Computations of Gross–Stark units via Shintani zeta-functions

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Hilbert's 12th Problem

$F =$ totally real field
$H =$ finite abelian extension of F

Can we construct H analytically from information intrinsic to F?

H itself will be specified via information instrinsic to F, e.g. let $H = H_{\mathfrak{f}}$, the narrow ray class field associated to a conductor $\mathfrak{f} \subset \mathcal{O}_F$.

Can we construct Stark units analytically? Can we implement these constructions in practice?

Partial zeta-functions

$\mathfrak{p}$ = prime ideal of F, splits completely in H
S = set of primes of F, with $S \supset \{\mathfrak{p}$, archimedean primes, those ramifying in $H\}$.
Assume $\#S \geq 3$, let $R = S - \{\mathfrak{p}\}$.

For $\sigma \in G = \mathrm{Gal}(H/F)$ and $\mathrm{Re}(s) > 1$, define

$$\zeta_R(\sigma, s) = \sum_{\substack{(\mathfrak{a},R)=1 \\ \sigma_{\mathfrak{a}}=\sigma}} \mathrm{N}\mathfrak{a}^{-s}.$$

Note that

$$\zeta_S(\sigma, s) = (1 - \mathrm{N}\mathfrak{p}^{-s})\zeta_R(\sigma, s).$$

In particular $\zeta_S(\sigma, 0) = 0$.

Auxiliary set T

T = set of primes of F disjoint from S containing two primes of different residue characteristic or one prime whise absolute ramification degree is at most its residue characteristic minus 2.

Define $\zeta_{S,T}(\sigma, s)$ by the group ring equation

$$\sum_{\sigma \in G} \zeta_{S,T}(\sigma, s)[\sigma] = \prod_{\eta \in T} (1 - [\sigma_\eta]\mathrm{N}\eta^{1-s}) \sum_{\sigma \in G} \zeta_S(\sigma, s)[\sigma],$$

for example

$$\zeta_{S,\{\eta\}}(\sigma, s) = \zeta_S(\sigma, s) - \mathrm{N}\eta^{1-s}\zeta_S(\sigma\sigma_\eta^{-1}, s).$$

Condition on T implies $\zeta_{S,T}(\sigma, 0) \in \mathbf{Z}$. It also implies there are no nontrivial roots of unity $\equiv 1 \pmod{T}$ in H.

Stark's Conjecture

Fix a prime $\mathfrak{P}$ of H above $\mathfrak{p}$.

Conjecture 1. *There exists a (unique)* $u_T \in H^\times$ *such that:*

1. $|u_T|_{\mathfrak{P}'} = 1$ *if* $\mathfrak{P}' \nmid \mathfrak{p}$.

2. *For all* $\sigma \in G$*, we have* $\zeta'_{S,T}(\sigma, 0) = \log |u_T^\sigma|_{\mathfrak{P}}$.

3. $u_T \equiv 1 \pmod{T}$.

The second condition can be restated

$$\operatorname{ord}_{\mathfrak{P}} u_T^\sigma = \zeta_{R,T}(\sigma, 0).$$

Gross's Conjecture

Let K be an auxiliary abelian extension of F containing H and unramified outside S. Let rec : $F_{\mathfrak{p}}^{\times} \to \mathbf{A}_F \to \mathrm{Gal}(K/F)$ be the Artin reciprocity map of local class field theory.

Note $H \subset H_{\mathfrak{P}} \cong F_{\mathfrak{p}}$.

Conjecture 2. *Conjecture 1 is true, and for all $\sigma \in G$ we have*

$$\mathrm{rec}(u_T^{\sigma}) = \prod_{\substack{\tau \in \mathrm{Gal}(K/F) \\ \tau|_H = \sigma}} \tau^{\zeta_{S,T}(K/F,\tau,0)}$$

in $\mathrm{Gal}(K/H)$.

Reformulating Gross's Conjecture

For expositional reasons, assume $\mathfrak{p} = (p)$ and $H =$ narrow Hilbert class field of F.
Let $S = \{\mathfrak{p}, \text{archimedean primes}\}$.
Class field theory:

$$\text{rec} : \mathcal{O}_{\mathfrak{p}}^{\times}/\widehat{E} \cong \text{Gal}(H_{\mathfrak{p}^\infty}/H),$$

where $\mathcal{O}_{\mathfrak{p}} =$ completion of $\mathcal{O}_F$ at $\mathfrak{p}$, and
$E =$ group of totally positive units of F.

Let $\mathfrak{b} =$ fractional ideal of F, relatively prime to S and T. Let U be a compact open subset of $\mathcal{O}_{\mathfrak{p}}^{\times}/\widehat{E}$. Define

$$\zeta_S(\mathfrak{b}, U, s) = \sum_{\substack{\mathfrak{a} \subset \mathcal{O}, (\mathfrak{a}, S) = 1 \\ \sigma_{\mathfrak{a}} \in \sigma_{\mathfrak{b}} \cdot \text{rec}(U)}} \text{N}\mathfrak{a}^{-s} = \text{N}\mathfrak{b}^{-s} \sum_{\substack{\alpha \in (\mathfrak{b}^{-1}/E) \cap U \\ \alpha \gg 0}} \text{N}\alpha^{-s},$$

using the change of variables $\mathfrak{a}\mathfrak{b}^{-1} = (\alpha)$.
Define $\zeta_{S,T}$ from ζ_S as before.

