Group Embeddings: Geometry and Representations

Michel Brion (Universite de Grenoble),
Stephen Doty (Loyola University Chicago),
Lex Renner (University of Western Ontario),
Ernest Vinberg (Moscow State University).

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1 Overview of the Field

The theory of algebraic group embeddings has developed dramatically over the last twenty-five years, based on work of Brion, DeConcini, Knop, Luna, Procesi, Putcha, Renner, Vust and others. It incorporates torus embeddings and reductive monoids, and it provides us with a large and important class of spherical varieties. The interest in these, and related, topics has led to vigorous and celebrated mathematical activity in Europe, Asia and North America. This BIRS workshop provided a timely opportunity to enhance the visibility of this highly interdisciplinary industry, by bringing together some of the principal players in group embeddings and representation theory.

Certain aspects of representation theory are well connected to the geometry of group embeddings, especially through the examples of linear algebraic monoids. The study of representations of reductive monoids has links with quasihereditary algebras and highest weight categories, important topics in the theory of finite dimensional algebras. Important work in this area includes that of Cline, Parshall, and Scott, Donkin, Erdmann, Green, Ringel and others. There are also significant interactions with the representation theory of finite groups, Hecke algebras, and quantum groups, and combinatorics.

The widespread interest in group embeddings results from the inherent richness and depth of the results; combining techniques from commutative algebra, algebraic geometry, representation theory, convex geometry, linear algebra, spherical embeddings, semigroup theory, and combinatorics. This subject is mature yet still growing, and there are many interesting open questions.

The "wonderful compactification" stands out among all others as the one embedding of a semisimple group that serves as the role model for further development. DeConcini and Procesi [**?**] originally calculated the Betti numbers and cell decomposition using the method of Bialinycki-Birula. Since then, important work has been done by Brion [**?**] and Springer [**?**] on the geometry of orbit closures. Kato [**?**] and Tchoudjem [**?**] have found an analogue of the Borel-Weil-Bott theorem, and Kato [**?**] described all the equivariant vector bundles. The wonderful compactification can also be described in terms of certain reductive monoids [**?**]. Several authors have discovered explicit cell decompositions of the wonderful compactification [**?**], [**?**], [**?**], [**?**]. Luna has recently identified a more general class of wonderful embeddings.

There will be generalized Schur algebras (in the sense of Donkin [**?**]) implicit in the coordinate bialgebra of any reductive monoid, and the representation theory of the monoid breaks up into a direct sum of the representation theories of the various generalized Schur algebras, which are finite dimensional quasihereditary algebras. All of this extends the classic motivating example of polynomial representation theory of general linear groups. Although this goes back to Schur's dissertation, at the very inception of representation theory as a mathematical discipline, it remains in positive characteristic a vigorous and intensively studied aspect of modern representation theory, with important work in the past twenty-five years by Green, James, Donkin, Erdmann, Cline, Parshall, and Scott, Dlab, Ringel, and others.

Solomon has studied certain finite monoids (the Renner monoids), analogues of the Weyl group of a reductive group. The simplest examples are the *rook monoids* which are natural generalizations of the symmetric groups. There is recent interesting work of Halverson and Ram (see [**?**], [**?**]) on a q-analogue of the rook monoid. This is just the tip of a large unchartered iceberg; many intriguing structures exist in abundance, and little is known in general. Recent results of Steinberg [**?**] have brought some much-needed clarity to the representation theory of inverse semigroups.

2 Recent Developments and Open Problems

2.1 Recent Developments

There are a great many of these. We give a sketch of the some of the important ones.

1. Complexity of group actions [**?**], [**?**].
2. Stable reductive varieties [**?**], [**?**].
3. Analogues of the Bruhat decomposition for spherical varieties [**?**], [**?**], [**?**].
4. Equivariant compactifications of spherical homogeneous spaces [**?**], [**?**], [**?**].
5. Cartier and Weil divisors on spherical varieties [**?**], [**?**].
6. Universal G_m torsors on certain moduli spaces [**?**], [**?**].
7. Embeddings of G_a^n; an important beginning with many interesting examples [**?**].
8. Compactification of Jacobian varieties [**?**].
9. Among group embeddings and spherical varieties there should be many opportunities to identify and study examples of ”Cox rings” [**?**, **?**].
10. Moduli spaces of group compactifications; generalizations of the toric Hilbert scheme [**?**, **?**, **?**, **?**, **?**].
11. More general wonderful embeddings [**?**]. See also recent work of P. Bravi and G. Pezzini, and P. Bravi and S. Cupit-Foutou.
12. Counterexample to Renner’s conjecture regarding blocks of algebraic monoids [**?**]
13. The issue of *normality* for symplectic and orthogonal monoids now settled in [**?**].
14. Putcha’s penetrating assessment of conjugacy classes on a reductive monoid [**?**]
15. Steinberg’s recent results [**?**] on the representation theory of inverse semigroups, using Moebius inversion and Munn-Ponizovski to obtain information about multiplicities and character formulas.

2.2 Open Problems

1. Describe the cohomology rings of smooth, complete group embedddings, by generators and relations. For toric varieties, the answer is given by a result of Jurkiewicz and Danilov. For ”regular” group embeddings, the equivariant cohomology ring has been described by Bifet, De Concini and Procesi as a subring of a larger ring. But no generators of the cohomology ring are known in general.

