?**],) among others, has clarified the structure of Einstein metrics which are asymptotic (in a precise sense) to real hyperbolic space. Biquard has extended some of the existence and uniqueness results to the case of Einstein metrics which are asymptotic to other rank one symmetric spaces([**?**]), but there remain many interesting questions. Explicit constructions of calibrated submanifolds have been found by Harvey-Lawson, Bryant, Joyce, and many others(c.f. [**?**]). These also tend to have high degrees of symmetry, reducing the complexity of the equations to something which is exactly solvable. The anti-self-dual connections on $\mathbb{R}^4$ were completely classified by Atiyah-Drinfeld-Hitchin-Manin and there has also been more recent work by Hurtubise and others for the cases of $S^k \times \mathbb{R}^{4-k}$.

On compact manifolds, the elliptic, fully non-linear partial differential equations which characterize such special structures are much more difficult to solve. Usually the best we can hope for are non-constructive existence results involving very hard analysis. For instance, Yau's solution of the Calabi conjecture to prove the existence of Ricci-flat Kahler metrics on compact Kahler manifolds with trivial canonical bundle([**?**]). The existence of compact examples of G_2 and $Spin(7)$ manifolds was first shown by Joyce([**?**]). A general theorem similar to the Calabi-Yau theorem for classifying which 7 or 8 dimensional manifolds admit such Ricci-flat metrics is still missing. In fact, there are not even any obstructions in dimensions greater than 4 to manifolds admitting ordinary Einstein metrics([**?**]).

Manifolds with special holonomy exhibit the phenomenon of mirror symmetry, which is better understood in the hyperKahler and Calabi-Yau cases, but which for the exceptional cases is at present still mysterious. It is also expected (Strominger-Yau-Zaslow, Gukov-Yau-Zaslow) that understanding mirror symmetry will involve studying the moduli spaces of calibrated submanifolds that these manifolds possess, as well as the moduli spaces of ASD connections or their exceptional holonomy analogues. It seems that at least in some cases, these three categories of 'special structures' are all interrelated in some way whose precise formulation is still far from clear.

Another important aspect is the role of spinors and Dirac operators in these settings. Spin geometry seems to be natural for describing many of these structures. For example, the Ricci-flat manifolds that have special holonomy admit parallel spinors, but there also exist classes of Einstein metrics which have a form of 'weak' holonomy, and admit non-trivial Killing spinors. Similarly there exist characterizations of minimal or CMC submanifolds in certain cases using spinors.

Finally, it is important to note that all of these problems involve similar analytic issues. The partial differential equations are all elliptic, often with similar types of non-linearity. Many techniques used for minimal or CMC surfaces, for example, have also been applied to Yamabe metrics, and then to Einstein metrics (such as gluing techniques or understanding of the structure of spaces of conformally compact Einstein metrics; compare for example the gluing constructions in [**?**], [**?**] and [**?**].) Differential geometers are finding more and more connections between these research areas all the time. Therefore it is vital to bring together researchers from the individual areas who up to now may not be familiar with the techniques and results from the other areas.

2 Objectives of the Workshop

Our main goal was to foster new research collaborations by bringing together mathematicians studying special metrics and their special submanifolds with those working in the area of gauge theory to promote cross-fertilization of their different techniques and approaches. Specifically we hoped that many analytic techniques which have been useful in each individual area should prove to be applicable to the other area as well. These research groups have not worked together enough, and through this workshop we hoped to encourage more interaction.

Additionally, most of these special geometric structures play a central role in general relativity and superstring theories. We hoped that by bringing in some physicists, as well as the participation of mathematical physicists, there would be a further opportunities for discussion and collaboration. Theoretical physics has proven time and again to dictate many of the directions of research in differential geometry and global analysis.

It was very important for us that the event be a workshop, and not just a conference. Our hope was that there would be a significant number of collaborations arising from the workshop. To this end, we arranged the schedule so that there would be a fair number of lectures, but also a large amount of open time, with the specific goal of encouraging informal discussions during this time. In the lectures, experts gave talks discussing their recent research, thereby allowing participants working in other fields to get an idea of the state of the given disciplines. Our hope was that these lectures would serve as the launching points for the informal discussions. We also organised a problem session, allowing experts to share the main open problems in their fields with colleagues.

Additionally, we made an effort to invite a significant number of young researchers, both graduate students and recent Ph.D.s, to the meeting in order that they could benefit from contact with more senior scholars.

