Symbolic computer algebra in theoretical physics

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1 Overview of the Field

Computers have assisted many fields of theoretical research as a tool for numerical calculations and simulations. However, it is a relatively recent development that computers can also manipulate algebraic expressions, and in particular that they can handle really huge formulas. This has become possible thanks to an increased power of the hardware, in particular the availability of large memory. Computer algebra requires larger resources than most numerical calculations because it needs, in general, more complex data structures.

Atomic and subatomic physics have benefited from this progress perhaps more than other fields. This is so because they study very clean systems where experiments can reach very high precision which, in principle, can be matched by the theory. However, the theoretical studies are limited by the complexity of calculations which can be carried out by humans.

One measure of the complexity of such calculations is the number of "loops". Represented by so-called Feynman diagrams, subatomic processes have closed loops of interacting particles – a purely quantum phenomenon. Although the first successful one-loop calculations were performed already in late 1940s, the number of loops that can be studied today rarely exceeds three. This slow progress on the side of theory is due to challenges that one has to overcome in each order in the number of loops.

On the other hand, the experimental progress has been huge. New technologies in laser spectroscopy and electronics have enabled studies of increasingly complex atomic systems with ever larger precision. A recent illustration is the 2005 Nobel Prize in physics awarded for the development of the so-called frequency comb, a new tool for optical frequency measurements. In Canada, Eric Hessels at York University has achieved the highest accuracy in helium fine structure measurements. If only a better theoretical knowledge of this spectrum can be achieved, his results will lead to a new determination of the fine structure constant, a fundamental characteristic of electromagnetic interactions.

Symbolic computation allows us now to close the gap between theory and experiment and take full advantage of the recent progress in measurements. Computer algebra has been successfully applied to determine properties of simple atoms and elementary particles. At the University of Alberta the Centre for Symbolic Computation has been created: a laboratory equipped with powerful hardware and software dedicated to manipulating large algebraic expressions.

However, to make further progress it is not enough to rely on the increasing power of computers. To meet the requirements of experiments, four-loop accuracy has to be achieved for a variety of subatomic observables, for example the Lamb shift and the anomalous magnetic moment of the electrons. This next level can be reached only with new mathematical methods.

A typical multi-loop calculation requires an evaluation of a large number of very difficult multiple integrals. The difficulty consists in part in divergences with plague individual integrals and cancel only in a the final sum of partial results. Thus, numerical evaluations are not a good option. An approach which has become standard consists in constructing a system of recurrence relations which, in principle, enable a reduction of all needed integrals to an irreducible basis of so-called master integrals.

Two aspects of this program are challenging:

(a) the solution of the recurrence relations; and

(b) the evaluation of master integrals.

The purpose of this short workshop was to review recent progress in mathematical approaches to the evaluation of Feynman diagrams, with the emphasis on methods of reducing the large number of diagrams to a set of master integrals. Some promising methods of evaluating master integrals were also reviewed.

2 Recent Developments and Open Problems

Recent work of many talented people has resulted in an emergence of new links between areas of mathematics such as perturbation theory, differential equations, and special functions. Indeed, a whole new class of special functions has been identified, harmonic polylogarithms [1]. These functions help to express many master integrals in an exact analytical form.

The real breakthrough in the approach to multi-loop diagrams was the discovery of the so-called Laporta algorithm [2]. It allows to use a computer algebra system to reduce the number of integrals which must be evaluated. Whereas in the past a problem was considered intractable analytically if it involved say a thousand Feynman integrals in total, now we attack problems which contain a similar number of master integrals. In practice, this means that the cutting edge has shifted, roughly speaking, from two- to four-loop problems. This is a huge progress.

However, the experimental techniques have made similar or even greater advances, and we really need to develop methods to tackle the four-loop challenge. The bottleneck turns out to be the evaluation of master integrals, for which an algorithm has yet to be developed.

3 Presentation Highlights

The opening talk of the workshop was given by Matthias Steinhauser (University of Karlsruhe, Germany). It was devoted to automatic generation of Feynman diagrams and their asymptotic expansions. The Karlsruhe group has created a powerful system of programs [3, 4] which can handle a wide class of processes. It is however important to extend it to other cases, especially to the practically important threshold kinematics. Such an extension would also enable one to solve a variety of atomic physics problems.

We had a special talk by a young and very talented PhD student, Alexey Pak. The intention of that talk was to be somewhat provocative, namely to present a number of new ideas from a newcomer to the field, how master integrals can be evaluated. This was a great success and a lively discussion followed (the talk was scheduled as the last one in a session). In addition, Alexey presented new ways of visualizing Feynman diagrams, which will be very helpful when connected with automatic generators of diagrams.

Andrey Grozin (Russian Academy of Sciences, Novosibirsk) reviewed applications of Groebner bases for the reduction of Feynman integrals. The original work by Buchberger provides a prescription for a unique reduction of any multi-variable polynomial. Very recently this has been extended to handle also *non-commuting* objects, such as operators raising and lowering powers of terms in the integrand of a Feynman integral [5]. Unfortunately, the implementation of that new algorithm is not a trivial task, and the one existing program is not publicly available.

4 Outcome of the Meeting

Much of the discussions at this workshop were devoted to various new ideas for evaluating master integrals. The most important outcome is an emerging collaboration between Alberta and Karlsruhe, whose purpose is to determine master integrals for the threshold production of heavy particles. We identified two promising approaches. One is based on differential equations, the other on asymptotic expansions. There is a very strong experimental motivation for this progress and we are confident that it will stimulate development of new mathematical methods.

References

[1] E. Remiddi and J. A. M. Vermaseren, Int. J. Mod. Phys. **A15**, 725 (2000).

[2] S. Laporta, Int. J. Mod. Phys. **A15**, 5087 (2000).

[3] T. Seidensticker, hep-ph/9905298 (unpublished).

[4] R. Harlander and M. Steinhauser, Prog. Part. Nucl. Phys. **43**, 167 (1999).

[5] A. V. Smirnov and V. A. Smirnov, JHEP **01**, 001 (2006).