New Applications and Generalizations of Floer theory

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May 13, 2007 – May 18, 2007

1 Overview of the Field

In the mid-eighties, Vladimir Arnold made several seminal conjectures which predicted that the number of fixed points of Hamiltonian diffeomorphisms, and more generally intersection points of Lagrangian submanifolds, should be greater than the bounds that one obtains from standard differential topology, [Ar]. These conjectures signaled the beginning of the study of symplectic topology and motivated much of the remarkable progress which has taken place in this active field during the intervening years.

One of the most important outcomes of this activity was Andreas Floer's development of his homology theory, [Fl1, Fl2, Fl3, Fl4, Fl5]. In essence, Floer theory is a set of techniques which makes it possible to extend certain aspects of the Morse theory of finite-dimensional manifolds to infinite dimensional examples. Within symplectic topology, Floer used Gromov's notion of pseudo-holomorphic curves to develop the Morse theory of the variational principles which underlie the Hamiltonian fixed-point and Lagrangian intersection problems. Instead of considering the total gradient flow of the corresponding functionals, Floer theory uses only those gradient trajectories with finite energy. It is Floer's great insight that these trajectories can be viewed as perturbed holomorphic curves, and that, under suitable assumptions, Gromov's compactness theorem provides the space of finite energy trajectories with the algebraic structure necessary to construct a chain complex whose homology is (an often rich) invariant. With this remarkable new tool, Floer proved some of the most significant of Arnold's conjectures under certain topological hypotheses.

The ground-breaking methods developed by Floer have had a profound influence on many areas of mathematics and mathematical physics. They have been applied to the Yang-Mills functional by Floer and Donaldson, and most recently to Seiberg- Witten theory in the work of Kronheimer and Mrowka. Another primary example of this influence is Heegaard Floer theory of Oszvath and Szabo which is a prominent new tool in low dimensional topology. Symplectic versions of Floer theory have also had a profound influence on other areas. For instance, the new and rapidly developing field of string topology is deeply related with the Floer homology of cotangent bundles and the symplectic field theory of unit cotangent bundles via the work of Viterbo, Salamon-Weber, Abbondandolo-Schwarz and Ceileibak-Latchev.

2 State of the field and open questions

Floer homology is still the most important tool in symplectic topology. Recently, there has been a great deal of activity in the field to generalize Floer theory and to apply it in new ways and in new settings. Many of these efforts focus on the analytic underpinnings of Floer theory with the intention of overcoming several natural restrictions of the original theory. Other efforts concern applications of the rich algebraic structures of Floer theories, which arise from its compactness statements. There is also a great effort to utilize the ideas of Floer theory to address new questions in Hamiltonian dynamics and symplectic rigidity. This workshop brought together researchers at the forefront of many of these developments.

3 Presentation Highlights

The following sixteen talks were delivered at the meeting.

A non-displaceable Lagrangian torus in T^*S^2
Peter Albers

Abstract: Leonid Polterovich exhibited a beautiful Lagrangian torus in T^*S^2 and asked if this torus is Hamiltonianly displaceable. In joint work with Urs Frauenfelder we prove that the Lagrangian Floer homology does not vanish, indeed equals the singular homology of the torus. In particular, this gives a negative answer to Polterovich's question. In the talk, we will describe the construction of the Lagrangian torus and present the computation of the Lagrangian Floer homology which is based on an symmetry argument.

An exact sequence for symplectic and contact homology
Frederic Bourgeois

Abstract: Given a symplectic manifold (W, ω) with contact type boundary (M, ξ), one can define the symplectic homology of (W, ω) and the linearized contact homology of (M, ξ) with respect to its filling. We introduce a Gysin-type exact sequence relating these invariants and describe one of the maps therein in terms of rational holomorphic curves in the symplectization of (M, ξ). This is joint work with Alexandru Oancea.

Floer-Novikov homology and Lagrangian embeddings in the cotangent bundle
Mihai Damian

Abstract: We use a non-Hamiltonian version of Floer theory to establish some obstructions on the existence of exact Lagrangian embeddings in the cotangent bundle of a manifold which fibers over the circle.