3 Overview of meeting

As we hoped, there were a large number of informal discussions, encouraged both by our scheduling large blocks of open time in between talks and by the excellent facilities at BIRS. Of course, it is in the nature of informal discussion that we don't have any records of them.

As for the talks, here are the abstracts, in alphabetical order by speaker surname:

Speaker: **Michael Anderson** (SUNY Stony Brook)
Title: "Spaces of Einstein metrics on bounded domains"

Abstract: There are two natural classes of Einstein metrics on bounded domains. (I) metrics which extend smoothly to the boundary, and (II) complete metrics which conformally extend to the boundary, (conformally compact metrics). We will discuss similarities and differences on the structure of these spaces of Einstein metrics, in particular in regard to the "natural" boundary value problems.

Speaker: **Adrian Butscher** (Stanford University)
Title: "Gluing Constructions for Constant Mean Curvature Surfaces"

Abstract: I will review the now classical Kapouleas gluing construction for CMC surfaces in Euclidean space and present some results and work in progress concerning the extensions of this theory to general ambient manifolds. An important feature which emerges is that the ambient Riemannian curvature seems to play a significant role in the existence of such surfaces; and exploiting this, it seems possible to construct examples of CMC surfaces having properties very different from their Euclidean analogues.

Speaker: **Benoit Charbonneau** (Duke University)
Title: "Existence of periodic instantons"

Abstract: Yang–Mills instantons on $S^1 \times \mathbb{R}^3$ (often called calorons) are in correspondence, via the Nahm transform, to solutions to Nahm's equations on the circle. In joint work with Jacques Hurtubise, we completed Nye and Singer's proof of this Nahm transform correspondence.

We also proved that the solutions on the circle are in correspondence, by a twistor transform, to certain classes of vector bundles on an associated twistor space. Those correspondence allow us to compute the moduli space of these objects, settling some very natural existence questions.

Speaker: **Andrew Dancer** (Jesus College, University of Oxford)
Title: "Symplectic versus hyperKähler geometry"

Abstract: We look at symplectic constructions such as cutting and implosion and investigate their analogues in hyperKähler geometry.

Speaker: **Robin Graham** (University of Washington)
Title: "The ambient metric beyond the obstruction in even dimensions"

Abstract: The ambient metric construction in conformal geometry will be reviewed, and also the equivalent formal construction of asymptotically hyperbolic Poincaré-Einstein metrics. A modification of the construction in even dimensions will be described which results in a family of smooth infinite order generalized ambient and Poincaré metrics. The generalized ambient metrics can be used to extend conformal invariant theory to all orders in even dimensions. This is joint work with Kengo Hirachi.

Speaker: **Marco Gualtieri** (Massachusetts Institute of Technology)
Title: "Constructions of generalized Kähler structures"

Abstract: I will describe a construction of generalized Kähler structures on holomorphic Poisson manifolds, which uses the concept of a holomorphic Poisson module. I will also describe some properties of the resulting generalized Kähler metric, which is a Riemannian metric admitting two different Hermitian complex structures.

Speaker: **Sergei Gukov** (University of California, Santa Barbara)
Title: "Deformations of Hyper-Kähler Metrics and Affine Hecke Algebras"

Abstract: I will explain how studying deformations of hyper-Kähler metrics on complex coadjoint orbits can provide a simple geometric explanation of certain deep results in representation theory, including categorification of the affine Hecke algebra. This talk is based on a joint work with Edward Witten.

Speaker: **Mark Haskins** (Imperial College London)
Title: "Gluing constructions of special Lagrangians"

Abstract: We describe joint work with Nicos Kapouleas that constructs infinitely many special Lagrangian cones whose link is an orientable surface of genus 4 or of any odd genus. These are the first special Lagrangian cones with links that are surfaces of genus greater than one. We use a geometric PDE 'gluing' method. Time permitting, we will sketch higher dimensional generalisations of these gluing constructions.

Speaker: **Marianty Ionel** (University of Toledo)
Title: "Austere submanifolds of dimension 4"

Abstract: An austere submanifold has the property that its second fundamental form in any normal direction has its eigenvalues symmetrically arranged around zero. The class of austere submanifolds was first introduced by Harvey and Lawson in 1982. The main motivation was their result showing that the conormal bundle of an austere submanifold in $\mathbb{R}^n$ is a special Lagrangian submanifold of $\mathbb{R}^{2n}$. The austere submanifolds of dimension 3 in Euclidean space were classified by R. Bryant. In this talk I will present some results towards a classification of austere submanifolds of dimension 4 in Euclidean space. Depending on the type of the second fundamental form, we get both non-existence results as well as new examples of austere submanifolds. This is joint work with Thomas Ivey.

