Discrete integrable systems in projective geometry

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The notion of integrability is one of the central notions in mathematics. Starting from Euler and Jacobi, the theory of integrable systems is among the most remarkable applications of geometric ideas to mathematics and physics in general.

Discrete integrable systems is a new and actively developing subject, hundreds of new articles in this field are written every year by mathematicians and physicists. However, geometric interpretation of most of the discrete integrable systems considered in the mathematical and physical literature is unclear.

The main purpose of this Workshop was to study one particular dynamical system called the *pentagram map*. The interest in this map is motivated by its natural geometric meaning and aestetical attractiveness. The pentagram map was introduced in [2], and further studied in [3] and [4]. Originally, the map was defined for convex closed n-gons. Given such an n-gon P, the corresponding n-gon $T(P)$ is the convex hull of the intersection points of consequtive shortest diagonals of P. Figure 1 shows the situation for a convex pentagon and a convex hexagon.

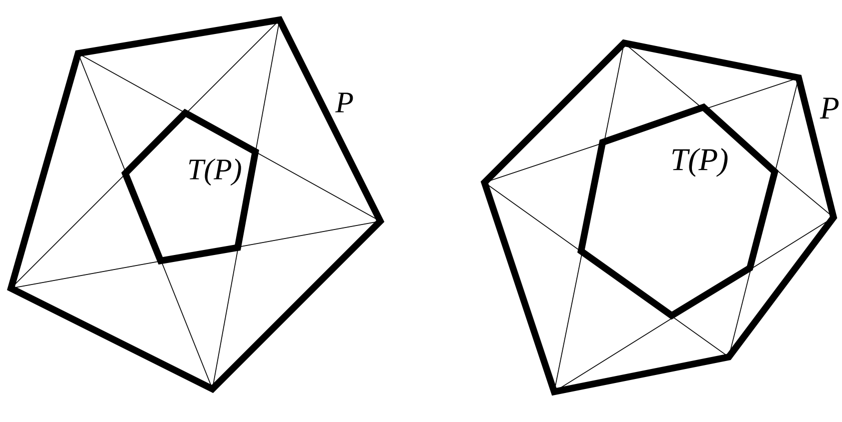

Figure 1: The pentagram map defined on a pentagon and a hexagon

Computer experiments suggested that the pentagram map is a completely integrable systems. Indeed, this was conjectured in [4].

The goal of this Workshop was to prove the integrability conjecture, but first one had to develop an adequate framework. Rather than work with closed n-gons, we worked with what we call *twisted n-gons*. A twisted n-gon is a map $\phi : \mathbb{Z} \to \mathbb{R}P^2$ such that that

$$\phi(k+n) = M \circ \phi(k); \quad \forall k.$$

Here M is some projective automorphism of $\mathbb{R}P^2$ called the *monodromy*.

It is a powerful general idea of projective differential geometry to represent geometrical objects in an algebraic way. It turns out that the space of twisted n-gons is naturally isomorphic to a space of difference equations. Given two arbitrary n-periodic sequences (a_i), (b_i) with $a_i, b_i \in \mathbb{R}$ and $i \in \mathbb{Z}$, such that $a_{i+n} = a_i$, $b_{i+n} = b_i$, one associates to these sequences a difference equation of the form

$$V_{i+3} = a_i \, V_{i+2} + b_i \, V_{i+1} + V_i,$$

A solution $V = (V_i)$ is a sequence of numbers $V_i \in \mathbb{R}$ satisfying this equation. Such an interpretation provides a global coordinate system $(a_i, \, b_i)$ on the space of twisted n-gons.

The main result obtained during and in the summer after the Workshop is as follows. It is proved that there exists a Poisson structure on the space of twisted n-gons, invariant under the pentagram map. The monodromy invariants Poisson-commute. This provides the classical Arnold-Liouville complete integrability of the pentagram map.

The pentagram map is expressed in the coordinates $(a_i, \, b_i)$ by a beautiful combinatorial formula:

$$T : a_i \mapsto a_{i+2} \prod_{k=1}^{m} \frac{1 + a_{i+3k+2} \, b_{i+3k+1}}{1 + a_{i-3k+2} \, b_{i-3k+1}}, \qquad T : b_i \mapsto b_{i-1} \prod_{k=1}^{m} \frac{1 + a_{i-3k-2} \, b_{i-3k-1}}{1 + a_{i+3k-2} \, b_{i+3k-1}}.$$

The T-invariant Poisson bracket is defined on the coordinate functions as follows.

$$\begin{array}{rcl} \{a_i, a_j\} &=& \displaystyle\sum_{k=1}^{m} (\delta_{i,j+3k} - \delta_{i,j-3k}) \, a_i \, a_j, \\ \{a_i, b_j\} &=& 0, \\ \{b_i, b_j\} &=& \displaystyle\sum_{k=1}^{m} (\delta_{i,j-3k} - \delta_{i,j+3k}) \, b_i \, b_j. \end{array}$$

It is also proved that the continuous limit of the pentagram map is precisely the classical Boussinesq equation which is one of the most studied infinite-dimensional integrable systems. Moreover, the above Poisson bracket is a discrete analog of the well known first Poisson structure of the Boussinesq equation.

The results obtained during the Workshop and developed after led to a preprint [1].

References

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