A Formula mod $\widehat{E}$

Define a $\mathbf{Z}$-valued measure on $\mathcal{O}_{\mathfrak{p}}^{\times}/\widehat{E}$ by

$$\mu_{\mathfrak{b}}(U) := \zeta_{S,T}(\mathfrak{b}, U, 0).$$

Proposition. *Conjecture 2 implies that*

$$u_T^{\sigma_{\mathfrak{b}}} = p^{\zeta_{R,T}(H/F,\sigma_{\mathfrak{b}},0)} \cdot \times\!\!\!\!\!\!\int_{\mathcal{O}_{\mathfrak{p}}^{\times}/\widehat{E}} x \; d\mu_{\mathfrak{b}}(x)$$

in $F_{\mathfrak{p}}^{\times}/\widehat{E}$.

Here

$$\times\!\!\!\!\!\!\int_{\mathcal{O}_{\mathfrak{p}}^{\times}/\widehat{E}} x \; d\mu_{\mathfrak{b}}(x) = \lim_{||\mathcal{U}||\to 0} \prod_{U\in\mathcal{U}} x_U^{\mu_{\mathfrak{b}}(U)} \in \mathcal{O}_{\mathfrak{p}}^{\times}/\widehat{E},$$

as $\mathcal{U}$ ranges over uniformly finer covers of $\mathcal{O}_{\mathfrak{p}}^{\times}/\widehat{E}$ by disjoint compact opens U.

Lifting the Measure

Let $\pi : \mathcal{O}_\mathfrak{p}^\times \to \mathcal{O}_\mathfrak{p}^\times/\widehat{E}$ denote the projection. Suppose we can define a $\mathbf{Z}$-valued measure $\tilde{\mu}_\mathfrak{b}$ on $\mathcal{O}_\mathfrak{p}^\times$ such that

$$\tilde{\mu}_\mathfrak{b}(\pi^{-1}(U)) = \mu_\mathfrak{b}(U)$$

for all $U \subset \mathcal{O}_\mathfrak{p}^\times/\widehat{E}$.

Then the image of

$$p^{\zeta_{R,T}(H/F,\sigma_\mathfrak{b},0)} \cdot \times\!\!\!\!\!\!\int_{\mathcal{O}_\mathfrak{p}^\times} x \; d\tilde{\mu}_\mathfrak{b}(x)$$

in $F_\mathfrak{p}^\times/\widehat{E}$ equals the value proposed by Gross's conjecture for the image of $u_T^{\sigma_\mathfrak{b}}$. Therefore, if this element of $F_\mathfrak{p}^\times$ depends only on the narrow ideal class of $\mathfrak{b}$, it is a good candidate for $u_T^{\sigma_\mathfrak{b}}$.

Lifted Measure via Fundamental Domain

For $U \subset \mathcal{O}_{\mathfrak{p}}^{\times}/\widehat{E}$, recall the formula

$$\zeta_S(\mathfrak{b}, U, s) = \mathrm{N}\mathfrak{b}^{-s} \sum_{\substack{\alpha \in (\mathfrak{b}^{-1}/E) \cap U \\ \alpha \gg 0}} \mathrm{N}\alpha^{-s}.$$

If we fix a fundamental domain $\mathfrak{b}^{-1}/E$ for the action of E on $\mathfrak{b}^{-1}$, then the subscript in the sum makes sense for $U \subset \mathcal{O}_{\mathfrak{p}}^{\times}$!

Which fundamental domain? Answer given by Shintani.

Shintani Domains

Let Q denote the positive orthant in $F \otimes \mathbf{R}$. A *simplicial cone* in Q is a subset of the form