Intersection rigidity in symplectic topology: some rigid sets are more rigid than the others
Michael Entov

Abstract: A central and well-known rigidity phenomenon in symplectic topology says that certain sets in symplectic manifolds cannot be displaced by a Hamiltonian isotopy (even though they can be displaced by a smooth one). The talk will concern an hierarchy of intersection rigidity properties of sets beyond such a non-displaceability by a Hamiltonian isotopy: as it turns out, some sets cannot be displaced by symplectomorphisms (including non-Hamiltonian ones!) from more sets than the others. I will also present new examples of rigidity of intersections involving, in particular, specific fibers of moment maps of Hamiltonian torus actions, monotone Lagrangian submanifolds (following the previous work of Peter Albers) as well as certain, possibly singular, sets defined in terms of Poisson-commutative subalgebras of smooth functions.

The results are based on the machinery of partial symplectic quasi-states. These are certain real-valued non-linear functionals on the space of all continuous functions on a closed symplectic manifold which are constructed by means of the Hamiltonian Floer theory and which conveniently encode a part of information contained in it.

This is a joint work with Leonid Polterovich.

Trivial Curves in Symplectic Field Theory
Oliver Fabert

Abstract: Unlike trivial cylinders themselves, branched covers of trivial cylinders might come with the right index to contribute to the algebraic invariants of SFT. However, one has to add abstract perturbations to the Cauchy-Riemann operator, using e.g. the polyfold theory by Hofer, Wysocki and Zehnder, before counting these curves. Using obstruction bundles we prove that the resulting relative virtual moduli cycles are zero. In particular, we show how to deal with the codimension one boundary of moduli spaces of punctured curves in order to define Euler numbers for Fredholm problems. It follows that the differential in SFT is indeed strictly (!) decreasing with respect to the natural action filtration.

Rabinowitz's action functional for very negative line bundles
Urs Frauenfelder

Abstract: This is joint work with Kai Cieliebak. The motivation for this work comes from an alternative attempt to prove the Arnold conjecture, avoiding abstract perturbation theory. We consider a very negative line bundle over an integral symplectic manifold. Such a line bundle is itself a symplectic manifold allowing a Hamiltonian circle action coming from rotations in the fibres. Its Marsden-Weinstein quotient is conformally symplectomorphic to the base symplectic manifold. Rabinowitz's action functional is a Lagrange multiplier functional whose critical points lie in the Marsden-Weinstein quotient and project down to critical points of the action functional of classical mechanics. If the line bundle is negative enough there are generically no holomorphic spheres in the line bundle so that Rabinowitz's action functional has better compactness properties than the action functional of classical mechanics.

Normally polynomial perturbation : revisited
Kenji Fukaya

Abstract: In this talk I will explain the first half of Chapter 8 of our book on Lagrangian Floer theory written jointly with Oh, Ohta, and Ono. There the construction of filtered A-infinity algebras over the integers is given for semi-positive Lagrangian submanifolds. The main part of the construction is to use single-valued abstract perturbations of the Kuranishi map of the Kuranishi structure constructed on the moduli space of pseudo-holomorphic discs. Various techniques are used for this purpose including representation theory of groups, real algebraic geometry, and the Whitney stratification of vector bundles over a stack.

The Generalized Weinstein-Moser Theorem and Periodic Orbits of Twisted Geodesic Flows
Basak Gurel

Abstract: The Weinstein–Moser theorem asserts the existence of a certain number of distinct periodic orbits of an autonomous Hamiltonian flow on a symplectic Euclidean space on every energy level near a non-degenerate extremum. A similar question is of interest and has been extensively studied in the case where the Euclidean space is replaced by any symplectic manifold and the non-degenerate extremum at a point is replaced by a Morse-Bott non-degenerate symplectic extremum.

