SYZYGIES and HILBERT FUNCTIONS

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1. Introduction

Throughout k stands for a field. For simplicity, we assume that k is algebraically closed and has characteristic 0. However, many of the open problems and conjectures make sense without these assumptions. The polynomial ring $S = k[x_1, \dots, x_n]$ is graded by $\deg(x_i) = 1$ for all i. A polynomial f is homogeneous if $f \in S_i$ for some i, that is, if all monomial terms of f have the same degree. An ideal I is graded (or homogeneous) if it satisfies the following equivalent conditions:

- ◦ if $f \in I$, then every homogeneous component of f is in I.
- ◦ if $I_i = S_i \cap I$, then $I = \oplus_{i \in \mathbf{N}} I_i$.
- ◦ I has a system of homogeneous generators.

Throughout the paper, I stands for a graded ideal in S and R stands for S/I. The quotient R inherits the grading by $(S/I)_i = S_i/I_i$ for all i.

1.1. Syzygies. A homological method for studying the structure of a finitely generated R-module T is to describe it by a resolution. A free resolution of T is an exact sequence

$$\mathbf{F}: \quad \dots \longrightarrow F_2 \xrightarrow{d_2} F_1 \xrightarrow{d_1} F_0$$

of homomorphisms of free R-modules such that $F_0/\mathrm{Im}(d_1) \cong T$. The idea to associate a resolution to T was introduced in Hilbert's famous 1890, 1893 – papers. He proved that if R is a polynomial ring, then every finitely generated R-module has a finite resolution (that is $F_n = 0$ for $n \gg 0$). If the module is graded, then there exists a minimal free resolution; it is unique up to an isomorphism and is contained in any free resolution of T. Thus, the rank of the n'th free module F_n in the minimal free resolution is less than or equal to the rank of the corresponding free module in any free resolution. The invariants of T are closely related to the properties of its minimal free resolution; the reason for this is that the first differential d_1 describes the minimal relations among the minimal generators of T, the second differential d_2 describes the relations on these relations, etc. Thus, the resolution is a way of describing the structure of T.

Let $\mathbf{F}$ be a minimal free resolution of T. The submodule $\mathrm{Ker}(d_{i-1}) = \mathrm{Im}(d_i)$ of F_{i-1} is called the i'th syzygy module of T, and its elements are called i'th syzygies. The rank of F_i is called the i'th Betti number of T. The Betti numbers are among the most studied and important invariants of T.

Another point of view: in essence constructing a resolution (that is, constructing syzygies) consists of repeatedly solving systems of R-linear equations. The first step is to solve the system of equations $d_1 x = 0$. If R is a field, the process stops here. Otherwise the solutions might be R-linearly dependent and in order to find the dependence relations we have to solve a new R-linear system of equations. This process might never terminate and then we have an infinite resolution. If R is a field, then Linear Algebra describes the computation and properties of the solutions of a system of linear equations. When R is not a field, then studying the properties of the solutions of a system of R-linear equations is usually very difficult.

Recent computational methods have made it possible to compute free resolutions by computer.

1.2. Hilbert functions. A very interesting and important numerical invariant of a graded finitele generated R-module M is its Hilbert function. It gives the sizes of the graded components of M. A case of particular importance is the Hilbert function of a projective algebraic variety X; this function gives the dimensions of the spaces P_i of forms of degree i vanishing on X (for all i). Hilbert's motivation for studying Hilbert functions came from another source: Invariant Theory.

The Hilbert function of R is $\mathrm{Hilb}_R(i)$ $=\dim(R_i)$ for $i \in \mathbf{N}$. Hilbert's insight was that Hilb_R is determined by finitely many of its values. He proved that there exists a polynomial (called the Hilbert polynomial) $h_R(t) \in \mathbf{Q}[t]$ such that $\mathrm{Hilb}_R(t) = h_R(t)$ for $t \gg 0$. If R is the coordinate ring of a projective algebraic variety X, then a classical result states that the degree of the Hilbert polynomial equals the dimension of X, and the leading coefficient determines yet another important invariant – the degree of X.

