MULTIPLICITIES OF GALOIS REPRESENTATIONS OF WEIGHT 1

GABOR WIESE

1. Galois representations of newforms and multiplicities

Let N be a positive integer. Let $p > 2$ be a prime with $p \nmid N$. Let $f \in S_k(\Gamma_1(N))$ be a newform of level N, with associated character $\chi\colon (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}^\times$. Let $\overline{\mathbb{Z}}$ be the integral closure of $\mathbb{Z}$ in an algebraic closure $\overline{\mathbb{Q}}$ of $\mathbb{Q}$. Choose a reduction map $\overline{\mathbb{Z}} \to \overline{\mathbb{F}}_p$. Suppose $3 \leq k \leq p$. Let $\mathbb{T}_\mathbb{Z}$ be the subring of $\operatorname{End} S_2(\Gamma_1(Np))$ generated by T_n for all n. Let $\mathbb{T}'_\mathbb{Z}$ be the subring of $\operatorname{End} S_2(\Gamma_1(Np))$ generated by T_n for all n not divisible by p. Let $\mathfrak{m}$ be the kernel of the ring homomorphism $\mathbb{T}_\mathbb{Z} \to \overline{\mathbb{F}}_p$ sending T_n to $\bar{a}_n$. Let $\mathfrak{m}'$ be the kernel of the ring homomorphism $\mathbb{T}'_\mathbb{Z} \to \overline{\mathbb{F}}_p$ sending T_n to $\bar{a}_n$ for all n not divisible by p. Let $\mathbb{F} := \mathbb{T}_\mathbb{Z}/\mathfrak{m}$. We get a representation

$$\rho_f\colon \operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\mathbb{F})$$

that is semisimple, odd, unramified outside Np. For $\ell \nmid Np$, the characteristic polynomial of $\rho_f(\mathrm{Frob}_\ell)$ is $X^2 - \bar{a}_\ell X + \ell^{k-1}\chi(\ell)$. Assume that ρ_f is irreducible.

Fact: There exists $f_2 \in S_2(\Gamma_1(Np))$ such that $f_2 \equiv f \pmod{p}$.

Let $J_1(Np)_\mathbb{Q}$ be the Jacobian of $X_1(Np)_\mathbb{Q}$. The group $J := J_1(Np)_\mathbb{Q}(\overline{\mathbb{Q}})$ has an action of $\operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ and T_n.

Theorem 1.1 (Boston-Lenstra-Ribet)**.**

(a) *$J[\mathfrak{m}] \simeq \rho_f^r$ for some $r \geq 1$. Call r the multiplicity of ρ_f on $J[\mathfrak{m}]$.*

(b) *$J[\mathfrak{m}'] \simeq \rho_f^{r'}$ for some $r' \geq r \geq 1$.*

Theorem 1.2 (Mazur, Ribet, Gross, Wiles, Buzzard, ...)**.**

(a) *If ρ_f is ramified at p, then $r = 1$.*

(b) *If ρ_f is unramified at p and $\rho(\mathrm{Frob}_p)$ is not scalar, then $r = 1$.*

Remark 1.3. Kilford found an example with $r = 2$.

Theorem 1.4.

(a) *If ρ_f is unramified at p, and $\rho(\mathrm{Frob}_p)$ is scalar, then $r > 1$.*

(b) *(This is a reformulation of (a).) We have that ρ_f is ramified at p if and only if $r' = 1$.*

2. Relation between multiplicity and Gorenstein defect

Let $\overline{\mathbb{T}} := \mathbb{T}_\mathbb{Z} \otimes \mathbb{F}_p$. Let $\overline{\mathfrak{m}}$ be the kernel of $\overline{\mathbb{T}} \to \overline{\mathbb{F}}_p$.

The first key theorem, 95% due to Buzzard is

Theorem 2.1. *Suppose that $T_p \notin \overline{\mathfrak{m}}$ (ordinary). Then*

$$0 \to \overline{\mathbb{T}}_{\overline{\mathfrak{m}}} \to J[p]_{\overline{\mathfrak{m}}} \to \overline{\mathbb{T}}^\wedge_{\overline{\mathfrak{m}}} \to 0$$

is an exact sequence of $\overline{\mathbb{T}}_{\overline{\mathfrak{m}}}$-modules.

Date: June 6, 2007.

We get

$$0 \to \overline{\mathbb{T}}_{\overline{\mathfrak{m}}}[\overline{\mathfrak{m}}] \to J[p]_{\overline{\mathfrak{m}}}[\overline{\mathfrak{m}}] \to \overline{\mathbb{T}}^{\vee}_{\overline{\mathfrak{m}}}[\overline{\mathfrak{m}}] \to 0.$$

The last term is $(\mathbb{T}_{\overline{\mathfrak{m}}}/\overline{\mathfrak{m}})^{\vee}$, which is a 1-dimensional $\mathbb{F}$-vector space. The first two terms are of dimensions $2r-1$ and $2r$, respectively. The middle term is isomorphic to $\rho_f^{\vee}$.

Proposition 2.2.

(a) $r = \frac{1}{2}(\dim \mathbb{T}_{\overline{\mathfrak{m}}}[\overline{\mathfrak{m}}] - 1) + 1$. *The term in parentheses is the* Gorenstein defect *and called* d.