2. Construct $B \times B$-equivariant desingularizations of the closures of $B \times B$-orbits in G-embeddings (G a connected reductive group, B a Borel subgroup). Does there exist such a desingularization with only finitely many fixed points of the maximal torus $T \times T$? This reduces easily to constructing a $B \times B$-equivariant desingularization of the closure of B; for regular embeddings, this closure is almost always singular. An affirmative answer to this question implies that the intersection cohomology of $B \times B$-orbit closures vanishes in all odd degrees. This was proved by Springer for the wonderful compactification, by combinatorial arguments.

3. Study the topology of hypersurfaces in smooth complete group embeddings: determine their numerical invariants, by generalizing the known results for toric varieties.

4. (Related to problem 1). In the case of a smooth, projective group embedding X, find an explicit cell decomposition of X, similar to what was done by Brion, Springer and Renner for the wonderful compactification. How is each cell made up of $B \times B$-orbits? Can one find an explicit Bialinycki-Birula 1-parameter subgroup? Is there a monoid-theoretic way to do it in some cases?

5. Compute the characters of simple modules (in the describing characteristic) for reductive normal algebraic monoids. This remains the most fundamental problem in representation theory.

6. Describe the blocks for reductive normal algebraic monoids. Donkin has a combinatorial description of the blocks of the monoid of all $n \times n$ matrices, but there are no results in any other case thus far. DeVisscher's recent counterexample in Type C shows that the answer is not easily predicted from the (known) description of the blocks of reductive algebraic groups.

7. Establish characteristic-free, double-centralizer results for classical groups and other related situations. In a sense, this goes back to early work of DeConcini and Procesi [**?**].

8. Describe the structure and representation theory of various centralizer algebras arising from double centralizer situations. Such algebras tend to be "diagram" algebras with a strong combinatorial flavour, and the modular representation theory of such algebras is wide open.

9. Obtain a class of embeddings for nonreductive groups that is well-behaved geometrically. Such embeddings could help lead to numerical/cohomological information about representations of nonreductive groups.

3 The Workshop Program

MONDAY

9:00–10:00	Mohan Putcha, *Decompositions of reductive monoids*
10:30–11:30	Valentina Kiritchenko, *Euler characteristic of complete intersections in reductive groups*
13:00–14:00	Guided Tour of The Banff Centre; meet in the 2nd floor lounge, Corbett Hall
14:00–15:00	Xuhua He, *G-stable-piece decomposition of a wonderful compactification*
15:30–16:20	Brian Parshall, *Some new highest weight categories with applications to filtrations*
16:30–17:30	Dan Nakano, *Cohomology for algebraic groups and Frobenius kernels*

TUESDAY

9:00–10:00	Claus Mokler, *The face monoid associated to a Kac-Moody group*
10:30–11:30	Alvaro Rittatore, *The structure of algebraic monoids: the affine case*
13:30–14:20	Jürgen Hausen, *Cox rings and combinatorics*
14:30–15:30	Ivan Arzhantsev, *Geometric invariant theory via Cox rings*
16:00–16:40	Benjamin Steinberg, *Möbius functions and semigroup representation theory*
16:50–17:30	Volodmyr Mazorchuk, *Schur-Weyl dualities for symmetric inverse semigroups*

WEDNESDAY

9:00–10:00	D. Luna, *Examples of wonderful varieties*
10:30–11:30	Zinovy Reichstein, *Essential dimension and group compactifications*

THURSDAY

9:00–10:00 Lizhen Ji, *Borel-Serre compactification of locally symmetric spaces and applications*
10:30–11:30 Kiumers Kaveh, *Newton polytopes for flag and spherical varieties*
13:30–14:20 Leonard L. Scott, *Semistandard filtrations in highest weight categories*
14:30–15:30 Nicolas Ressayre, *Geometric invariant theory and eigenvalue problem*
16:00–16:40 V. Uma, *Equivariant K-theory of compactifications of algebraic groups*
16:50–17:30 Jon Kujawa, *Cohomology and support varieties for Lie superalgebras*

FRIDAY

9:00–10:00 Henning Haahr Andersen, *Combinatorial categories and Kazhdan-Lusztig theories*
10:30–11:30 Stephen Donkin, *Calculating the cohomology of line bundles on flag varieties in characteristic p*

4 Abstracts of Talks Given

Speaker: **Henning Haahr Andersen** (Aarhus)
Title: *Combinatorial categories and Kazhdan-Lusztig theory*
Abstract: In joint work with Jantzen and Soergel [1] in the early 1990's we constructed a combinatorial category $\mathcal{K}$. We used it to compare representations of (small) quantum groups to modular representations of the corresponding (infinitesimal) semisimple algebraic group. In recent work Peter Fiebig [2] considers another combinatorial category $\mathcal{B}$ and he gives a functor $\mathcal{B} \to \mathcal{K}$. This he then applies to the related Kazhdan-Lusztig theories.

We shall discuss the constructions of the two categories, the functor between them, and the consequences in representation theory.

[1] H. H. Andersen, J. C. Jantzen and W. Soergel, *Representations of quantum groups at a p-th root of unity and of semisimple groups in characteristic p: Independence of p*, Asterisque **220** (1994), pp. 1–321.

[2] P. Fiebig, *Sheaves on affine Grassmannians, Projective Representations and Lusztig's Conjectures,* Preprint (Universität Freiburg 2007).

