RM abelian surfaces over $\mathbf{Q}$

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Discriminants of RM fields of $J_0(N)$ quotients in online Modular Forms Database [levels are "all $N < 5135$ (and many more up to 7248)"]

D	#
5	3552
8	2490
12	1410
13	779
17	1080
21	221
24	329
28	225
29	40
33	278
37	34
40	95
41	110
44	21
53	0

D	#
56	6
57	48
60	13
61	12
65	9
69	3
73	37
76	11
77	0
85	3
88	5
89	1
92	0
93	0
97	7

D	#
101	0
104	0
105	0
109	6
113	2
120	0
124	0
129	0
133	0
···	
145	2
···	
184	2
···	
201	2

Comparison with type (rational, blown-up K3, blown-up “honestly ellipic”, or general) of the relevant moduli surface $Y_-(D)$ for ppas with endomorphisms by the alg. integers in $\mathbf{Q}(\sqrt{D})$ [Hirzebruch-Zagier 1977; D prime: Hirzebruch-van de Ven 1974]

D	Y_-	#	D	Y_-	#	D	Y_-	#
5	R	3552	56	G	6	101	G	0
8	R	2490	57	E	48	104	G	0
12	R	1410	60	G	13	105	G	0
13	R	779	61	E	12	109	G	6
17	R	1080	65	E	9	113	G	2
21	R	221	69	G	3	120	G	0
24	$K3$	329	73	G	37	124	G	0
28	$K3$	225	76	G	11	129	G	0
29	$K3$	40	77	G	0	133	G	0
33	$K3$	278	85	G	3	$\cdots$		
37	$K3$	34	88	G	5	145	G	2
40	$K3$	95	89	G	1	$\cdots$		
41	$K3$	110	92	G	0	184	G	2
44	E	21	93	G	0	$\cdots$		
53	E	0	97	G	7	201	G	2

Why no examples with $D = 53$?

A birational model for $Y_{-}(53)/\mathbf{Q}$ is $y^2 = P(s) = \sum_{i=0}^{4} p_i\,(r) s^{4-i}$ where: $p_0 = -27/8$,

$$\begin{aligned} p_1 &= -9s^2 + 63s - 13, \\ p_2 &= -32s^6 + 240s^5 - 706s^4 \\ &\qquad + 1120s^3 - 1198s^2 + 432s - 22, \\ p_3 &= 16s(18s^6 - 143s^5 + 458s^4 \\ &\qquad - 781s^3 + 779s^2 - 380s + 63), \\ p_4 &= 32(s^2 - s)(-27s^6 + 207s^5 - 634s^4 \\ &\qquad + 1002s^3 - 855s^2 + 323s - 44). \end{aligned}$$

The rational function r on $Y_{-}(53)$ gives the genus-1 fibration, probably with no section defined over $\mathbf{Q}$ (can this be proved?), but still a Zariski-dense set of points starting from the rational curve $s = 1$, $r(224 - 54r) = d^2$.

Some examples:

$(r,s,y) = (25/9, 3/4, 155/12^3)$: the resulting Clebsch-Igusa invariants are obstructed over $\mathbf{Q}$.

$(r,s,y) = (116/25, 7/5, 2584/5^5)$: invariants

$$[\frac{11141}{30}, \frac{662041}{64}, \frac{21436460999}{23040}, -180]$$

are unobstructed — but give $Y^2 = S(X)$ with disc(S) equal

$$-2^{32}3^{12}5^6 17^5 (4861 \cdot 1804921 \cdot 16882356217)^{15},$$

apparently not just RM-53 but some QM defined over $\mathbf{Q}(\sqrt{\text{disc}(S)})$ (QM = Quaternionic Multiplication)!

But we also get some actual RM-53 examples, with conductors too large for the online database. For instance:

$(r, s, y) = (141/32, 19/16, 7293/2^{15})$ yields

$$Y^2 = 2332X^6 + 902X^5 + 5060X^4 + 17111X^3 + 5995X^2 + 17545X + 27951$$

(disc. $= 2^{10}3^{12}11^{12}13^4 17^{10}$);

$(r, s, y) = (115/32, 29/16, 53^2 13/2^{15})$ yields

$$Y^2 = 140450X^5 + 168540X^4 + 55703X^3 - 6572X^2 + 8706X + 5584$$

(yes, a quintic), disc. $= 2^8 3^{24} 5^8 13^8 53^6$;

"etc."

Some other examples: $Y_{-}(24)$ is birational to the surface with equation

$$y^2 = -s^4 + (9r^2+2)s^3 - (24r^4 - 25r^2)s^2 \\ + (16r^6 - 36r^4 + 22r^2 - 2)s + (r^2-1)^2$$

("singular K3 surface", i.e. Néron-Severi group over $\mathbf{C}$ has rank 20 [=max.]); the pair of rat'l points $(r,s,y) = (3/2, 1, \pm 9/4)$ yields genus-2 curves with isogenous RM Jacobians:

$$Y^2 = 9X^6 + 9X^4 - 60X^3 - 45X^2 + 132X - 53,$$

$$Y^2 = 9X^6 + 198X^4 + 1884X^3, \\ + 4680X^2 + 4200X + 1000$$

(disc. $-2^{18}3^{20}$ and $-2^{18}3^{20}5^{12}$); twisting by $\mathbf{Q}(\sqrt{2})$ gives point-counts mod small primes that match the tabulated coefficients of a modular form of weight 2 and level $2592 = 2^5 9^2$.

Likewise the K3 surface $Y_{-}(33)$ is "singular" (with a more complicated equation); a sample pair of rational points yields curves with isogenous RM Jacobians:

$$Y^2 = (X^3 - 3X - 1)(4X^3 - 3X + 5),$$

$$Y^2 = -(X^3 - 3X - 1)\,(8X^3 + (3X - 1)^2),$$

matching a databased modular form of weight 2 and level $1296 = 2^4 3^4$.