$$C(v_1, \ldots, v_r) = \left\{ \sum_{i=1}^{r} c_i v_i : c_i > 0 \right\}$$

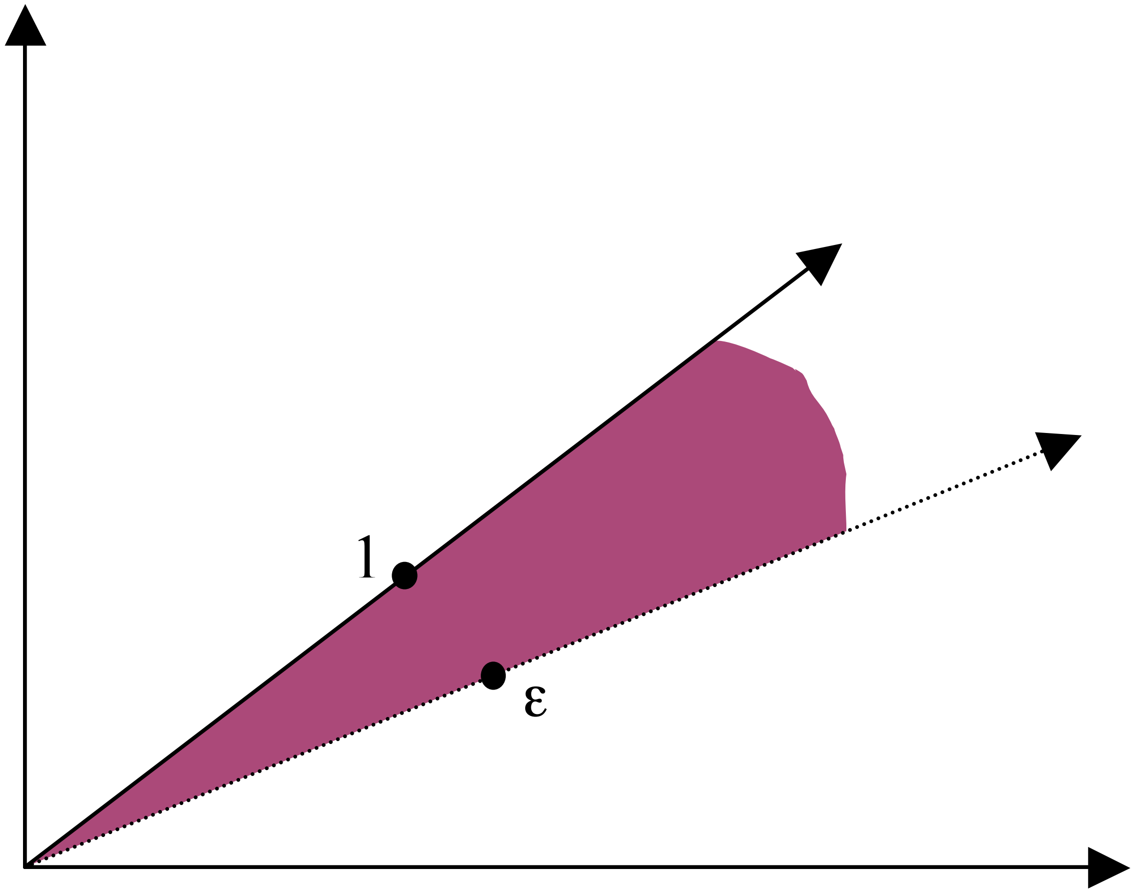
for r linearly independent elements $v_i \in Q$.

Proposition. *There exists a fundamental domain $\mathcal{D}$ for the action of E on Q which consists of a union of simplicial cones generated by elements of F.*

Such a set $\mathcal{D}$ is called a Shintani domain.

Shintani Domain for $n = 2$

If $n = 2$ and $E = \langle \epsilon \rangle$, then $\mathcal{D} = C(1) \cup C(1, \epsilon)$ is a Shintani domain.



Shintani Zeta-Functions

Let $\mathcal{D}$ be such a Shintani domain, and define for $U \subset \mathcal{O}_{\mathfrak{p}}^{\times}$:

$$\zeta_S(\mathfrak{b}, \mathcal{D}, U, s) = \mathrm{N}\mathfrak{b}^{-s} \sum_{\alpha \in \mathfrak{b}^{-1} \cap \mathcal{D} \cap U} \mathrm{N}\alpha^{-s}.$$

Define $\zeta_{S,T}(\mathfrak{b}, \mathcal{D}, U, s)$ from $\zeta_S(\mathfrak{b}, \mathcal{D}, U, s)$ as before, and let

$$\tilde{\mu}_{\mathfrak{b},\mathcal{D}}(U) := \zeta_{S,T}(\mathfrak{b}, \mathcal{D}, U, 0).$$

Two formulas for $\tilde{\mu}_{\mathfrak{b},\mathcal{D}}(U)$: one as the trace of an algebraic integer, and one as a generalized Dedekind sum (sums of products of $B_1(x)$ for various rational x).

A formula for u_T?

Theorem. *If $\mathcal{D}$ and T are chosen to satisfy a certain technical condition, then $\tilde{\mu}_{\mathfrak{b},\mathcal{D}}$ is $\mathbf{Z}$-valued, and*

$$u_T(\mathfrak{b},\mathcal{D}) := p^{\zeta_{R,T}(H/F,\sigma_{\mathfrak{b}},0)} \cdot \times\!\!\!\!\!\!\int_{\mathcal{O}_{\mathfrak{p}}^{\times}} x \; d\tilde{\mu}_{\mathfrak{b},\mathcal{D}}(x) \in F_{\mathfrak{p}}^{\times}$$

depends only on the narrow ideal class of $\mathfrak{b}$ (and in particular not on the choice $\mathcal{D}$), up to a root of unity.

The root of unity ambiguity does not occur when $n = 2$ (and the technical condition is quite simple in this case).

The refined conjecture

Fix an embedding $H \subset F_{\mathfrak{p}}^{\times}$.

Conjecture 3. *The root of unity ambiguity in the theorem does not hold, so we may write $u_T(\mathfrak{b}, \mathcal{D})$ as $u_T(\mathfrak{b})$. Furthermore,*

1. *$u_T(\mathfrak{b}) \in \mathcal{O}_H[1/p]^{\times}$ and has absolute value 1 at all archimedean places.*

2. *$u_T(\mathfrak{b}) \equiv 1 \pmod{T}$.*

3. *(Shimura Reciprocity Law) $u_T(\mathfrak{a}\mathfrak{b}) = u_T(\mathfrak{b})^{\sigma_{\mathfrak{a}}}$.*

Conjecture 3 $\Rightarrow$ Conjecture 2 (Gross) $\Rightarrow$ Conjecture 1 (Stark).

