VISIBILITY OF THE SHAFAREVICH-TATE GROUP FOR ANALYTIC RANK 1

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Let E be an elliptic curve over $\mathbb{Q}$ with conductor N (often squarefree). Then we have $\pi\colon J_0(N) \to E$. Assume that $\ker \pi$ is connected. Then there exists a newform $f \in S_2(\Gamma_0(N), \mathbb{C})$ such that $E = A_f := J_0(N)/I_f J_0(N)$ where $I_f := \operatorname{Ann}_{\mathbb{T}} f$. We have $E^\vee \hookrightarrow J_0(N)$. An element of $Ш_{E^\vee}$ is said to be *visible* in $J_0(N)$ if it is in the kernel of the map $Ш_{E^\vee} \to Ш_{J_0(N)}$.

Goal: Use visibility to account for BSD conjectural $Ш_E$.

Conjecture 0.1 (Stein-Jetchev)**.** Given $\sigma \in Ш_{E^\vee}$, there exist M and a quotient C of $J_0(NM)$ and an injection $E^\vee \hookrightarrow C$ such that $\sigma \in \ker(Ш_{E^\vee} \to Ш_C)$.

Suppose that there exists an elliptic curve $F \subseteq J_0(N)$ such that $E[p] = F[p]$ in $J_0(N)$ for some prime p. Then

$$0 \to E(\mathbb{Q})/pE(\mathbb{Q}) \to H^1(\mathbb{Q}, E[p]) \to H^1(\mathbb{Q}, E)[p] \to 0$$

and

$$0 \to F(\mathbb{Q})/pF(\mathbb{Q}) \to H^1(\mathbb{Q}, F[p]) \to H^1(\mathbb{Q}, F)[p] \to 0$$

with the middle terms being equal. Suppose that $E(\mathbb{Q})[p] = 0$ and $\operatorname{rank} F(\mathbb{Q}) > \operatorname{rank} E(\mathbb{Q})$.

Theorem 0.2 (Dummigan-Stein-Watkins)**.** *Suppose that N is prime. Let p be a prime such that $p \nmid N(N-1)$. Suppose that there exists an eigenform $g \in S_2(\Gamma_0(N), \mathbb{C})$ with $f \equiv g$ modulo a maximal ideal of $\mathbb{T}$ over p and* $\operatorname{rank} A_g(\mathbb{Q}) > 0 = \operatorname{rank} A_f(\mathbb{Q})$ *(with $A_f = E$). Then p divides $\#Ш_{A_f^\vee}$.*

1. Analytic rank zero case

Suppose that $L(E,1) \neq 0$. Let $H = H_1(X_0(N), \mathbb{Z})$. We have $H_1(X_0(N), \mathbb{Z}) \otimes \mathbb{R} \xrightarrow{\sim} \operatorname{Hom}_{\mathbb{C}}(H^0(X_0(N), \Omega/\mathbb{C}), \mathbb{C})$ mapping the class of a cycle γ to $(\omega \mapsto \int_\gamma \omega)$. The winding element e is the element mapping to $(\omega \mapsto -\int_0^{i\infty} \omega)$. Let $\mathcal{I} = \operatorname{Ann}_{\mathbb{T}}((0) - (\infty))$. Then $\mathcal{I}e \in H$. Let $K = \ker(H \to H_1(E, \mathbb{Z}))$. Fact:

$$\frac{L(E,1)}{\Omega_E} = \frac{|H_1(E,\mathbb{Z})^+/\pi(\mathcal{I}e)|}{c_\infty(E) c_E |\pi(\mathbb{T}e/\mathcal{I}e)|}$$

where c_E is the Manin constant. Let $I_e = \operatorname{Ann}_{\mathbb{T}} e$. Then, up to powers of 2,

$$\frac{L(E,1)}{\Omega_E} = \frac{\left|\frac{H^+}{K^+ + H[I_e]^+}\right| \left|\frac{H[I_e^+]}{K^+ \mathcal{I}e}\right|}{|\pi(\mathbb{T}e/\mathcal{I}e)|} \stackrel{?}{=} \frac{|Ш_E| \prod_{\ell | N} c_\ell(E)}{|E(\mathbb{Q})_{\text{tors}}|^2}.$$

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2. Analytic rank one case

Let K be an imaginary quadratic field satisfying the Heegner hypothesis. Let $x \in X_0(N)(H)$ be a Heegner point, where H is the Hilbert class field of K. Then we get

$$P = \left[\sum_{\sigma \in \mathrm{Gal}(H/K)} ((x) - (\infty))^{\sigma} \right] \in J_0(N)(H).$$

BSD II becomes Gross-Zagier:

$$[E(K)/\mathbb{Z}\pi(P)] \stackrel{?}{=} c_E \prod_{\ell | N} c_\ell(E) \sqrt{Ш(E/K)}.$$

Kolyvagin proved $\geq$.

Given $0 \to B \to J \to E \to 0$, where $B := I_f J$. Then we have exact sequences

$$0 \to B(K) \to J(K) \to E(K) \to H^1(K, B) \to H^1(K, J)$$

and

$$0 \to \frac{J(K)}{B(K) + \mathbb{Z}P} \to \frac{E(K)}{\mathbb{Z}\pi(P)} \to \ker(H^1(B) \to H^1(J)) \to 0.$$

Let p be a prime that divides $|E(K)/\mathbb{Z}\pi(P)|$. Suppose that $p \nmid |J_0(N)(\mathbb{Q})_{\mathrm{tors}}|$. Let A be the sum of $A_g^\vee$ in $J_0(N)$ such that g has analytic rank 1. We have

$$\left| \frac{J(K)}{B(K) + \mathbb{Z}P} \right| = \left| \frac{J(K)}{B(K) + A(K)} \right| \left| \frac{B(K) + A(K)}{B(K) + \mathbb{Z}P} \right|.$$

Theorem 2.1. *Suppose that N is prime and $p \nmid N(N-1)$. Assume BSD I. Recall $p \nmid |J_0(N)(\mathbb{Q})_{\mathrm{tors}}|$. If p divides either $\left|\frac{J(K)}{B(K)+A(K)}\right|$ or $|\ker(H^1(B) \to H^1(J))|$, then p divides $|Ш_E|$.*

Let's sketch the proof in the case that p divides $\left|\frac{J(K)}{B(K)+A(K)}\right|$. The long exact sequence associated to

$$0 \to A \cap B \to A \oplus B \to J \to 0$$

gives

$$0 \to \frac{J(K)}{A(K) + B(K)} \to \ker(H^1(A \cap B) \to H^1(A) \oplus H^1(B)) \to 0.$$

Let $C = (A \cap B)^0$ and $Q = (A \cap B)/C$. Let $m = |Q|$.