Open problems on rank–k numerical ranges

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This is a summary of open problems raised in connection with my talk at the BIRS workshop "Operator Structures in Quantum Information Theory", 15 February 2007. The main references are [CHKŻ] and [CGHK]. Given $T \in M_N(\mathbb{C})$ (the algebra of $N \times N$ complex matrices), we use $\Lambda_k(T)$ to denote the "rank–k numerical range of T":

$$\Lambda_k(T) = \{\lambda \in \mathbb{C} : \text{ there exists } P \in \mathcal{P}_k \text{ such that } PTP = \lambda P\},$$

where $\mathcal{P}_k$ denotes the space of rank–k orthogonal projections in $M_N(\mathbb{C})$.

Problem 1: Is $\Lambda_k(T)$ always a convex subset of $\mathbb{C}$?

Remarks: This is perhaps the key problem, and others mentioned in what follows are related to it. There is considerable theoretical and experimental evidence suggesting a positive answer. Such an answer would be valuable in at least two ways:

(1) The venerable Toeplitz–Hausdorff Theorem says that the classical numerical range $W(T)$ (ie $\{(Tu, u) : \|u\| = 1\}$) is convex, and it is easy to see that $\Lambda_1(T) = W(T)$. Thus a positive answer to Problem 1 for any $k \geq 2$ would be a striking extension of the Toeplitz–Hausdorff Theorem.

(2) The problem is important in connection with quantum information theory (see [CKŻ1], [CKŻ2]), particularly in the light of the criteria for correctable subspaces due to Knill–Laflamme and Bennett et al. For example, a positive answer would extend (and perhaps complete) our understanding of the CKŻ conjecture:

CKŻ Conjecture: If T is normal, then $\Lambda_k(T) = \Omega_k(T)$, where

$$\Omega_k(T) = \bigcap_{\#(J)=N-k+1} \text{conv}(\{\lambda_j : j \in J\}).$$

Here λ_j are the eigenvalues of the normal T and J runs over subsets of $\{1, 2, \ldots, N\}$. Since $\Omega_k(T)$ is convex, the CKŻ conjecture implies a positive answer to Problem 1 for normal matrices. On the other hand, it is often relatively easy to verify that the extreme points of $\Omega_k(T)$ (for normal T) lie in $\Lambda_k(T)$; in such cases a positive answer to Problem 1 yields the CKŻ conjecture as well. It may well be that the combinatorial techniques of [CHKŻ] can be extended to give a positive answer to the following problem.

PROBLEM 2: When T is a normal matrix, is each extreme point (vertex) of $\Omega_k(T)$ also an element of $\Lambda_k(T)$?

REMARKS: In [CHKŻ] the CKŻ conjecture is reduced to the unitary case and for unitary T with distinct eigenvalues the conjecture is established except when $2k < N < 3k$; it is also established by nonconstructive methods for $N = 3k - 1$ and $N = 5k/2$. In applications to quantum information theory, the structure of the quantum systems may require us to deal with unitary or normal T having multiple eigenvalues. Hence the importance of the next problem.

PROBLEM 3: Extend the results of [CHKŻ] on the CKŻ conjecture for normal T to the case of multiple eigenvalues.

In [CGHK] Problem 1 is reduced (for a given value of k) to a variety of "simpler" but equivalent forms. We mention just two of them as Problems 4 and 5.

PROBLEM 4: (equivalent to Problem 1) Do we have, for any given $X, Y \in M_k(\mathbb{C})$, the existence of $Z \in M_k(\mathbb{C})$ such that $I_k + XZ + Z^*Y - Z^*Z = 0_k$?

PROBLEM 5: (equivalent to Problem 1) Do we have, for any given $M, R \in M_k(\mathbb{C})$, where R is positive definite, a Hermitian fixed point H for the map $f_{M,R}$ defined by

$$f_{M,R}(H) = I_k + MH + HM^* - HRH?$$

REMARK: Evidently Problem 4 suggests that the methods of algebraic geometry (over the real field) may be brought to bear on Problem 1, while Problem 5 suggests that topological methods may be helpful. In [CGHK] some special cases are handled successfully.

In [CGHK] we present experimental (graphic) results that suggest the following more ambitious form of Problem 1.

PROBLEM 6: Do we have, for any $T \in M_N(\mathbb{C})$,

$$\Lambda_k(T) = \bigcap \{W(P_L T|_L) : L \text{ is a subspace of dimension } N-k+1\}?$$

REMARKS: A positive answer to Problem 6 would imply a positive answer to Problem 1 since the Toeplitz–Hausdorff Theorem ensures that each $W(P_L T|_L)$ is convex. Note that the CKŻ conjecture can be restated in an analogous form: for normal T,

$$\Lambda_k(T) = \bigcap \{W(P_L T|_L) : L \text{ is a } T\text{–invariant subspace of dimension } N-k+1\}.$$

Experiments indicate that invariant subspaces do *not* suffice for nonnormal T.

While it is a standard technique to trace the boundary of the classical numerical range $W(T)$ $(=\Lambda_1(T))$ by plotting $(Tu_\theta^1, u_\theta^1)$ where u_θ^1 is a unit eigenvector corresponding to the largest eigenvalue of $\mathrm{Re}(e^{i\theta}T)$, the phenomenon seen in Figure 1 and 2 remains to be explained. There it appears that $(Tu_\theta^k, u_\theta^k)$ where u_θ^k is a unit eigenvector corresponding to the k–th largest eigenvalue of $\mathrm{Re}(e^{i\theta}T)$ traces the boundary of $\Lambda_k(T)$ (plus "wings" at the "corners" of $\Lambda_k(T)$). Thus we include the following somewhat vague problem.