Computing u_T

Recall

$$\begin{aligned} u_T(\mathfrak{b}, \mathcal{D}) &:= p^{\zeta_{R,T}(H/F,\sigma_{\mathfrak{b}},0)} \cdot \times\!\!\!\!\!\!\int_{\mathcal{O}_{\mathfrak{p}}^{\times}} x \; d\tilde{\mu}_{\mathfrak{b},\mathcal{D}}(x) \\ &:= p^{\zeta_{R,T}(H/F,\sigma_{\mathfrak{b}},0)} \cdot A. \end{aligned}$$

We have $\mathcal{O}_{\mathfrak{p}}^{\times} \cong (\mathcal{O}_{\mathfrak{p}}/\mathfrak{p})^{\times} \times (1 + \mathfrak{p}\mathcal{O}_p)^{\times}$. For $a \in (\mathcal{O}_{\mathfrak{p}}/\mathfrak{p})^{\times}$, let

$$A_a := \times\!\!\!\!\!\!\int_{a+\mathfrak{p}\mathcal{O}_{\mathfrak{p}}} x \; d\tilde{\mu}_{\mathfrak{b},\mathcal{D}}(x).$$

Then $\log A = \sum \log A_a$.

Computing $\log A_a$ mod $\mathfrak{p}^M$

Write ν for $\tilde{\mu}_{\mathfrak{b},\mathcal{D}}$.

$$\begin{aligned}
\log A_a &\equiv \log \prod_{b \in (a+\mathfrak{p}\mathcal{O}_\mathfrak{p})/\mathfrak{p}^M} b^{\nu(b+\mathfrak{p}^M\mathcal{O}_\mathfrak{p})} \\
&= \sum_b \nu(b+\mathfrak{p}^M\mathcal{O}_\mathfrak{p}) \log(b) \\
&= \sum_b \nu(b+\mathfrak{p}^M\mathcal{O}_\mathfrak{p}) \left(\log\left(1 + \left(\frac{b}{a} - 1\right)\right) + \log(a) \right) \\
&= (\log a)\nu(a+\mathfrak{p}\mathcal{O}_\mathfrak{p}) \\
&+ \sum_b \nu(b+\mathfrak{p}^M\mathcal{O}_\mathfrak{p}) \sum_{i=1}^{\infty} \frac{(-1)^{i+1}}{i} \left(\frac{b}{a} - 1\right)^i
\end{aligned}$$

The moments of ν

Write

$$\sum_{i=1}^{k} \frac{(-1)^{i+1}}{i} \left(\frac{b}{a} - 1\right)^{i} = c_k(a)b^k + c_{k-1}(a)b^{k-1} + \cdots + c_0(a).$$

Define measures ν_i on $\mathcal{O}_\mathfrak{p}$ by

$$\nu_i(U) := \int_U x^i d\nu(x).$$

Then

$$\log A_a = (\log a)\nu(a + \mathfrak{p}\mathcal{O}_\mathfrak{p}) + \sum_{i=0}^{k} c_i(a)\nu_i(a + \mathfrak{p}\mathcal{O}_\mathfrak{p}).$$

Calculating ν_i

Fix $\tau : F \to \mathbf{C}$.

For $U \subset \mathcal{O}_\mathfrak{p}$, define

$$\zeta_{S,i}(\mathfrak{b}, \mathcal{D}, U, s) = \mathsf{N}\mathfrak{b}^{-s} \sum_{\alpha \in \mathfrak{b}^{-1} \cap \mathcal{D} \cap U} \frac{\tau(\alpha)^i}{\mathsf{N}\alpha^s},$$

which converges for $\mathrm{Re}(s) > i + 1$. Define $\zeta_{S,T,i}$ from $\zeta_{S,i}$ as before.

Proposition. *The function $\zeta_{S,T,i}(\mathfrak{b}, \mathcal{D}, U, s)$ extends to a meromorphic function on $\mathbf{C}$, and the value $\zeta_{S,T,i}(\mathfrak{b}, \mathcal{D}, U, 0) \in \mathbf{C}$ lies in the image of τ. Furthermore, we have*

$$\nu_i(U) = \tau^{-1}(\zeta_{S,T,i}(\mathfrak{b}, \mathcal{D}, U, 0)).$$

Digression — Shintani zeta-functions

Let $A = (a_{jk})$ be an $r \times n$ matrix with positive entries.

Consider the linear forms

$$L_k(z_1, ..., z_r) = \sum_{j=1}^{r} a_{jk} z_j, \quad 1 \leq k \leq n.$$

Let $x = (x_1, ..., x_r)$ with $x_j > 0$ and let $\chi = (\chi_1, ..., \chi_r)$ be an r-tuple of complex numbers with $|\chi_j| \leq 1$ for all $j = 1, ..., r$. Let $a_1, ..., a_r$ be nonnegative integers.

The Dirichlet series

$$Z_{a_1,...,a_r}(A, x, \chi, s) = \sum_{z_1,...,z_r=0}^{\infty} \frac{\chi_1^{z_1} \dots \chi_r^{z_r} z_1^{a_1} \dots z_r^{a_r}}{\prod_{k=1}^{n} (L_k(z+x))^s}$$

converges absolutely for Re(s) $> \frac{r(1+max(a_1,...,a_r))}{n}$.