Speaker: **Ivan V. Arzhantsev** (Moscow State University)
Title: *Geometric invariant theory via Cox rings*
Abstract: (Joint work with Jürgen Hausen.) The passage to a quotient by an algebraic group action is often an essential step in classical moduli space constructions of Algebraic Geometry, and it is the task of Geometric Invariant Theory (GIT) to provide such quotients. Starting with Mumford's approach of constructing quotients for actions of reductive groups on projective varieties via linearized line bundles and their sets of semistable points [7], the notion of a "good quotient" became a central concept in GIT, compare [10] and [3].

A good quotient for an action of a reductive group G on a variety X is an affine morphism $\pi : X \to Y$ of varieties such that Y carries the sheaf of invariants $\pi_*(\mathcal{O}_X)^G$ as its structure sheaf. In general, a G-variety X need not admit a good quotient, but there may be many (different) invariant open $U \subseteq X$ with a good quotient; we will call them the good G-sets. In this talk, we present a combinatorial construction of good G-sets $U \subseteq X$, which are maximal with respect to the properties either that the quotient space $U//G$ is quasiprojective or, more generally, that it comes with the A2-property, i.e., any two of its points admit a common affine neighbourhood.

Our first step is to consider actions of G on factorial affine varieties X. The basic data for the construction of good G-sets of X are *orbit cones*. They live in the rational character space $\mathbb{X}_{\mathbb{Q}}(G)$, and for any $x \in X$ its orbit cone $\omega(x)$ is the convex cone generated by all $\chi \in \mathbb{X}(G)$ admitting a semiinvariant f with weight χ such that $f(x) \neq 0$ holds. It turns out that there are only finitely many orbit cones and all of them are polyhedral.

To any character $\chi \in \mathbb{X}(G)$ we associate its GIT-*cone*, namely

$$\lambda(\chi) := \bigcap_{\chi \in \omega(x)} \omega(x) \subseteq \mathbb{X}_{\mathbb{Q}}(G).$$

We say that a collection Φ of orbit cones is *2-maximal*, if for any two members their relative interiors overlap and Φ is maximal with respect to this property.

Theorem. *Let a connected reductive group G act on a factorial affine variety X.*

(i) The GIT-cones form a fan in $\mathbb{X}_{\mathbb{Q}}(G)$, and this fan is in a canonical order reversing bijection with the collection of sets of semistable points of X.

(ii) There is a canonical bijection from the set of 2-maximal collections of orbit cones onto the collection of A2-maximal good G-sets of X.

For the case of a torus G this result was known before. The first statement is given in [2]. Moreover, a result similar to the second statement was obtained in [4] for linear torus actions on vector spaces, and for torus actions on any affine factorial X, statement (ii) is given in [1].

To obtain the general statement, we reduce to the case of a torus action as follows. Consider the quotient $Y := X//G^s$ by the semisimple part $G^s \subseteq G$. It comes with an induced action of the torus $T := G/G^s$, and the key observation is that the good T-sets in Y are in a canonical bijection with the good G-sets in X. This approach turns out to be as well helpful for computing GIT-fans, because Classical Invariant Theory in many cases provides enough information on the algebra $\mathbb{K}[X]^{G^s}$ of invariants.

Our next aim is to study quotients of certain non-affine G-varieties X, e.g., the classical case of X being a product of projective spaces. More precisely, we consider normal varieties X with a finitely generated Cox ring

$$\mathcal{R}(X) = \bigoplus_{D \in \mathrm{Cl}(X)} \Gamma(X, \mathcal{O}(D)),$$

where the divisor class group $\mathrm{Cl}(X)$ is assumed to be free and finitely generated. The "total coordinate space" $\overline{X}$ of X is the spectrum of the Cox ring $\mathcal{R}(X)$. This $\overline{X}$ is a factorial affine variety [2] acted on by the Neron-Severi torus H having the divisor class group $\mathrm{Cl}(X)$ as its character lattice. Moreover, X can be reconstructed from $\overline{X}$ as a good quotient $q \colon \widehat{X} \to X$ by H for an open subset $\widehat{X} \subseteq \overline{X}$.

After replacing G with a simply connected convering group, its action on X can be lifted to the total coordinate space $\overline{X}$. The actions of H and G on $\overline{X}$ commute, and thus define an action of the direct product $\overline{G} := H \times G$. Given a good $\overline{G}$-set $W \subseteq \overline{X}$, we introduce a "saturated intersection" $W \sqcap_G \widehat{X}$. The main feature of this construction is the following.

Theorem. *The canonical assignment $W \mapsto q(W \sqcap_G \widehat{X})$ defines a surjection from the collection of good $\overline{G}$-sets in $\overline{X}$ to the collection of good G-sets in X.*

So this result reduces the construction of good G-sets on X to the construction of good $\overline{G}$-sets in $\overline{X}$, and the latter problem, as noted before, is reduced to the case of a torus action. Again, this allows explicit computations. Note that our way to reduce the construction of quotients to the case of a torus action has nothing in common with the various approaches based on the Hilbert-Mumford Criterion, see [3], [5], [7], [9] and [11].

As a first application of this result, we give an explicit description of the ample GIT-fan, i.e., the chamber structure of the linearized ample cone, for a given normal projective G-variety X with finitely generated Cox ring. Recall that existence of the ample GIT-fan for any normal projective G-variety was proven in [5] and [11]. As an example, we compute the ample GIT-fan for the diagonal action of $\mathrm{Sp}(2n)$ on a product of projective spaces $\mathbb{P}^{2n-1}$.