For many years, Hilbert functions have been both central objects and fruitful tools in many fields, including Algebraic Geometry, Combinatorics, Commutative Algebra, and Computational Algebra. Two major applications of Hilbert functions in Algebraic Geometry are the celebrated Riemann-Roch Formula (proved using a Hilbert polynomial) and Chern classes. Hilbert functions of monomial ideals in $S/(x_1^2, \ldots, x_n^2)$ have been extensively studied in Combinatorics since each such Hilbert function counts the number of faces in a simplicial complex.

Gröbner Basis Theory (from Computational Algebra) reduces many questions on properties of Hilbert functions to properties of Hilbert functions of quotients of S modulo ideals generated by monomials.

Algorithms for the computation of Hilbert functions are implemented in computer algebra systems as COCOA, MACAULAY, MACAULAY2, and SINGULAR. Hilbert functions are used in some algorithms to speed up computation or to compute other invariants.

A classical result of Macaulay provides a characterization of the sequences of numbers that are Hilbert functions of graded ideals in S.

2. Notation

The generating function $i \mapsto \dim_k (S/I)_i$ is called the Hilbert function of R. Denote by $\mathbf{F}$ the graded minimal free resolution of S/I over S. It is unique up to an isomorphism. Let

$$\mathbf{F}: \quad \ldots \longrightarrow F_i \xrightarrow{d_i} F_{i-1} \longrightarrow \ldots \longrightarrow F_1 \xrightarrow{d_1} F_0 \longrightarrow S/I \to 0\,.$$

The *i'th Betti number* of S/I is $b_i = \mathrm{rank} F_i$.

The modules in the resolution are graded and the differential has degree 0. For $p \in \mathbf{Z}$ denote by $S(-p)$ the free graded S-module such that $S(-p)_i = S_{i-p}$; the module $S(-p)$ is generated by one element in degree p. Since each module F_i is a free finitely generated S-module, we can write it as $F_i = \oplus_{p\in\mathbf{Z}} S(-p)^{b_{i,p}}$. Therefore,

$$\mathbf{F}: \ \ldots \ \to \ F_i = \oplus_{p\in\mathbf{Z}}\, S(-p)^{b_{i,p}} \xrightarrow{d_i} F_{i-1} = \oplus_{p\in\mathbf{Z}} S(-p)^{b_{i-1,p}} \ \to \ \ldots$$

The numbers $b_{i,p}$ are called the *graded Betti numbers* of S/I. We say that $b_{i,p}$ is the Betti number in homological degree i and inner degree p, or the i'th Betti number in degree p.

3. Conjectures and Open Problems

In this section we list a number of open problems and conjectures on Hilbert functions and syzygies. Due to space limitations, we do not provide a survey; we just list the problems. Our list is certainly not complete. We have focused on problems that we see as most exciting, or important, or popular. The books [Ei] and [Pe] contain expository papers on some of the problems and related topics.

3.1. Characterizations of Hilbert Functions. The following question is very natural and important: "What sequences of numbers are Hilbert functions of ideals (subject to some property)?".

The characterization of all Hilbert functions of graded ideals in S was discovered by Macaulay. The characterization of all Hilbert functions of graded ideals containing $x_1^2, \ldots, x_n^2$ was obtained by Kruskal-Katona; such Hilbert functions are often studied by counting faces of simplicial complexes via the Stanley-Reisner theory. Furthermore, Clements and Lindström generalized Macaulay's idea and provided a characterization of all Hilbert functions of graded ideals containing $x_1^{a_1}, \ldots, x_n^{a_n}$ for $a_1 \leq \ldots \leq a_n \leq \infty$.

The following very challenging conjecture aims to answer the question "What sequences of numbers are Hilbert functions of ideals containing a regular sequence?".

Conjecture 3.1.1. (Eisenbud-Green-Harris) [EGH] If I contains a regular sequence of homogeneous elements of degrees $a_1, \ldots, a_j$, then there exists a monomial ideal containing $x_1^{a_1}, \ldots, x_j^{a_j}$ with the same Hilbert function.

Using Clements-Lindström's Theorem, it is easy to see that if $j = n$ then Conjecture 3.1.1 is equivalent to the conjecture stated in [EGH]. The following special case is open, and is perhaps the main case of interest:

Conjecture 3.1.2. [EGH] If I contains a regular sequence of n quadrics, then there exists a monomial ideal containing $x_1^2, \ldots, x_n^2$ with the same Hilbert function.