Along these lines, in a recent joint work with Viktor Ginzburg, we prove the existence of periodic orbits of an autonomous Hamiltonian flow on all energy levels near a Morse–Bott non-degenerate symplectic extremum of the Hamiltonian, provided that the ambient manifold meets certain topological conditions.

As an immediate application of the generalized Weinstein–Moser theorem, we establish the existence of periodic orbits of a twisted geodesic flow on all low energy levels, provided that the "magnetic field" form is symplectic and spherically rational.

In this talk I will discuss these results and outline the proof of the main theorem.

Gluing pseudoholomorphic curves along branched covered cylinders
Michael Hutchings

Abstract: We discuss how to glue together two pseudoholomorphic curves in the symplectization of a contact 3-manifold together with an index zero branched cover of an $\mathbb{R}$-invariant cylinder between them. The number of such gluings is given by a count of zeroes of a certain section of an obstruction bundle over a noncompact moduli space of branched covers. We obtain a combinatorial formula for this count. We deduce that $d^2 = 0$ in embedded contact homology. (joint work with Cliff Taubes)

From Monopoles to curves on manifolds with cylindrical ends
Yi-Jen Lee

Abstract: I will describe an extension of Taubes's "$SW \to Gr$" theorem in the context of manifolds with cylindrical ends. The key new ingredient of the proof is an estimate of the Seiberg- Witten "topological energy", which grows linearly with r–the parameter of perturbation–for large r. In fact, its asymptotic slope is precisely $1/4\pi$ of a Gromov notion of energy by Bourgeois-Eliashberg-Hofer-Wysocki-Zehnder. This technical result is useful for a program towards the proof of the equivalence of Seiberg-Witten and Heegaard Floer homologies.

Spectral invariants in Lagrangian Floer homology
Remi Leclercq

Abstract : We generalize the spectral invariants introduced by Y.-G. Oh and M. Schwarz to the case of Lagrangian intersections Floer theory. They are the homological counterpart of Lagrangian spectral invariants of higher order that we also introduce. We provide a way to distinguish one from the other via a purely topological object and estimate their differences in terms of a geometric quantity. We show that this property induces interesting corollaries regarding the homological invariants. Finally, we show that they carry strictly more information than their homological counterpart, even in the Morse case, by making explicit computations in a particular case.

A structure equation for moduli spaces of holomorphic discs
Klaus Mohnke

Abstract: The moduli spaces of holomorphic discs with boundary on a Lagrangian submanifold define an element in the String Topology algebra of the submanifold. This element satisfies an equation similar to the Maurer-Cartan equations. This was observed and exploited by K. Fukaya. He pointed out that this element can be viewed as a flat connection, and applications arise if one applies gauge theoretic arguments. I will try to shed some light on these ideas.

Monotone Lagrangian tori in $\mathbb{C}P^n$
Felix Schlenk

Abstract: We construct many (about 2^n) different monotone Lagrangian tori in $\mathbb{C}P^n$. This is work joint with Yuri Chekanov.

The ring isomorphism between the pair-of-pants and the Chas-Sullivan product
Matthias Schwarz

Abstract: This joint work with Alberto Abbondandolo establishes an explicit ring isomorphism between Floer homology on cotangent bundles with the pair-of-pants product and the Chas-Sullivan product on the free loop space. We consider also the more general case of Floer homology for paths with conormal boundary conditions.

Floer field theory and Lagrangian correspondences
Katrin Wehrheim

Abstract: In joint work with Chris Woodward we prove an isomorphism of Floer homologies for embedded composition of Lagrangian correspondences. This isomorphism is obstructed by a novel type of bubble, which can however be excluded in monotone settings. This isomorphism is the crucial step in building a symplectic 2-category whose morphism spaces are Donaldson/Fukaya-type categories of (generalized) Lagrangian correspondences and Floer homology classes. The algebraic structures are defined by counts of "pseudoholomorphic quilts". We also obtain a "categorification 2-functor". One application is a general machinery for constructing new topological invariants by associating smooth Lagrangian correspondences to "simple morphisms" (e.g. 3-cobordisms or tangles with one critical point) and checking that the Cerf moves (which connect equivalent decompositions into simple morphisms) correspond to embedded composition of Lagrangian correspondences.