A second application of the above result are Gelfand-MacPherson type correspondences. Classically [6], this correspondence relates orbits of the diagonal action of the special linear group G on a product of projective spaces to the orbits of an action of a torus T on a Grassmannian. Kapranov [8] extended this correspondence to isomorphisms of certain GIT-quotients and used it in his study of the moduli space of point configurations on the projective line. Similarly, Thaddeus [12] proceded with complete collineations. We put these correspondences into a general framework, relating GIT-quotients and also their inverse limits. As examples, we retrieve a result of [12] and also an isomorphism of GIT-limits in the setting of [8].

Finally, we use our approach to study the geometry of quotient spaces of a connected reductive group G on a normal variety X with finitely generated Cox ring. The basic observation is that in many cases our quotient construction provides the Cox ring of the quotient spaces. This allows to apply the language of bunched rings developed in [2], which encodes information on the geometry of a variety in terms of combinatorial data living in the divisor class group.

[1] I.V. Arzhantsev, J. Hausen: On embeddings of homogeneous spaces with small boundary. J. Algebra 304, No. 2, 950–988 (2006), math.AG/0507557

[2] F. Berchtold, J. Hausen: GIT-equivalence beyond the ample cone, Michigan Math. J. 54, No. 3, 483–516 (2006), math.AG/0503107

[3] A. Bialynicki-Birula: Algebraic Quotients. In: R.V. Gamkrelidze, V.L. Popov (Eds.), Encyclopedia of Mathematical Sciences, Vol. 131., 1–82 (2002)

[4] A. Bialynicki-Birula, J. Świci ecka: A recipe for finding open subsets of vector spaces with a good quotient. Colloq. Math. 77, 97–114 (1998)

[5] I.V. Dolgachev, Y. Hu: Variation of geometric invariant theory quotients. (With an appendix: "An example of a thick wall" by N. Ressayre). Publ. Math., Inst. Hautes Etud. Sci. 87 (1998), 5–56.

[6] I.M. Gelfand, R.W. MacPherson: Geometry in Grassmanians and a generalization of the dilogarithm. Adv. in Math. 44, 279–312 (1982)

[7] D. Mumford, J. Fogarty, F. Kirwan: Geometric Invariant Theory. 3rd enl. ed.. Ergebnisse der Mathematik und ihrer Grenzgebiete. Berlin: Springer-Verlag. (1993)

[8] M.M. Kapranov: Chow quotients of Grassmannians. I. Advances in Soviet Math. 16, Part 2, 29–110 (1993)

[9] N. Ressayre: The GIT-equivalence for G-line bundles. Geom. Dedicata 81, No. 1–3, 295–324 (2000)

[10] C.S. Seshadri: Quotient spaces modulo reductive algebraic groups. Ann. of Math. (2) 95, 511–556 (1972)

[11] M. Thaddeus: Geometric invariant theory and flips. J. Amer. Math. Soc. 9, 691–723 (1996)

[12] M. Thaddeus: Complete collineations revisited. Math. Ann. 315, 469–495 (1996)

Speaker: **Stephen Donkin** (York)
Title: *Calculating the cohomology of line bundles on flag varieties in characteristic* p
Abstract: Let G be a connected reductive group over an algebraically closed field of characteristic p and let B be a Borel subgroup. The character of the cohomology of a the line bundle on the flag variety G/B is not well understood (by contrast with the situation in characteristic zero where this is given by Weyl's character formula, via the Borel-Weil-Bott Theorem). We describe some general methods of calculation and a complete solution for the case $G = SL_3(k)$.

Speaker: **Jürgen Hausen** (Tübingen)
Title: *Cox rings and combinatorics*
Abstract: (Joint work with I.V. Arzhantsev and F. Berchtold.) Suppose that X is normal variety with $\Gamma(X, \mathcal{O}^*) = \mathbb{K}^*$ and free, finitely generated divisor class group $\mathrm{Cl}(X)$. Fix a subgroup $K \subset \mathrm{WDiv}(X)$ of the group of Weil divisors mapping isomorphically onto $\mathrm{Cl}(X)$. The *Cox ring* $\mathcal{R}(X)$ is the algebra of global sections of a sheaf of K-graded algebras:

$$\mathcal{R}(X) := \Gamma(X, \mathcal{R}), \qquad \text{where} \quad \mathcal{R} := \bigoplus_{D \in K} \mathcal{O}(D).$$

Note that multiplication in the Cox ring is just multiplication of rational functions on X. Up to isomorphy, the Cox ring does not depend on the choice of K. A basic observation is that Cox rings are unique factorization domains.

The sheaf $\mathcal{R}$ defines moreover a generalized *universal torsor* $'X \to X$. Suppose that $\mathcal{R}$ is locally of finite type; this holds for example, if X is locally factorial or if $\mathcal{R}(X)$ is finitely generated. Then we may consider the relative spectrum $'X := \mathrm{Spec}_X\, \mathcal{R}$, which turns out to be a quasiaffine variety. The K-grading of R defines an action of the torus $H := \mathrm{Spec}\, \mathbb{K}[K]$ on $'X$, and the canonical morphism $p\colon {'X} \to X$ is a good quotient, i.e., it is an H-invariant affine morphism satisfying $\mathcal{O}_X = (p_*\mathcal{O}_{'X})^H$.

If X has a finitely generated Cox ring $\mathcal{R}(X)$, then $'X$ is an invariant open subvariety of the *total coordinate space* $\overline{X} := \mathrm{Spec}\, \mathcal{R}(X)$. Thus, varieties with finitely generated Cox ring are obtained as good quotient spaces of certain affine torus actions on factorial affine varieties. Such quotients in turn admit a description by combinatorial data, which we call "bunches of cones". We describe basic geometric properties of X in terms of its defining bunch of cones, for example, we discuss singularities, the ample cone, Fano criteria, and modifications. Moreover, we give some applications to almost homogeneous spaces.