Define polynomials $Q_a(q)$ for integers $a \geq 0$ by

$$\sum_{n=0}^{\infty} n^a q^n = \frac{Q_a(q)}{(1-q)^{a+1}} \quad \text{for} \quad |q| < 1.$$

Following Shintani, Slavov proved:

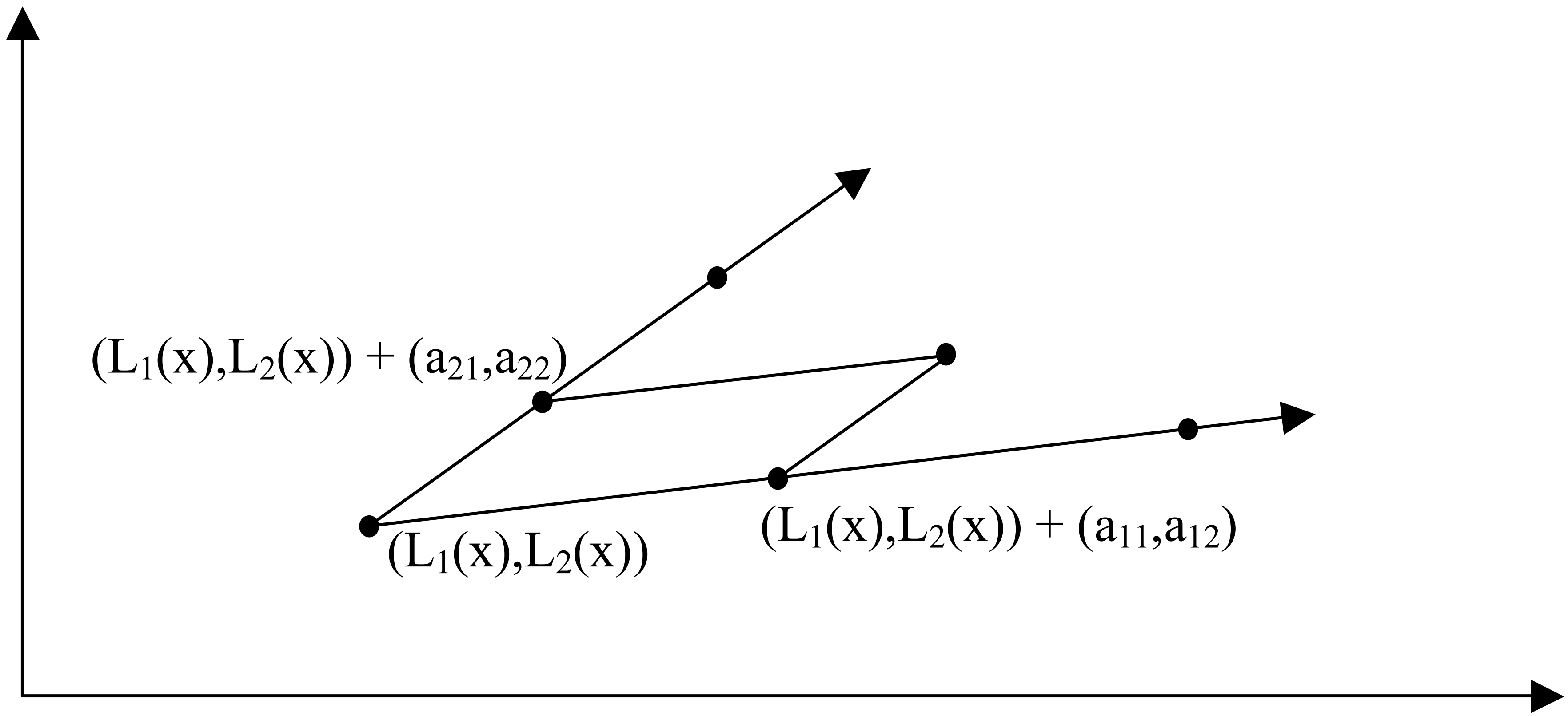
Proposition. *The function $Za_1, ..., a_r$ extends to a meromorphic function on $\mathbb{C}$. If $\chi_j \neq 1$ for all j, then*

$$Za_1, ..., a_r(A, x, \chi, 0) = \frac{Q_{a_1}(\chi_1)}{(1-\chi_1)^{a_1+1}} \cdots \frac{Q_{a_r}(\chi_r)}{(1-\chi_r)^{a_r+1}}.$$

In other words, the value at $s = 0$ is obtained by formally plugging in $s = 0$ in the series

$$Z_{a_1,...,a_r}(A, x, \chi, s) = \sum_{z_1,...,z_r=0}^{\infty} \frac{\chi_1^{z_1} \cdots \chi_r^{z_r} z_1^{a_1} \cdots z_r^{a_r}}{\prod_{k=1}^{n} (L_k(z+x))^s}.$$

Picture of $L_k(z+x)$ when $r = n = 2$



A "lattice cone"

Final step — reducing $\zeta_{S,T,i}$ to $Z_{a_1,\ldots,a_r}$

Let $T = \{\eta\}$, with $\mathrm{N}\eta = \ell$, a prime in $\mathbf{Z}$.

Let $\chi : \mathfrak{b}^{-1}/\mathfrak{b}^{-1}\eta \to \mathbf{C}^\times$ denote a non-trivial character.