Another challenging conjecture aims to answer the question "What sequences of numbers are Hilbert functions of ideals generated by generic forms?". The conjecture is that if I is generated by generic forms, then I_i is expected to generate in degree $i+1$ as much as possible; the numerical form of the conjecture is:

Conjecture 3.1.3. (Fröberg) [Fr] Let $f_1, \ldots, f_r$ be generic forms of degrees $a_1, \ldots, a_r$. Set $I = (f_1, \ldots, f_r)$. The Hilbert series of S/I is

$$\mathrm{Hilb}_{S/I}(t) = \left| \frac{\prod_{1 \leq i \leq r} (1 - t^{a_i})}{(1-t)^n} \right|,$$

where $||$ means that a term $c_i t^i$ in the series is omitted if there exists a later term $c_j t^j$ with $j \geq i$ and negative coefficient $c_j \leq 0$.

3.2. Regularity. The Castelnuovo-Mumford regularity (or simply regularity) of S/I is

$$\mathrm{reg}(S/I) = \max\{ j \mid b_{i,i+j}(S/I) \neq 0 \text{ for some } i \}$$

and $\mathrm{reg}(I) = \mathrm{reg}(S/I) + 1$. [Ch] is an expository paper on the properties and open problems on regularity. The following conjecture has been open for about 25 years and is the most exciting (currently) open conjecture on syzygies:

The Regularity Conjecture 3.2.1. (Eisenbud-Goto) [EG], [BM] If $P \subset (x_1, \ldots, x_n)^2$ is a prime graded ideal, then

$$\mathrm{reg}(P) \ \leq \ \deg(S/P) - \mathrm{codim}(S/P) + 1\,.$$

The following particular case is very interesting; it is open for toric ideals.

Conjecture 3.2.2. If $P \subset (x_1, \ldots, x_n)^2$ is a prime graded ideal, then the maximal degree of an element in a minimal set of homogeneous generators is $\leq \deg(P)$.

Conjecture 3.2.1 is sharp: for example, the equality holds for the defining ideal of the twisted cubic curve. It will be interesting to explore when the equality holds:

Problem 3.2.3. Find (classes of) graded prime ideals so that for every such ideal $P \subset (x_1, \ldots, x_n)^2$ we have $\mathrm{reg}(P) = \deg(S/P) - \mathrm{codim}(S/P) + 1$.

There is only one known family of ideals – the Mayr-Meyer's examples – where the regularity is doubly exponential in the number of variables, while the maximum degree of an element in a minimal system of homogeneous generators of the ideal is fixed (it is 4). Eisenbud has pointed recently that it will be of interest to construct and study more such examples:

Problem 3.2.4. Find families of graded ideals with regularity, which is doubly exponential (or exponential, or polynomial) in the number of variables, while the maximum degree of an element in a minimal system of homogeneous generators of an ideal is bounded (by a constant).

In the spirit of works by Bertram-Ein-Lazarsfeld and Chardin-Ulrich, we consider:

Problem 3.2.5. Let $a_1 \geq \ldots \geq a_p \geq 2$ be the degrees of the elements in a minimal system of homogeneous generators of I. Set $r = \mathrm{codim}(S/I)$. Find nice sufficient conditions on I so that

$$\mathrm{reg}(S/I) \leq a_1 + \ldots + a_r - r - 1\,.$$

3.3. Lex ideals. The key idea in Macaulay's Theorem, which provides a characterization of the Hilbert functions of graded ideals in S, is that every Hilbert function is attained by a lex ideal. If I is a monomial or toric ideal, then we can define the notion of a lex ideal in the quotient ring R.

Problem 3.3.1. (Mermin-Peeva) [MP] Find classes of either monomial or projective toric ideals I so that Macaulay's Theorem holds over R, that is, every Hilbert function over R is attained by a lex ideal.

Toric varieties are an important class of varieties which occur at the intersection of Algebraic Geometry, Commutative Algebra, and Combinatorics. They might provide many examples of interesting rings in which all Hilbert functions are attained by lex ideals.

