Laplacian Growth and Random Matrices

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Acknowledgments

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- Mihai Putinar (UCSB)

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Area law: $\boxed{t_0 = r^2 - \sum_k k|u_k|^2}$

Schwarz functions and Schottky doubles

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- Complex curve $f(z,\zeta) = 0$, $\Gamma : \zeta = \bar{z}$ - Schottky double

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$$\int f(z)\rho(z)\mathrm{d}^2 z = \int f(z)\rho_s(z)\mathrm{d}^2 z, \quad f(z) \text{ integrable}$$

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- Continuum limit = Laplacian Growth variational formulation

Resolving finite-time singularities of Hele-Shaw flows (Saffman,Taylor, Sakai, Kadanoff, Bensimon, Howison, King, Tanveer, Crowdy, ...)

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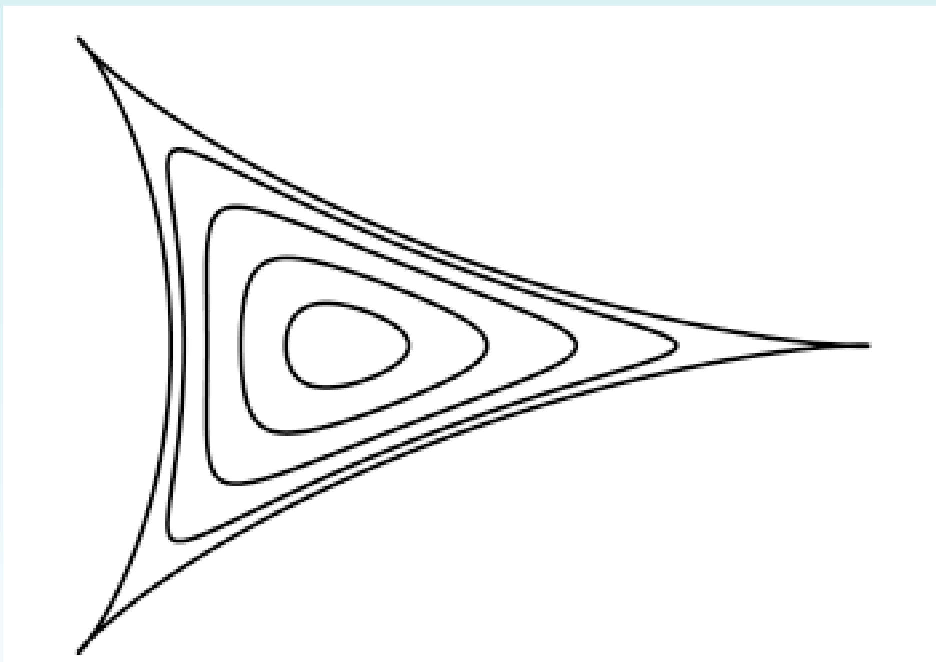

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$$\boxed{w'(z) \to \infty, \quad z \in \partial D}$$

How to make a boundary cusp

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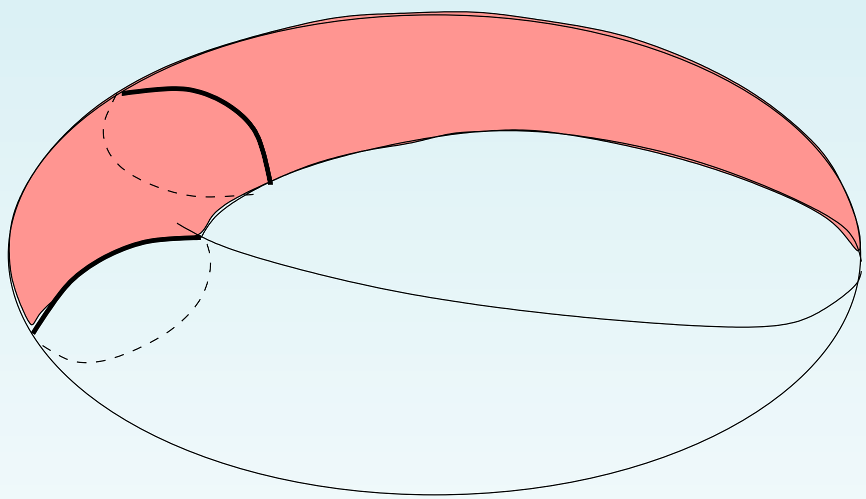
Actually, ... interior branch point $w'(z) \to \infty$ meets exterior double point $S_1(z) = S_2(z)$



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- Often dynamics remains under-determined

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Problem: find the equivalent of Rankine-Hugoniot and Lax-Oleinik conditions for Laplacian Growth dynamics in a weak sense, from stochastic (RMT) formulation.

Normal matrix model and biorthogonal polynomials

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Proper measures – biorthogonal polynomials

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Equivalent to spectral theory of Putinar and Gustafsson.

Circular symmetry: electrons in uniform magnetic field

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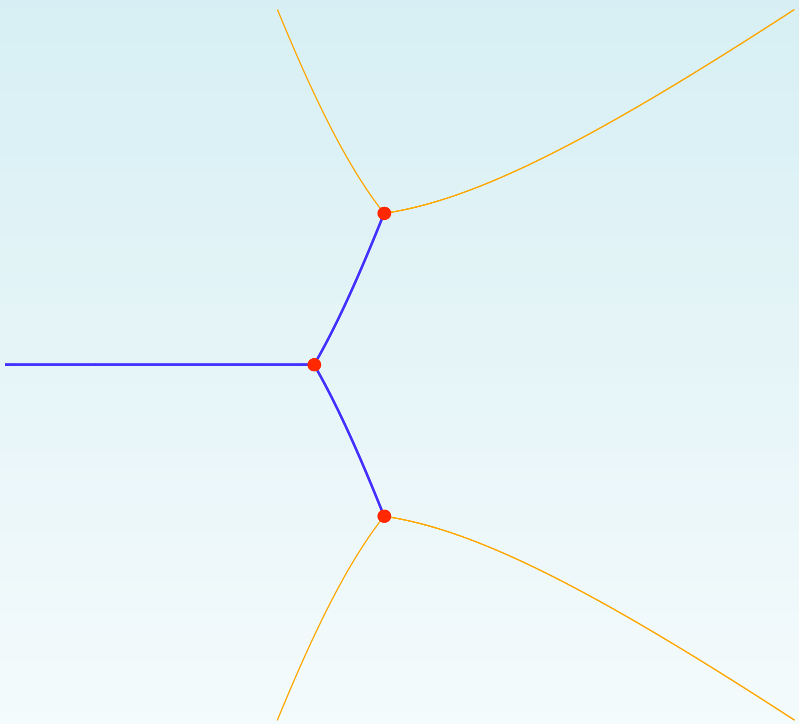

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