Speaker: **Jim Isenberg** (University of Oregon)
Title: "Constructing solutions of the Einstein constraint equations"

Abstract: The first step in finding a spacetime solution to the Einstein gravitational field equations via the inital value formulation is to construct initial data which satisfy the Einstein constraint equations. There are three ways of carrying out this construction which have been found to be useful: the conformal and conformal thin sandwich methods, the gluing techniques, and the quasi-spherical approaches. We describe each of these, we discuss their advantages and disadvantages, we outline some of their recent successful applications, and we present some of the outstanding questions remaining to be solved from each of these perspectives.

Speaker: **John Loftin** (Rutgers University Newark)
Title: "Affine Hermitian-Einstein Metrics"

Abstract: A special affine manifold is a manifold with an atlas whose gluing maps are all constant affine maps in $\mathbb{R}^n$ preserving the standard volume form. The tangent bundle to a special affine manifold has the structure of a complex manifold with holomorphic volume form. We develop a theory of stable bundles and affine Hermitian-Einstein metrics for flat vector bundles over a special affine manifold. The proof involves adapting the proof of Uhlenbeck-Yau on the existence of Hermitian-Einstein metrics on Kähler manifolds, and the extension of this theorem by Li-Yau to the non-Kähler complex case of Gauduchon metrics. Our definition of stability involves only flat vector subbundles (and not singular subsheaves), and so is simpler than the complex case in some places.

Speaker: **Dan Pollack** (University of Washington)
Title: "Singular Yamabe metrics and Space-times with Positive Cosmological Constant"

Abstract: The Delaunay (aka Fowler) metrics form the asymptotic models for isolated singularities of conformally flat metrics of constant positive scalar curvature metrics. The Kottler-Schwarzschild-de Sitter spacetimes form the model family for black hole solutions of the Einstein field equations with a positive cosmological constant. We will show why the former coincides with the time-symmetric initial data sets for that latter. We will then demonstrate how to construct large families of initial data sets for the vacuum Einstein equations with positive cosmological constant which contain exactly Delaunay ends; these are non-trivial initial data sets whose ends coincide with those for the Kottler-Schwarzschild-de Sitter metrics. From the purely Riemannian geometric point of view, this produces complete, constant positive scalar curvature metrics with exact Delaunay ends which are not globally Delaunay. The construction provided applies to more general situations where the asymptotic geometry may have non-spherical cross-sections consisting of Einstein metrics with positive scalar curvature. This is joint work with Piotr Chrusciel.

Speaker: **Martin Reiris** (Massachusetts Institute of Technology)
Title: "The Einstein flow and the Yamabe invariant of three-manifolds"

Abstract: We will explain how the long time evolution of the Einstein flow is related with the Yamabe invariant of three-manifolds. We will set the main conjectures and elaborate on partial results. The discussion will be based on the importance of volume in General Relativity.

Speaker: **Andrew Swann** (University of Southern Denmark)
Title: "Intrinsic Torsion and Curvature"

Abstract: For a Riemannian G-structure a large part of the Riemannian curvature is determined by the intrinsic torsion. Representation theoretic techniques lead to a number of constraints and relations and in many situations much can be gleaned from the exterior algebra. This talk will discuss recent work in this area, including particular results for almost Hermitian and almost quaternion-Hermitian structures.

Speaker: **Christina Tønnesen-Friedman** (Union College)
Title: “Hamiltonian 2-forms in Kähler geometry”

Abstract: Hamiltonian 2-forms, introduced in [1], induce isometric Hamiltonian torus actions and underpin many explicit constructions in Kähler geometry. This talk will take off as a survey and discussion of the techniques developed in [1] and [2]. Some of these techniques have already been applied in subsequent works with my co-authors, but we believe that there are still exciting avenues to take – in particular in the case of higher order Hamiltonian 2-forms and, in general, higher order toric bundles.
[1] “Hamiltonian 2-forms in Kähler geometry, I General Theory”; V. Apostolov, D.M.J. Calderbank, and P. Gauduchon; *J. Diff. Geom.*, **73** (2006), 359–412.
[2] “Hamiltonian 2-forms in Kähler geometry, II Global Classifications”; V. Apostolov, D.M.J. Calderbank, P. Gauduchon , and C. Tønnesen-Friedman; *J. Diff. Geom.*, **68** (2004), 277–345.