[1] I.V. Arzhantsev, J. Hausen: On embeddings of homogeneous spaces with small boundary. J. Algebra 304, No. 2, 950–988 (2006), `math.AG/0507557`

[2] F. Berchtold, J. Hausen: Cox rings and combinatorics. Transactions of the AMS 359, No. 3, 1205–1252 (2007), `math.AG/0311105`

Speaker: **Xuhua He** (SUNY - Stony Brook)
Title: *G-stable-piece decomposition of a wonderful compactification*
Abstract: Let G be a connected semisimple algebraic group of adjoint type over an algebraically closed field. Let us consider the diagonal G-action on the wonderful compactification X of G. The classification of the G-orbits were obtained by Lusztig in terms of G-stable pieces. He also used the G-stable-piece decomposition to construct certain simple perverse sheaves on X (which are called character sheaves on X). In this talk, I will discuss some geometric properties of the G-stable pieces. First, we will talk about some relation between the G-stable pieces and the $B \times B$-orbits in X, where B is a Borel subgroup of G. We will then use this relation to study the closure relation of the G-stable pieces and some algebro- geometric properties of the closures of G-stable pieces. Although the closures are not normal in general, they do have "nice" singularities (for example, they admit a Frobenius splitting and as a consequence, they are all weakly normal). If time allows, we will also discuss some generalization to complete symmetric varieties.

Speaker: **Lizhen Ji** (Michigan)
Title: *Borel-Serre compactification of locally symmetric spaces and applications*
Abstract: Let $\mathbf{G}$ be a semisimple linear algebraic group defined over $\mathbb{Q}$, and $\Gamma \subset \mathbf{G}(\mathbb{Q})$ an arithmetic subgroup. Let $X = G/K$ be the symmetric space of noncompact type associated with the real locus $G = \mathbf{G}(\mathbb{R})$. Assume that the $\mathbb{Q}$-rank of $\mathbf{G}$ is positive, or equivalently, the locally symmetric space $\Gamma\backslash X$ is noncompact. In studying both Γ and $\Gamma\backslash X$, an important role is played by the Borel-Serre compactification $\overline{\Gamma\backslash X}^{BS}$, which is the quotient by Γ of a partial compactification $\overline{X}^{BS}$ of X. For example, together with the Solomon-Tits Theorem for Tits building of $\mathbf{G}$, $\overline{X}^{BS}$ can be shown that Γ is a virtual duality group, but not a virtual Poincare duality group.

In this lecture, I will explain this and other applications, together with the following topics:

1. $\overline{X}^{BS}$ is a Γ-cofinite universal space for proper actions of Γ.
2. A uniform Borel-Serre method to construct compactifications of both symmetric and locally symmetric spaces, in particular, the reductive Borel-Serre compactification.
3. Analogues for Teichmuller spaces and mapping class groups and applications.

Speaker: **Kiumars Kaveh** (Toronto)
Title: *Newton polytopes for flag and spherical varieties*
Abstract: The goal of the talk is to give a natural geometric description of the string polytopes for flag varieties and spherical varieties analogous to the definition of the Newton polytopes for toric varieties. This will be a generalization of a result of Okounkov for Gelfand-Cetlin polytopes of $SP(2n, \mathbb{C})$.

The classical construction of Gelfand and Cetlin associates a convex polytope to each irreducible representation of $GL(n, \mathbb{C})$ in such a way that the integral points in the polytope parameterize the elements of a natural basis for the representation. Equivalently one can think of them as polytopes associated to the ample line bundles on the flag variety. A main feature of the G-C polytopes is that the self-intersection number of a generic section of the line bundle is given by the volume of the corresponding polytope. This can be viewed as the flag variety analogue of the well-known Kushnirenko theorem in toric geometry. Since then G-C polytopes have been generalized to all reductive groups, called "string polytopes", by the works of Littelmann, Bernstein, Zelevinsky and others. Even further, it has been generalized to spherical varieties by Okounkov (for classical groups) and by Alexeev-Brion for all reductive groups.

After an introduction to G-C and string polytopes, I will discuss the main result of the talk. Namely, we see that the integral points in the string polytope of a dominant weight λ can be interpreted as the highest terms of the elements of the corresponding irreducible representation V_λ (regarded as polynomials on the big cell) with respect to a natural valuation or term order.

In the second part of the talk, I generalize this construction to any algebraic variety equipped with an ample line bundle (even without a group action!). That is, we associate a convex set (which in most cases turns

out to be a polytope) to an ample line bundle, hence arriving at a far reaching generalization of Kushnirenko theorem in toric geometry. In particular the Okounkov-Brion-Alexeev polytope of a spherical variety can be obtained in this way. Part of this work is joint with A. G. Khovanskii.

Speaker: **Valentina Kiritchenko** (Jacobs University Breman)
Title: *The Euler characteristic of complete intersections in reductive groups*
Abstract: Consider the following class of hypersurfaces in a complex reductive group: for each representation of the group take all generic hyperplane sections corresponding to this representation. I will present an explicit combinatorial formula for the Euler characteristic of complete intersections of such hypersurfaces. The Euler characteristic is expressed in terms of the weight polytopes of the corresponding representations. In particular, this formula extends the formulas of Bernstein and Khovanskii (all complete intersections in a complex torus) and of Brion and Kazarnovskii (zero-dimensional complete intersections in an arbitrary reductive group).