It is easy to find rings over which Macaulay's Theorem does not hold. Sometimes, the trouble is in the degrees of the minimal generators of I. Thus, it makes sense to relax the problem to:

Problem 3.3.2. (Mermin-Peeva) [MP] Let d be the maximal degree of an element in a minimal homogeneous system of generators of I. Find classes of either monomial or projective toric ideals I so that every Hilbert function over R of a homogeneous ideal generated in degrees $\geq d$ is attained by a lex ideal.

Furthermore, in view of Hartshorne's Theorem [Ha] that every homogeneous ideal in S is connected by a sequence of deformations to a lex ideal, it is natural to ask:

Problem 3.3.3. Let J be a homogeneous ideal in R, where I is either a monomial or a projective toric ideal, and let L be a lex ideal with the same Hilbert function. When is J connected to L by a sequence of deformations? What can be said about the structure of the Hilbert scheme that parametrizes all homogeneoues ideals in R with the same Hilbert function as L?

Peeva and Stillman have studied these over an exterior algebra, but the problem is already open over Clements-Lindström rings (when $I = (x_1^{a_1}, \ldots, x_n^{a_n})$ and $a_1 \leq \ldots \leq a_n \leq \infty$).

A consequence of the proof of Hartshorne's Theorem is that the lex ideal attains maximal graded Betti numbers among all graded ideals in S with the same Hilbert function; it should be noted that there are examples where no ideal attains minimal Betti numbers. In the same spirit we can consider:

Conjecture 3.3.4. (Mermin-Peeva) [MP] Let J be a homogeneous ideal in R, where I is either a monomial or a toric ideal. Suppose that L is a lex ideal with the same Hilbert function in R. Then:

(1) The Betti numbers of J over R are less than or equal to those of L.
(2) The Betti numbers of $J + I$ over S are less than or equal to those of $L + I$.

In the case when I is generated by powers of the variables, Conjecture 3.3.4(1) coincides with a conjecture of Gasharov, Hibi, and Peeva [GHP]. Conjecture 3.3.4(2) was inspired by work of G. Evans and his conjecture, cf. [FR]:

The Lex-plus-powers Conjecture 3.3.5. (Evans) [FR] If a homogeneous ideal J in S contains a regular sequence of homogeneous elements of degrees $a_1, \dots, a_j$, and if there exists a lex-plus-$(x_1^{a_1}, \dots, x_j^{a_j})$ ideal L with the same Hilbert function as J, then the Betti numbers of L are greater or equal to those of J.

Conjecture 3.3.5 was inspired by the Eisenbud-Green-Harris Conjecture 3.2.1. It proved to be quite challenging. The following interesting special case is open:

Conjecture 3.3.6. Suppose that $I = (x_1^{a_1}, \dots, x_n^{a_n})$ and $a_1 \leq \dots \leq a_n \leq \infty$. Suppose that J is a monomial ideal in R and L is the lex ideal with the same Hilbert function (it exists by Clement's-Lindström's Theorem) in R. Then the Betti numbers of $L + I$ over S are greater than or equal to those of $J + I$.

3.4. Extremal Betti numbers. In the spirit of the above conjectures that the lex ideal attains maximal Betti numbers in various settings, there are several difficult problems on minimal/maximal Betti numbers for specific classes of ideals. We list three of them:

Problem 3.4.1. (Peeva-Stillman) [PS] Let T be a projective toric ideal. Is it true that S/T has the smallest Betti numbers among all ideals with the same multigraded Hilbert function as T?

Conjecture 3.4.2. (Herzog-Hibi) cf. [HM] . Let M be a square-free monomial ideal in S. Let P be the square-free monomial ideal in S such that P is the generic initial ideal of M over the exterior algebra (on the same variables as S). The Betti numbers of M over S are less than or equal to those of P.

Problem 3.4.3. (Geramita-Harima-Shin) [GHS] Does there exist an ideal which has greatest graded Betti numbers among all Gorenstein artinian graded ideals with a fixed Hilbert function?

3.5. The linear strand. The subcomplex

$$\dots \to S(-i-1)^{b_{i,i+1}} \xrightarrow{d_i} S(-i)^{b_{i-1,i}} \to \dots \to S(-2)^{b_{1,2}}$$

of the graded minimal free resolution $\mathbf{F}$ is called the 2-linear strand of S/I. All entries in the differential matrices in the 2-linear strand are linear forms. The length of the 2-linear strand is $p = \max\{i \mid b_{i,i+1} \neq 0\}$.