Speaker: **Guofang Wei** (University of California at Santa Barbara)
Title: “Comparison Geometry for the Smooth Metric Measure Spaces”

Abstract: For a smooth metric measure space $(M, g, e^{-f}dvol_g)$ the Bakry-Emery Ricci tensor is a natural generalization of the classical Ricci tensor. It occurs naturally in the study of diffusion processes, Ricci flow, the Sobolev inequality, warped products, and conformal geometry. We prove mean curvature and volume comparison results when the ∞-Bakry-Emery Ricci tensor is bounded from below and f is bounded or $\partial_r f$ is bounded from below, generalizing the classical ones (i.e. when f is constant.) This leads to extensions of many theorems for Ricci curvature bounded below to the Bakry-Emery Ricci tensor. In particular, we give extensions of all of the major comparison theorems when f is bounded. Simple examples show the bound on f is necessary for these results. This is a joint work with W. Wylie.

Here is a list of the problems suggested during the open problem sessions:

First Problem: Conan Leung asked the following question related to G_2-manifolds: Let L be a special Lagrangian submanifold of $\mathbb{C}^3$, or more generally of some Calabi-Yau 3-fold X. Does there exist a G_2-manifold M^7 such that there exists a map $\pi : M^7 \longrightarrow \mathbb{C}^3$ with the property that $\pi^{-1}(x) = S^1$ if x is a point not lying on L, and $\pi^{-1}(x)$ is a point if $x \in L$. The motivation for this problem is that given a Calabi-Yau 3-fold X, closed string theories predict a duality between $X^6 \times \mathbb{R}^{3,1}$ and $X \times S^1 \times \mathbb{R}^{3,1}$, and $X^3 \times S^1$ admits a canonical G_2-structure. In the case of an open string theory, the string boundary in the Calabi-Yau manifold X would lie in a special Lagrangian submanifold L, which explains the above condition on π.

Second Problem: Benoit Charbonneau presented a conjecture of Jardim. Consider an anti-self dual(ASD) connection A on a $SU(n)$-bundle E over $\mathbb{R}^2 \times T^2$. Jardim has conjectured that if F_A, the curvature of A, is in L^2, then F_A must be $O(r^{-2})$ or perhaps even $o(r^{-2})$.

The motivation for this arises from the Nahm transform

$$\mathcal{N} : \mathcal{M}_{ASD/QD}\left(\mathbb{R}^2 \times T^2\right) \longrightarrow \mathcal{M}_{SHP}\left(T^2\right),$$

where $\mathcal{M}_{ASD/QD}\left(\mathbb{R}^2 \times T^2\right)$ is the space of ASD connections with quadratic curvature decay on $\mathbb{R}^2 \times T^2$, and $\mathcal{M}_{SHP}\left(T^2\right)$ is the space of singular Hitchin pairs on T^2. It turns out that if you start off with A with non-trivial limit at infinity and satisfying $F_A = O(r^{-1-\epsilon})$ for small ϵ, then $\mathcal{N}(A)$ exists, and $\mathcal{N}^{-1}(\mathcal{N}(A))$ has quadratic decay. Moreover, being L^2 is the usual condition on curvature for the domain of the Nahm transform on $\mathbb{R}^4$ and other spaces.

Third Problem: Spiro Karigiannis suggested the following problem: let M be a compact spin manifold, and let $\not{D}$ be its associated Dirac operator. Do there exist natural flows $\frac{\partial s}{\partial t} = \not{D}s$? This is a spinor analogue of the heat flow on forms. This question is motivated by the fact that such a flow exists on manifolds with special holonomy, with $\not{D}$ being replaced by another first-order operator which is similar to a Dirac operator.

Since this operator is first-order, there is no maximum principle available to understand this flow. Studying the above flow on spin manifolds could allow a better understanding of these more complicated flows on manifolds with special holonomy. Here is an example of one of these flows. It gives an impression of how complicated they are, and of how they resemble Dirac operators. Consider $\mathbb{R}^7$ with its standard G_2 structure, and let φ be the associated parallel three-form. Let $\psi = *\varphi$. Then an oriented three-dimensional submanifold $L \subset \mathbb{R}^7$ is associative iff the restriction of φ to L is the volume form of L. This is in turn equivalent (up to a change of orientation) to the vanishing of χ on L, where $\chi \in \Lambda^3(T^*) \otimes T$ is given by $\chi^i_{klm} = g^{ij}\psi_{jklm}$. Now consider an embedded 3-dimensional submanifold $f : U \longrightarrow \mathbb{R}^7$ where $U \subset \mathbb{R}^3$. Then we obtain the flow

$$\frac{\partial f}{\partial t} = \chi\left(\frac{\partial f}{\partial u_1}, \frac{\partial f}{\partial u_2}, \frac{\partial f}{\partial u_3}\right).$$