PROBLEM 7: Explain what is going on in Figures 1 and 2. More specifically, when is $(Tu_\theta^k, u_\theta^k)$ an element of $\Lambda_k(T)$?

Finally, we point out yet another way of looking at Problem 1. The "k–th spatial numerical range" has been defined as follows:

$$W_s^k(T) = \{X^*TX : X \text{ is a } N \times k \text{ matrix with } X^*X = I_k\}.$$

These sets have been studied in some detail (see [Li–Tsing], [Farenick], for example). They are not convex in general but it seems possible that the

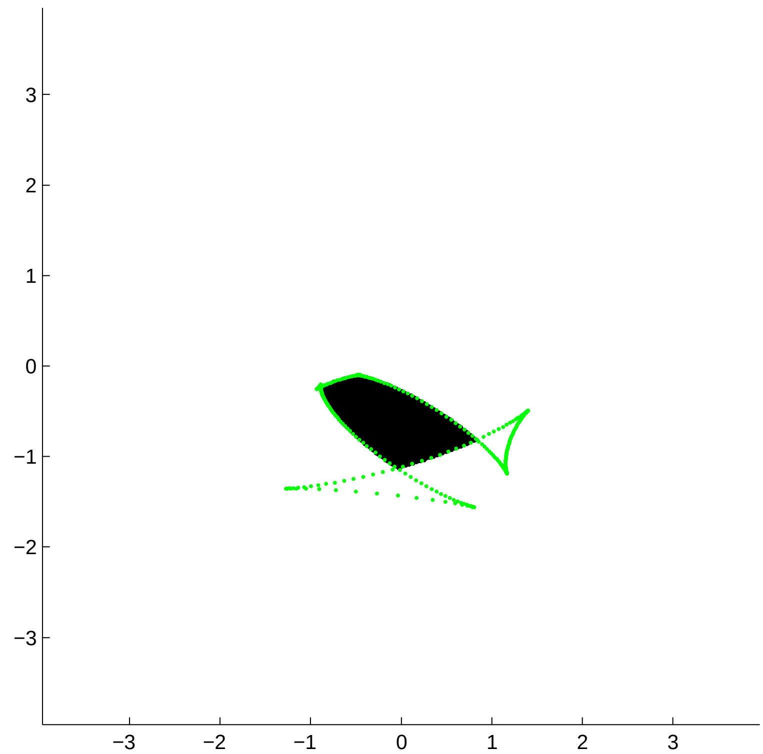

Figure 1: $\Lambda_2(T)$ for a random $T \in M_4(\mathbb{C})$ (black; via a Newton–Raphson technique) bounded (with additional "wings") by the curve $(Tu_\theta^2, u_\theta^2)$ where u_θ^k is a unit eigenvector corresponding to the k–th largest eigenvector of $\mathrm{Re}(e^{i\theta}T)$; this mysterious phenomenon is discussed in connection with Problem 7

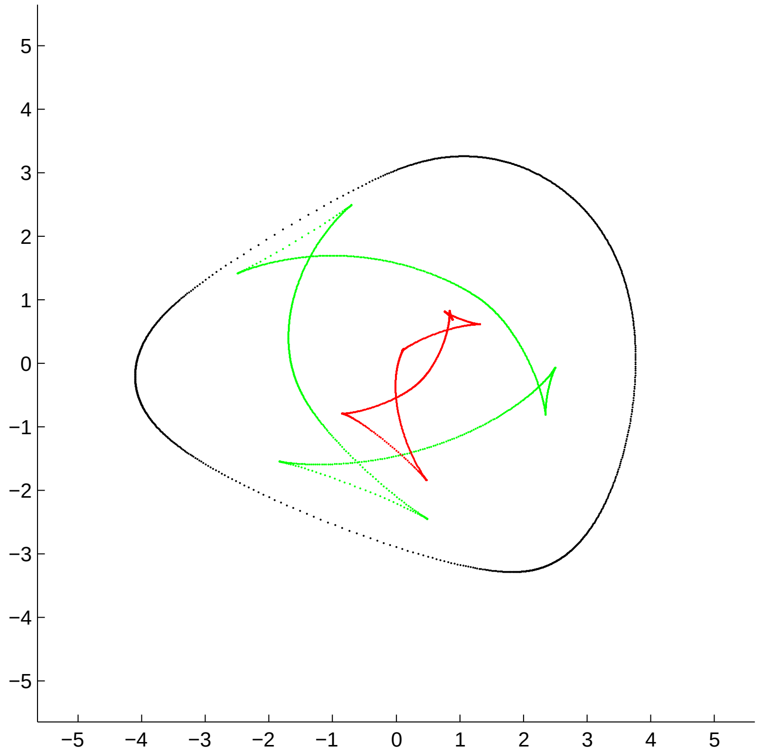

Figure 2: The boundries of $\Lambda_k(T)$ $(k = 1, 2, 3)$ are plotted for a random $T \in M_6(\mathbb{C})$ by the (poorly understood) method described in connection with Problem 7; the "wings" should be ignored; the curves plot $(Tu_\theta^k, u_\theta^k)$ where u_θ^k is a unit eigenvector corresponding to the k–th largest eigenvector of $\mathrm{Re}(e^{i\theta}T)$

scalar matrices $\lambda I_k \in W_s^k(T)$, which correspond to $\Lambda_k(T)$, do nevertheless form a convex set. Hence we have the following problem.

PROBLEM 8: (equivalent to Problem 1) Do the scalar matrices in $W_s^k(T)$ form a convex set?

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