The main ingredients of my formula are Chern classes of a reductive group. These classes are related to the usual Chern classes of regular compactifications of the group. An adjunction formula involving these Chern classes allows to express the Euler characteristic via the intersection indices of the Chern classes with hyperplane sections. The latter are then computed using the De Concini-Procesi algorithm, which was originally devised for the intersection indices of divisors in wonderful compactifications of symmetric spaces. I will show how to refine this algorithm so that it produces explicit formulas for the intersection indices.

Speaker: **Jon Kujawa** (University of Georgia)
Title: *Cohomology and Support Varieties for Lie Superalgebras*
Abstract: (Joint work with Brian D. Boe and Daniel K. Nakano.) Let $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ be a simple classical Lie superalgebra over the complex numbers as classified by Kac [3]. The classical Lie superalgebras are the simple Lie superalgebras whose $\mathfrak{g}_{\bar{0}}$-component is a reductive Lie algebra. Let $G_{\bar{0}}$ be the reductive algebraic group such that $\operatorname{Lie} G_{\bar{0}} = \mathfrak{g}_{\bar{0}}$.

This project entails developing a support variety theory for Lie superalgebras much like the theory for representations in prime characteristic. We first construct detecting subalgebras of $\mathfrak{g}$ and show that these subalgebras arise naturally by using results from invariant theory of reductive groups by Luna and Richardson [5]. In particular, if $R = H^{\bullet}(\mathfrak{g}, \mathfrak{g}_{\bar{0}}; \mathbb{C})$ is the relative cohomology for the Lie superalgebra $\mathfrak{g}$ relative to $\mathfrak{g}_{\bar{0}}$ then there exists a Lie subsuperalgebra $\mathfrak{e} = \mathfrak{e}_{\bar{0}} \oplus \mathfrak{e}_{\bar{1}}$ such that

$$R \cong S^{\bullet}(\mathfrak{g}_{\bar{1}}^{*})^{G_{\bar{0}}} \cong S^{\bullet}(\mathfrak{e}_{\bar{1}}^{*})^{W} \cong \mathrm{H}^{\bullet}(\mathfrak{e}, \mathfrak{e}_{\bar{0}}; \mathbb{C})^{W},$$

where W is a finite pseudoreflection group. By using the finite generation of R we develop a theory of support varieties for modules over the Lie superalgebra (cf. [2]). This description allows us to conclude that the representation theory for the superalgebra over $\mathbb{C}$ has similar features to looking at modular representations of finite groups over fields of characteristic two.

One of our main objectives was to uncover deeper results about combinatorics of the blocks for finite dimensional representations of the Lie superalgebra $\mathfrak{g}$. The "defect" of a Lie superalgebra and the "atypicality" of a simple module (due to Kac-Wakimoto and Serganova) are combinatorial invariants used to give a rough measure of the complications involved in the block structure. We can now provide cohomological and geometric interpretations of the defect of a Lie superalgebra. In particular, this suggests that one could give a more general and functorial definition of defect.

A focus of recent work is the calculation of support varieties in specific cases. We calculate the support varieties for the finite dimensional universal highest weight supermodules (ie. Kac supermodules) for several infinite families of classical Lie superalgebras. When $\mathfrak{g} = \mathfrak{gl}(m|n)$ we are able to use powerful results of Serganova [7] to calculate the support varieties of the simple supermodules. In particular, this allows us to confirm our "atypicality conjecture" discussed in the previous paragraph in the case of $\mathfrak{gl}(m|n)$. These calculations also show that there are striking differences between this theory and the classical theory of support varieties for finite groups.

Let us also mention recent joint work of Irfan Bagci, Jonathan Kujawa, and Daniel K. Nakano on the type W simple Lie superalgebra which suggests that the theory extends to the Lie superalgebras of Cartan type.

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Speaker: **D. Luna** (Fourier Institute - Grenoble)
Title: *Examples of wonderful varieties*
Abstract: Wonderful varieties of rank bigger than 2, under a semi-simple group G, are difficult to describe explicitely. So by "examples" I mean couples (H, S), where H is a subgroup of G such that G/H has a wonderful completion, and where S is the "spherical system" of this completion (i.e. its main combinatorial invariant). I will concentrate on examples for groups of type D_4 and F_4.

Speaker: **Volodymyr Mazorchuk** (Uppsala)
Title: *Schur-Weyl dualities for symmetric inverse semigroups*
Abstract: In this talk I would like to present new Schur-Weyl type dualities which connects the classical symmetric inverse semigroup on $\{1, 2, \ldots, n\}$ (the rook monoid) and the relatively young dual symmetric inverse semigroup on $\{1, 2, \ldots, n\}$. This generalizes both the classical Schur-Weyl duality, the Schur-Weyl type duality between the symmetric group and the partition algebra, and the Schur-Weyl type dualities for the rook monoid discovered by Solomon. An interesting point here is the fact that the dual symmetric inverse semigroup, which was originally defined via a dual categorical construction, now appears as the dual object for the symmetric inverse semigroup from the representation theoretical point of view.

Speaker: **Claus Mokler** (Wuppertal)
Title: *The face monoid associated to a Kac-Moody group*
Abstract: The face monoid and its coordinate ring are obtained from the category of integrable modules of the category O of a symmetrizable Kac-Moody algebra by a Tannaka reconstruction. The face monoid contains the Kac-Moody group as open dense unit group. Its idempotents are related to the faces of the Tits cone. It has similar structural properties as a reductive algebraic monoid. In my talk I will give an overview (on slides) of the algebraic and algebraic geometric results obtained for this monoid as well as for the complex-valued points of its coordinate ring.