(Note that $\chi|_{T(f(U))}$ is a normal vector field to the submanifold.) In the special case where $L^3 = \Sigma^2 \times \mathbb{R} \subset \mathbb{C}^3 \times \mathbb{R}$, the L^3 is associative iff $\Sigma^2 \subset \mathbb{C}^3$ is a complex curve. Then we can simplify the embedding above to $h : V \longrightarrow \mathbb{C}^3$, where $V \subset R^2$, and the corresponding flow also reduces to

$$\frac{\partial h}{\partial t} = \Omega\left(\frac{\partial h}{\partial u_1}, \frac{\partial h}{\partial u_2}\right),$$

where $\Omega = dz^1 \wedge dz^2 \wedge dz^3$ is the canonical holomorphic volume form on $\mathbb{C}^3$. If we write out $h(u_1, u_2) = (z^1(u_1, u_2), z^2(u_1, u_2), z^3(u_1, u_2))$, then in coordinates the flow looks like:

$$\frac{\partial z^k}{\partial t} = (\text{Cauchy Riemann Eq}) \cdot (\text{Other first-order terms}).$$

Fourth Problem: The next question was proposed by Dan Pollack. Consider complete metrics of the form

$$g = u^{\frac{4}{n-2}}(t, \theta)(dt^2 + d\theta^2)$$

on $\mathbb{R} \times S^{n-1}$ with constant scalar curvature $R(g) = n(n-1)$. It follows from the work of Gidas, Ni and Nirenberg, that all solutions are independent of θ and that $u = u(t)$ must satisfy an explicit ODE. This ODE has a one-parameter family of positive periodic solutions which corresponds to the set of ”Delaunay metrics”. Now, say that we replace S^{n-1} in the warped product above by some other Einstein manifold M^{n-1} with scalar curvature $R(g) = (n-1)(n-2)$, and we look once again the periodic solutions to the associated ODE. They are of course the same. What can we say about the possibly more general set of complete metrics with $R(g) = n(n-1)$ which are conformal to this product metric? For example, what is the dimension of the space of solutions? In the case of Delaunay metrics, as the parameter ϵ tends to 0, the metric degenerates to the incomplete metric on the sphere minus two antipodal points. In the case of more general M, the degenerations may be more complicated due to the topology.

One of the reasons for studying Delaunay metrics is that the ends of the complete constant scalar curvature metrics constructed by Schoen, Mazzeo and Pacard, and Mazzeo, Pollack and Uhlenbeck on $S^n - \{p_1, ..., p_n\}$ are asymptotically Delaunay. Moreover this behavior must hold in general about any isolated non-removable singular point (at least in the conformally flat case, or in low dimensions). It would be very interesting to know whether there are manifolds with constant positive scalar curvature whose ends are (asymptotically or exactly) generalized Delaunay, in the above sense, but are not diffeomorphic to the product manifold. The construction of such an example would probably involve a gluing argument.

Fifth Problem: The session ended with a conjecture from Robin Graham: suppose $I(g)$ is a scalar conformal invariant of weight $-n$ on an n-dimensional manifold, with n even. Further suppose that $I(g)$ can be written as a linear combination of contractions of covariant derivatives of the Ricci tensor. Then $I(g) \equiv 0$.

The background for this conjecture is the following: let $I(g)$ be a scalar Riemannian invariant which is a linear combination of contractions of covariant derivatives of the curvature tensor. If for every conformal rescaling $\hat{g} = \Omega^2 g$, we have

$$I(\hat{g}) = \Omega^w I(g),$$

then we say that I is a conformal invariant of weight w. For example, W, the Weyl tensor, is a pointwise conformal invariant, so its norm squared $\|W\|^2$ is a conformal invariant of weight -4. If $w = -n$, then

$$\int_M I(g)\, dV$$

is a global conformal invariant, since the scaling of the volume form under a conformal transformation is cancelled out by that of $I(g)$.

4 Outcome of the Meeting

As we mentioned above, there were many informal discussions, in particular a great deal of exchanges between experts in different fields. Although it is too early to say exactly what collaborations will arise from this workshop, we are confident that the meeting allowed a significant cross-fertilization between fields, and in particular allowed several of the younger participants to advance their research programme thanks to the advice of more senior scholars.

All of the participants were very enthusiastic about BIRS: the natural setting, the infrastructure and the warmth, hospitality and professionalism of the staff were all very much appreciated.

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