Using the orthogonality relation (for $a \in \mathfrak{b}^{-1}$):

$$\sum_{t=0}^{\ell-1} \chi(a)^t = \begin{cases} \ell, & \text{if} \quad a \in \mathfrak{b}^{-1}\eta \\ 0, & \text{if} \quad a \notin \mathfrak{b}^{-1}\eta, \end{cases}$$

the series

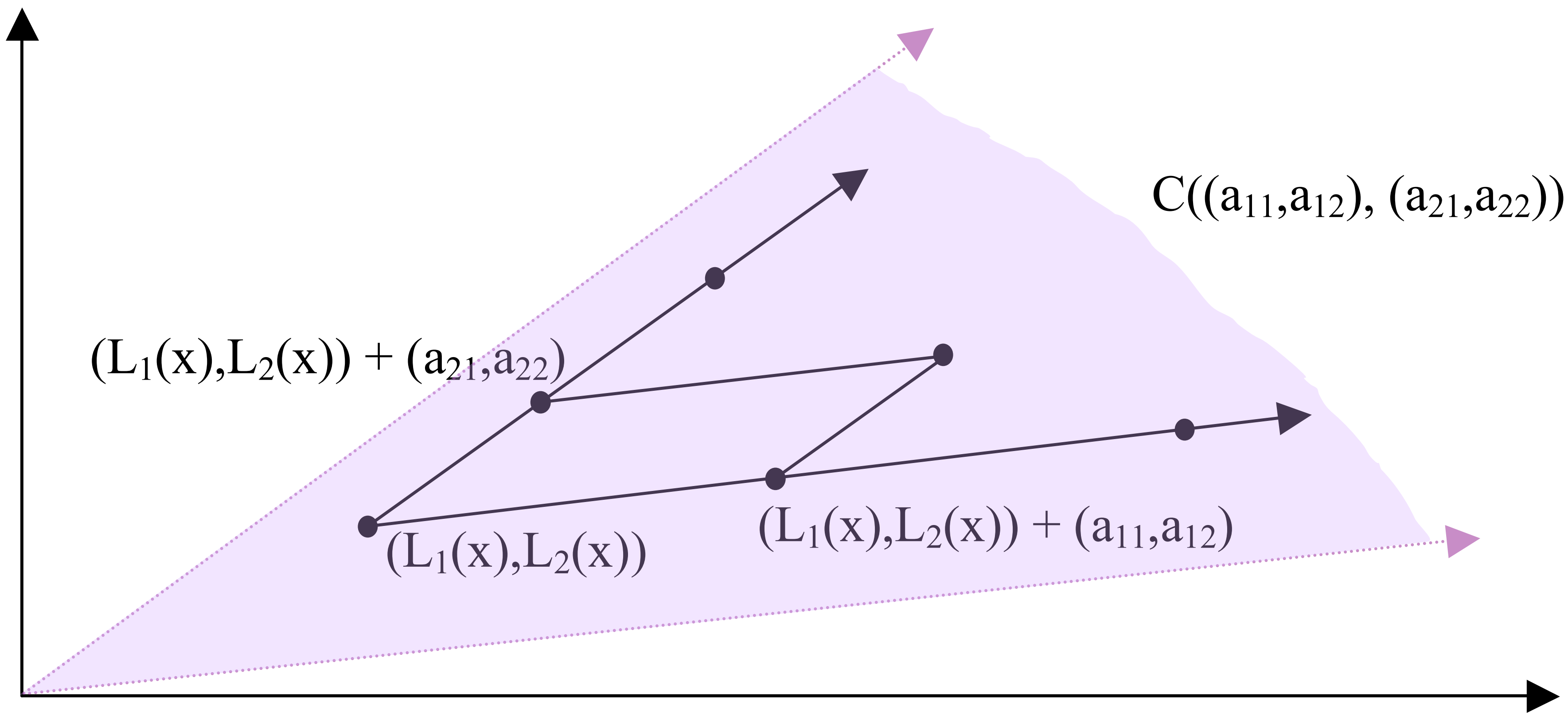
$$\mathrm{N}\mathfrak{b}^s \zeta_{S,T,i}(\mathfrak{b}, \mathcal{D}, U, s) = \sum_{\alpha \in \mathfrak{b}^{-1} \cap \mathcal{D} \cap U} \frac{\tau(\alpha)^i}{\mathrm{N}\alpha^s} - \ell \sum_{\alpha \in \mathfrak{b}^{-1}\eta \cap \mathcal{D} \cap U} \frac{\tau(\alpha)^i}{\mathrm{N}\alpha^s}$$

can be expressed as a finite linear combination of $Z_{a_1,\ldots,a_r}$ with coefficients in ℓth roots of unity, with $a_1 + a_2 + \cdots + a_r = i$.

Note: computing the indexing set of this finite sum involves the LLL algorithm.

A Shintani Domain intersected with a lattice translate

It is a finite union of "lattice cones" if the generators (a_{11}, a_{12}) and (a_{12}, a_{22}) lie in the lattice.



Results, $n = 2$

Let $F = \mathbf{Q}(\sqrt{11})$ with $\mathcal{O}_F = \mathbf{Z}[\sqrt{11}]$. We take $\mathfrak{p} = (3)$, and η over $\ell = 5$. Take $S = \{\infty_1, \infty_2, \mathfrak{p}\}$ and $T = \{\eta\}$.