Speaker: **Brian Parshall** (University of Virginia)
Title: *Some new highest weight categories with applications to filtrations*
Abstract: Let G be a semisimple, simply connected algebraic group defined over an algebraically closed field k of positive characteristic $p > h$ (the Coxeter number of G). Let $\mathcal{C}$ be the category of rational G-modules. Assume that for each restricted, dominant weight, the Lusztig character formula holds for the character of the irreducible G-module $L(\lambda)$. In this talk, we present two new highest weight categories $\mathcal{C}^{\rm reg}_{\rm even}$ and $\mathcal{C}^{\rm reg}_{\rm odd}$, which might be called the "even" and the "odd" categories of rational G-modules. These categories are (perhaps remarkably) full subcategories of $\mathcal{C}$. This fact depends on the use of the realization of the standard modules $\Delta^{\rm red}(\lambda)$ and costandard modules $\nabla^{\rm red}(\lambda)$ using quantum groups. We indicate some applications of the result; for example, we mention how it is related to a filtration conjecture.

This talk is based on:

[1] E. Cline, B. Parshall, and L. Scott, "Reduced standard modules and cohomology," Trans. Amer. Math. Soc., in press (2007).

[2] B. Parshall and L. Scott, "Some new highest weight categories," to appear in conference proceedings for ICRT-IV Tibet (2007).

Paper [1] initiates a cohomological study of the modules $\Delta^{\rm red}(\lambda)$, $\nabla^{\rm red}(\lambda)$ and applies this to the conjecture of Gurnalnick on 1-cohomology of finite groups. Paper [2] proves the main result mentioned above.

Speaker: **Mohan Putcha** (North Carolina State University)
Title: *Decompositions of reductive monoids*

Abstract: A reductive monoid M is the Zariski closure of a reductive group G. We will discuss three basic decompositions of of M, each leading to a finite poset via Zariski closure inclusion:

1. The decomposition of M into $G \times G$-orbits. The associated poset is the cross-section lattice Λ. This is a generalization of the face lattice of a polytope. While in general the structure of Λ is quite complicated, it is possible to compute the Möbius function on it.

2. The decomposition of M into $B \times B$-orbits, where B is the Borel subgroup of G. The associated poset R is the Renner monoid of M whose unit group W is the Weyl group of G. For $M_n(k)$, combinatorists know R as the *rook monoid* and semigroup theorists know R as the *symmetric inverse semigroup*. We will discuss the rich algebraic and combinatorial structure of R.

3. There is a decomposition of M related to conjugacy classes that is in betweeen the above two decompositions. The underlying finite *conjugacy poset* $\mathcal{C}$ is yet to be fully understood, but promises to have a very rich combinatorial structure. For the matrix monoid $M_n(k)$, $\mathcal{C}$ consists of partitions of m, $m \leqslant n$, ordered by a generalization of the dominance order on partitions of n. As an application of this decomposition we derive a description of the irreducible components of the nilpotent variety M_{nil} of M.

Speaker: **Daniel K. Nakano** (University of Georgia)
Title: *Cohomology for algebraic groups and Frobenius kernels*
Abstract: (joint work with Christopher P. Bendel, Cornelius Pillen.) Let G be a connected reductive algebraic group scheme, B be a Borel sub- group of G, and U be the unipotent radical of B. One of the outstanding open problems is to generalize the Bott-Borel-Weil theorem to understand the structure of the line bundle cohomology groups $H^\bullet(\lambda) := \mathcal{H}^\bullet(G/B, \mathcal{L}(\lambda))$ over fields of positive characteristic. A related question and significant part of this problem involves computing the rational B-cohomology groups $H^\bullet(B, \lambda)$ where λ is a one-dimensional character.

Let $F : G \to G$ be the Frobenius map and G_r (resp. B_r , U_r) be the r-th Frobenius kernels of G (resp. B, U). In this talk I will discuss recent progress in computing cohomology groups for algebraic groups and Frobenius kernels. My objectives for the talk are as follows:

1) Outline how the cohomology calculations for $H^\bullet(B, \lambda)$, $H^\bullet(G_r, H^0(\lambda))$, $H^\bullet(B_r, \lambda)$, $H^\bullet(U_r, k)$, and $H^\bullet(u, k)$ (ordinary Lie algebra cohomology for $u = \text{Lie } U$) are interrelated.

2) Briefly discuss connections with B-cohomology and computing cohomology for Specht modules for symmetric groups due to Hemmer-Nakano [HN]. This topic falls under Section 3 of the Conference Objectives.

3) Discuss two conjectures related to these cohomological calculations:

a) Donkin's Conjecture [D]: This conjecture has a counterexample which was discovered by van der Kallen [vdK]. However, a modified version will be explained and formulated.

b) Induction Conjecture: This conjecture connects the B_r-cohomology with G_r-cohomology.

4) Exhibit explicit cohomological calculations (via slides) for H^1 and H^2 even for small primes [BNP1, BNP2, W]. Generic behavior will be discussed.