Exact triangles for fibered Dehn twists
Chris Woodward

Abstract: Seidel proved an exact triangle for Dehn twists in Floer homology which is a key ingredient in his computational algorithm for computing the Fukaya category of Lefschetz fibrations. I will discuss a generalization of this triangle to the case of fibered Dehn twists, suggested by Seidel-Smith, and its applications to exact triangles for various Floer theories such as $SU(n)$ knot homology. The proof uses pseudoholomorphic quilts. This is joint work with Katrin Wehrheim.

4 Outcome of the Meeting

The talks given at the workshop represented a broad range of research programs which concern or utilize Floer homology.

Various aspects of the analytic underpinnings of Floer theory were addressed in the talks of Fabert, Fukaya, Hutchings, and Lee. Kenji Fukaya presented the latest developments in his joint work with Oh, Ohta and Ono to use Floer theory to associate a filtered A^{∞}-algebra to a semi-positive Lagrangian submanifold. This construction plays a major role in the field of Mirror Symmetry. In his talk, Fukaya described a new approach to the transversality problem for the underlying spaces of holomorphic curves, which uses a variety of techniques. Michael Hutchings presented a new procedure for enumerating the number of ways of gluing holomorphic curves in symplectizations along branched $\mathbb{R}$-invariant holomorphic cylinders of index zero. This is an important step in his joint project with Taubes to construct Embedded Contact Homology and to prove that it agrees with an appropriate version of Seiberg-Witten Floer homology. In particular, this gluing result implies that the square of the differential is zero. On a related topic, Oliver Fabert presented a proof that branched $\mathbb{R}$-invariant holomorphic cylinders of index zero do not contribute the differential Symplectic Field theory. One important implication of this result is that the differential preserves the action filtration Yi-Jen Lee presented new energy estimates which will play a crucial role in her program to relate the Seiberg-Witten and Ozsvath-Szabo Floer theories.

In another direction, Urs Frauenfelder described a joint project with Kai Cieliebak aiming at making use of Rabinowitz's action functional to overcome analytic difficulties in the proof of Arnold's conjecture.

The talks of Bourgeois, Leclercq, Mohnke and Schwarz described new applications and constructions involving the algebraic structure of Floer theory. Frederic Bourgeois described his joint work with Oancea to construct an exact sequence relating the symplectic homology of a symplectic manifold with the linearized contact homology of its boundary. Remi Leclercq outlined a construction of new spectral invariants for Lagrangian manifolds generalizing spectral invariants in Lagrangian Floer homology. The talk of Klaus Mohnke focused on the structure of the string theory algebra of a Lagrangian submanifold, generalizing its Lagrangian Floer homology. Matthias Schwarz described an isomorphism between the ring of Hamiltonian Floer homology of a cotangent bundle and the homology of the loop space of the base equipped with the Chas–Sullivan product.

The talk of Felix Schlenk concerned his joint project with Chekanov to construct and classify monotone Lagrangian tori, which are central objects in the study of Lagrangian Floer homology. Peter Albers reported on a new computation of Lagrangian Floer homology, obtained with Urs Frauenfelder, which is relevant to recent work of Eliashberg, Kim, and Polterovich.

There were also several talks concerning new applications of Floer homology to Hamiltonian dynamical systems and symplectic intersection phenomena. Basak Gurel presented a solution, obtained in collaboration with Viktor Ginzburg, of an old problem of Arnold concerning the existence of periodic orbits for low energy charged particles moving in a nondegenerate magnetic field. Mihai Damian described new obstructions to exact Lagrangian embeddings into cotangent bundles obtained using Floer homology for symplectic maps which are not isotopic to the identity. Michael Entov discussed a hierarchy of symplectic intersection results, obtained, together with Leonid Polterovich, using their newly developed theory of partial symplectic quasi-states.

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