With $\mathfrak{b}_1 = 1$ and $\mathcal{D} = C(1) \cup C(1, 10 - 3\sqrt{11})$, we compute

$$A = \times\!\!\!\!\!\!\int_{\mathcal{O}_\mathfrak{p}^\times} x \; d\nu(\mathfrak{b}_1, \mathcal{D}, x) \in \mathcal{O}_\mathfrak{p}^\times$$

up to $M = 9$ $\mathfrak{p}$-adic digits, and we obtain

$$A \equiv -118098 + 638972\sqrt{11} \pmod{3^9}.$$

Since

$$\zeta_{R,T}(H/F, \mathfrak{b}_1, 0) = -1$$

we have

$$u_T(\mathfrak{b}_1, \mathcal{D}) = \frac{A}{3}.$$

Next we take $\mathfrak{b}_2 = (\sqrt{11})$, and compute

$$A' = \times\!\!\!\!\!\!\int_{\mathcal{O}_\mathfrak{p}^\times} x \; d\nu(\mathfrak{b}_2, \mathcal{D}, x) \equiv \frac{1}{A} \pmod{3^9}.$$

Thus, $u_T(\mathfrak{b}_1, \mathcal{D})$ and $u_T(\mathfrak{b}_2, \mathcal{D})$ are roots of the polynomial in $F_\mathfrak{p}[x]$, whose coefficients are as follows up to 9 $\mathfrak{p}$-adic digits:

$$x^2 - \left(\frac{A}{3} + \frac{3}{A}\right) x + 1 \equiv x^2 + \frac{1}{3}\sqrt{11}x + 1 \pmod{3^9}.$$

Indeed, the narrow Hilbert class field of F, namely $F(\sqrt{-1})$, is the splitting field of the polynomial

$$x^2 + \frac{1}{3}\sqrt{11}x + 1 \pmod{3^9},$$

and the roots of this polynomial are the Gross–Stark units for the data $(H/F, S, T)$.

Shintani Domains for $n = 3$

If $n = 3$ and E has basis (ϵ_1, ϵ_2) as a free abelian group, Colmez proved that

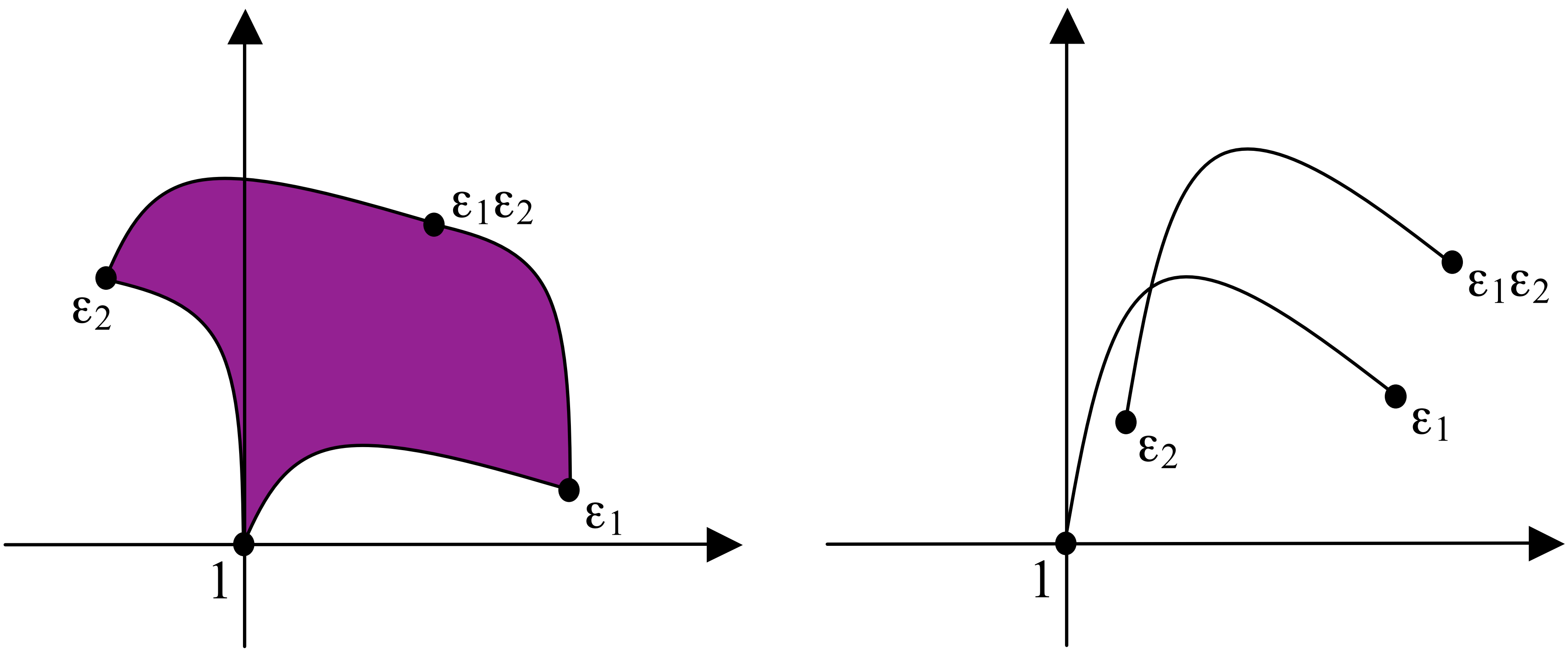
$$\begin{aligned}\mathcal{D} &= C(1) \cup C(1, \epsilon_1) \cup C(1, \epsilon_2) \cup C(1, \epsilon_1\epsilon_2) \cup \\ &\quad C(1, \epsilon_1, \epsilon_1\epsilon_2) \cup C(1, \epsilon_2, \epsilon_1\epsilon_2)\end{aligned}$$

is a Shintani domain, provided ϵ_1, ϵ_2 satisfy the sign condition

$$\det(1, \epsilon_1, \epsilon_1\epsilon_2) \det(1, \epsilon_2, \epsilon_1\epsilon_2) < 0,$$

where $\det(\alpha, \beta, \gamma) = \det \begin{pmatrix} \alpha^1 & \beta^1 & \gamma^1 \\ \alpha^2 & \beta^2 & \gamma^2 \\ \alpha^3 & \beta^3 & \gamma^3 \end{pmatrix}$ for $\alpha, \beta, \gamma \in F$.