(b) *The following are equivalent:*
  (i) $r = 1$.
  (ii) $d = 0$.
  (iii) $\overline{\mathbb{T}}_{\overline{\mathfrak{m}}}$ *is a Gorenstein ring.*

3. Weight 1 and proofs

The second key theorem is

Theorem 3.1 (Edixhoven, Gross, Coleman-Voloch)**.** *If ρ_f is unramified at p, then there exists a Katz modular form $g \in S_1(\Gamma(N), \mathbb{F}_p)_{\mathrm{Katz}}$ such that $\rho_g \simeq \rho_f$.*

There is a $\overline{\mathbb{T}}'$-equivariant injection

$$S_1(\Gamma(N), \mathbb{F}_p)^2_{\mathrm{Katz}} \hookrightarrow S_p(\Gamma_1(N), \mathbb{F}_p)$$

sending $(g = \sum b_n q^n,\ h = \sum c_n q^n)$ to $Ag + Fh = \sum b_n q^n + \sum c_n q^{np}$, where A is the Hasse invariant, and F is the Frobenius. Then $f \in \langle Ag, Fg \rangle \simeq S_p(\Gamma_1(N), \mathbb{F}_p)[\overline{\mathfrak{m}}']$. We should view f as an oldform, since it is in the sum of images of degeneracy maps.

The third key theorem is

Theorem 3.2 (Gross)**.** *Let ρ_f be of weight 1. The image of $\overline{\mathbb{T}}_{\overline{\mathfrak{m}}}[\overline{\mathfrak{m}}'] \hookrightarrow J[p]_{\overline{\mathfrak{m}}}[\overline{\mathfrak{m}}']$ is unramified at p, and on it $\rho_f(\mathrm{Frob}_p) = T_p^{-1}$.*

Proof of (a). Given that ρ_f is unramified at p, and $\rho_f(\mathrm{Frob}_p)$ scalar, we want to show that $\overline{\mathbb{T}}_{\overline{\mathfrak{m}}}$ is not Gorenstein. Assume that $\overline{\mathbb{T}}_{\overline{\mathfrak{m}}}$ is Gorenstein. Then $\overline{\mathbb{T}}^{\vee}_{\overline{\mathfrak{m}}} \simeq \overline{\mathbb{T}}_{\overline{\mathfrak{m}}}$ and $\overline{\mathbb{T}}_{\overline{\mathfrak{m}}'} \simeq \overline{\mathbb{T}}^{\vee}_{\overline{\mathfrak{m}}'} \simeq S_p(\Gamma_1(N), \mathbb{F}_p)_{\overline{\mathfrak{m}}}$. So $\overline{\mathbb{T}}^{\vee}_{\overline{\mathfrak{m}}}[\overline{\mathfrak{m}}] \simeq \overline{\mathbb{T}}_{\overline{\mathfrak{m}}}[\overline{\mathfrak{m}}]$ and $\overline{\mathbb{T}}_{\overline{\mathfrak{m}}'}[\overline{\mathfrak{m}}'] \simeq \overline{\mathbb{T}}^{\vee}_{\overline{\mathfrak{m}}'}[\overline{\mathfrak{m}}'] \simeq S_p(\Gamma_1(N), \mathbb{F}_p)_{\overline{\mathfrak{m}}}[\overline{\mathfrak{m}}']$. Because T_p is scalar, the $\overline{\mathfrak{m}}$-torsion equals the $\overline{\mathfrak{m}}'$-torsion, so all the vector spaces in the previous sentence are isomorphic. But the first is 1-dimensional, and the last is 2-dimensional. □

Proof of (b). We want to show that $r' = 1$ if and only if ρ_f is ramified at p.

Suppose that ρ_f is ramified at p. Then $r = 1$ by the theorem. Now $\overline{\mathbb{T}}_{\overline{\mathfrak{m}}'}/\overline{\mathbb{T}}'_{\overline{\mathfrak{m}}'}$ is a faithful module for $\mathbb{T}(S_1(\Gamma(N), \mathbb{F}_p)_{\overline{\mathfrak{m}}'})$, so it is zero and $\overline{\mathfrak{m}}' = \overline{\mathfrak{m}}$ so $r = r'$. Hence $r' = 1$.

Suppose that $r' = 1$. Then $r = 1$ too. We have that $\overline{\mathbb{T}}_{\overline{\mathfrak{m}}}$ is Gorenstein, and $\rho_f(\mathrm{Frob}_p)$ is not scalar. Assume that ρ_f is unramified at p. By the first key theorem, $\rho(\mathrm{Frob}_p)$ has two different eigenvalues. We get f_1 and f_2, corresponding to $\overline{\mathfrak{m}}_1$ and $\overline{\mathfrak{m}}_2$. We have

$$J[\overline{\mathfrak{m}}_1] \oplus J[\overline{\mathfrak{m}}_2] = J[\overline{\mathfrak{m}}],$$

so $r' = 2$, a contradiction. □

Assume $\rho(\mathrm{Frob}_p) = \begin{pmatrix} a & x \\ 0 & a \end{pmatrix}$ with $x \neq 0$. We have

$$0 \to \mathbb{T}_{\mathfrak{m}}[\overline{\mathfrak{m}}] \to J[\overline{\mathfrak{m}}] \to S_p[\overline{\mathfrak{m}}] \to 0$$

in which the dimensions are $1, 2, 1$. Now

$$0 \to \mathbb{T}_{\mathfrak{m}}[\overline{\mathfrak{m}}'] \to J[\overline{\mathfrak{m}}'] \to S_p[\overline{\mathfrak{m}}'] \to 0$$

where the dimensions are $2, 4, 2$.