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Speaker: **Zinovy Reichstein** (Univ. of British Columbia)
Title: *Essential dimension and group compactifications*
Abstract: The essential dimension of an algebraic object (e.g., of a finite-dimensional algebra, a polynomial, an algebraic variety or a group action) is the minimal possible number of independent parameters required to define the underlying structure. In recent years this notion has been studied by a number of algebraic, geometric and cohomological techniques. In the first part of this talk I will give an overview of this topic. In the second part, based on recent joint work with Ph. Gille, I will discuss a particular lower bound on the essential dimension, conjectured by J.-P. Serre. Our proof of this bound relies on the existence and properties of regular group compactifications.

Speaker: **Nicolas Ressayre** (Monpellier)
Title: *Geometric invariant theory and eigenvalue problem*
Abstract: Let A be an Hermitian matrix: it is diagonalizable with real eigenvalues. Let $\lambda(A)$ denote its increasing spectrum. Set

$$\Delta(l) = \{(\lambda(A_1), \cdots, \lambda(A_l)) \mid A_i \text{ Hermitian with } A_1 + \cdots + A_l = 0\}.$$

The set $\Delta(l)$ is actually a convex polyedral cone. The cone $\Delta(l)$ may also be described in terms of the tensor products of representations of $\mathrm{SL}(n)$ and there are has generalizations for all simple groups.

The question to determine explicitely the inequalities fullfilled by the points of $\Delta(l)$ began with H. Weyl in 1912. Recently, Belkale and Kummar have proposed a list of inequalities which characterize the cone $\Delta(l)$ (and its genralisations for the others simple groups) paramitrized by a condition expressed in terms of a new product on the cohomology group of the flag varieties. Here, we assert that the list of Belkale and Kummar is minimal. The proof is made by using GIT.

Speaker: **Alvaro Rittatore** (Universidad de la Republica)
Title: *The structure of algebriac monoids: the affine case*
Abstract: In the 80's, L. Renner asked the following questions: "Is it true that if an algebraic monoid M is such that its unit group is affine, then M is affine?"; "is it possible to extend Chevalley's Theorem on the structure of algebraic groups to the case of algebraic monoids?". Recently (M. Brion '07), such a structure theorem has been proved. In this talk we concentrate on the first step of this study, namely we show that the first question has a positive answer.

Speaker: **Leonard L. Scott** (Univesity of Virgina)
Title: *Semistandard filtrations in highest weight categories*
Abstract: A definition of semistandard filtration of an object in a highest weight category is given, assuming finiteness of the indexing weight set and of all composition series in the latter category. These filtrations are studied especially in maximal submodules of standard modules, and their behavior under exact functors, such as translation to a wall in an algebraic groups setting, is examined. Some applications are given to extension groups for irreducible modules.

Speaker: **Benjamin Steinberg** (Carleton University)
Title: *Möbius functions and semigroup representation theory*

Abstract: Using Rota's theory of Möbius inversion, we are able to make very explicit the work of Munn and Ponizovskii on representations of inverse semigroups. In particular, one can obtain a formula for multiplicities of representations using only knowledge of the characters of maximal subgroups and the Mobius function of the idempotent semilattice. Since most important inverse monoids, such as Renner monoids of algebraic monoids, have Eulerian semilattices, this leads to relatively simple formulas.

The results for inverse semigroups can be made to work for other classes of semigroups including semigroups of upper triangular matrices over a field. This leads to applications in computing spectra of random walks on such semigroups.

Speaker: **V. Uma** (Madras)
Title: *Equivariant K-theory of compactifications of algebraic groups*
Abstract: In this talk we shall describe the $G \times G$-equivariant K-ring of X, where X is a regular compactification of a connected complex reductive algebraic group G. Furthermore, in the case when G is a semisimple group of adjoint type, and X its wonderful compactification, we shall describe its ordinary K-ring $K(X)$. More precisely, we prove that $K(X)$ is a free module over $K(G/B)$ of rank the cardinality of the Weyl group. We further give an explicit basis of $K(X)$ over $K(G/B)$, and also determine the structure constants with respect to this basis.

The above results have recently appeared in my paper titled "Equivariant K-theory of compactifications of algebraic groups" in Transformation Groups, Vol. 12, No.2, 2007, pp. 371–406.

5 Some Comments on the Outcome of the Meeting

It is clear that "embedding theory", as portrayed in the activities of this conference, constitute a deep and important part of mathematics. Embedding theory has its roots in the 19th century work of Cayley, Klein, Schubert, Cartan and Hilbert. Since then it has been infused with the 20th century developments of Chevalley, Weil, Nagata, Borel, Tits and Mumford. Some of the main general questions are

1. What "is" symmetry?
2. How is it compactified?
3. What does it "look like" at infinity?
4. How is it measured?
5. How do singularities play a role?

Each of these fundamental questions involves some important statements from the geometry of embeddings, combined with some important statements from representation theory.

The first important outcome of this exciting meeting was to "reaquaint" some of the main players in representation theory with some of the main players in embedding geometry. Many important, developing themes in algebra stem from the interaction of the following general themes.

1. Schur algebras, Highest weight categories and character formulas,
2. embedding theory of reductive groups,
3. algebro-geometric methods in representation theory,
4. geometric-topological methods in representation theory, and
5. intersection homology.

The second major outcome of the meeting was to aquaint some of the established researchers with some of the developing young people. This was particularly successful. We had ten young and vigorous researchers participating (Can, Cupit-Foutou, He, Kiritchenko, Kaveh, Maffei, Parker, Tchoudjem, Therkelsen, Uma). Many of these young scholars have already made significant contributions to embedding theory. Unfortunately there was not sufficient time for all of them to give a presentation.

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