Heuristic Picture for $n = 3$



Results, $n = 3$

Let $F = \mathbf{Q}(w)$, where $w^3 + 2w^2 - 3w - 2 = 0$, with $\mathcal{O}_F = \mathbf{Z}[w]$.

We choose a conductor $\mathfrak{f} = \mathfrak{q}^2$, where $(2) = \mathfrak{q}\mathfrak{q}'$ with $\mathfrak{q}, \mathfrak{q}'$ prime ideals and $N(\mathfrak{q}) = 2$. The narrow ray class field $H_{\mathfrak{f}}$ over F has Galois group

$$G = \langle (3), \mathfrak{q}' \rangle \cong \mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}.$$

We take $\mathfrak{p} = (5)$ and η with $(11) = \eta\eta'$ in F, with $N\eta = \ell = 11$. We have $S = \{\infty_1, \infty_2, \infty_3, \mathfrak{q}, \mathfrak{p}\}$ and $T = \{\eta\}$.

Cheating — calculating the Gross-Stark unit knowing $H_{\mathfrak{f}}$

We compute

$$\zeta_{R,T}(H/F,\mathfrak{b},0) = \begin{cases} -10, & \text{if} \quad \mathfrak{b} = 1, \\ 10, & \text{if} \quad \mathfrak{b} = (3), \\ -10, & \text{if} \quad \mathfrak{b} = \mathfrak{q}', \\ 10, & \text{if} \quad \mathfrak{b} = (3)\mathfrak{q}'. \end{cases}$$

If $\mathfrak{P}_1, \dots, \mathfrak{P}_4$ denote the primes of $H_{\mathfrak{f}}$ above $\mathfrak{p}$, we compute the corresponding product

$$\mathfrak{P}_1^{-10}\mathfrak{P}_2^{10}\mathfrak{P}_3^{10}\mathfrak{P}_4^{-10} = (u).$$

Choosing u such that $u \equiv 1 \pmod{\eta}$ and $|u|_w = 1$ for any infinite w, we compute that its minimal polynomial over F is

$$x^2 + \frac{1}{5^{10}}(-1154763w^2 - 6369741w + 5739634)x + 1.$$

Attempting to calculate via Shintani domains

The Colmez sign condition is satisfied, so we take the corresponding Shintani domain. Fix $\mathfrak{b} = (\mathfrak{q}')^2$; it is a representative for the trivial class in G.

We set $M = 6$ $\mathfrak{p}$-adic digits and find

$$\begin{aligned} A &= \times\!\!\!\!\!\!\int_{\mathcal{O}_\mathfrak{p}^\times} x \; d\nu(\mathfrak{b}, \mathcal{D}, x) \\ &= 14138w^2 + 10366w + 10366 \pmod{5^6} \end{aligned}$$

in $\mathcal{O}_\mathfrak{p} = \mathbf{Z}_5[w]$.

The minimum polynomial of $u_T(\mathfrak{b}, \mathcal{D}) = 5^{-10}A$ should be

$$x^2 - \left(5^{-10}A + \frac{5^{10}}{A}\right)x + 1.$$

Indeed, we have that $-A$ is congruent to the middle coefficient of the calculated minimal polynomial of the Stark unit mod 5^6.

Connection with modular forms

In earlier work with Henri Darmon, I provided an alternate formula for the Gross–Stark units when F is a real quadratic field and H is a ring class field extension. Hugo Chapdelaine generalized this to ray class fields in his thesis. Both constructions use the modular symbols attached to Eisenstein series for $\mathbf{GL}_2(\mathbf{Q})$.

I proved that the formula above via Shintani domains agrees with that arising from the modular symbol method in the real quadratic case.

Question. *Is there a Darmon-type construction using modular forms for the Gross–Stark units attached to an arbitrary totally real field F? If so, we should be able to prove it agrees with the one presented here using Shintani domains. Does this suggest that we can define Stark–Heegner points over arbitrary totally real fields?*

Earlier Computations

Example. $p = 3$, $\mathbb{Q}(\sqrt{209})$, $h = 1$, $u(\mathfrak{P}, \mathcal{D})$ *satisfy* $3x^2 + 5x + 3$.

Example. $p = 7$, $\mathbb{Q}(\sqrt{321})$, $h = 3$, $u(\mathfrak{P}, \mathcal{D})$ *satisfy*

$$7^5 x^6 - \frac{2205\sqrt{D} + 53361}{2} x^5 + \frac{3465\sqrt{D} + 48699}{2} x^4 - \frac{4455\sqrt{D} + 21791}{2} x^3 + \frac{3465\sqrt{D} + 48699}{2} x^2 - \frac{2205\sqrt{D} + 53361}{2} x + 7^5.$$