Problem 3.5.1. (Eisenbud) [Ei2] Let P be a prime graded ideal containing no linear form and whose quadratic part is spanned by quadrics of rank ≤ 4. Suppose that the 2-linear strand of S/P has length p. Find nice sufficient conditions on P so that P contains the 2×2-minors of a $v \times w$-matrix A satisfying the following conditions:

(1) $v + w - 3 = p$
(2) A has linear entries
(3) no entry is zero, and no entry can be made zero by row and column operations.

The need of extra conditions on P in the above problem, is shown to be necesessary by recent examples by Schenk and Stillman. Green's conjecture covers a special case when the ideal P satisfies the following additional conditions:

(1) S/I is normal (that is, it is integrally closed)
(2) $\dim(S/I) = 2$ (that is, P defines a projective curve)
(3) S/I is Koszul
(4) S/I is Gorenstein
(5) $\deg(S/I) = 2(n-1)$.

Problem 3.5.2. How is the length of the linear strand related to the other invariants of S/I?

3.6. Lower bounds on Betti numbers. The following conjecture has been open for a long time, cf. [CE]:

Conjecture 3.6.1. (Buchsbaum-Eisenbud, Horrocks) If I is an artinian graded ideal, then $b_i \geq \binom{n}{i}$ for $i \geq 0$.

3.7. The multiplicity conjecture. [FS] is an expository paper on this conjecture. The set of i'th shifts in $\mathbf{F}$ is $\{p \,|\, b_{i,p} \neq 0\}$. Denote by t_i the minimal i'th shift, and by T_i the maximal i'th shift, that is

$$t_i = \min\{p \,|\, b_{i,p} \neq 0\} \quad \text{and} \quad T_i = \max\{p \,|\, b_{i,p} \neq 0\}\,.$$

The following conjecture provides a lower and upper bound on the multiplicity in terms of the shifts:

Conjecture 3.7.1. (Herzog-Huneke-Srinivasan) Let I be a homogeneous ideal and $c = \operatorname{codim}(S/I)$. The following inequality holds

$$\deg(S/I) \leq \frac{(\prod_{i=1}^{c} T_i)}{c!}\,.$$

If in addition S/I is Cohen-Macaulay, then

$$\frac{(\prod_{i=1}^{c} t_i)}{c!} \leq \deg(S/I)\,.$$

Problem 3.7.2. Explore the above conjecture in the case when I is a monomial ideal and the shifts $\{t_i,\, T_i \,|\, i \geq 0\}$ are the shifts in Taylor's (usually non-minal) free resolution.

Furthermore, it was recently conjectured in what cases one gets equalities in the conjecture:

Problem 3.7.3. (Herzog-Zheng) [HZ] Let I be a homogeneous ideal and $c = \operatorname{codim}(S/I)$. Is it true that one of the equalities

$$\deg(S/I) \leq \frac{(\prod_{i=1}^{c} T_i)}{c!}$$

$$\frac{(\prod_{i=1}^{c} t_i)}{c!} \leq \deg(S/I)\,.$$

holds if and only if both equalities hold, if and only if I is Cohen-Macaulay with a pure minimal free resolution?

3.8. Bounds on the projective dimension.

Problem 3.8.1. (Stillman) Fix a sequence of natural numbers $a_1, \ldots, a_r$. Does there exist a number p, such that

$$\operatorname{pd}(W/J) \leq p$$

if W is a polynomial ring and J is a graded ideal with a minimal system of homogeneous generators of degrees $a_1, \ldots, a_r$? Note that the number of variables in the polynomial ring W is not fixed.

G. Caviglia proved that an upper bound p on the projective dimension (in the above problem) exists if and only if there exists an upper bound on regularity.

3.9. Monomial ideals. Some of the known concepts/constructions useful for studying monomial resolutions are: the Stanley-Reisner correspondence, simplicial and cellular resolutions, Alexander duality, the lcm-lattice, discrete Morse theory, the Scarf complex. The main goal in this area is

Problem 3.9.1. Introduce new constructions and ideas on monomial resolutions.

A monomial ideal M is p-Borel fixed if it is invariant under the action of the general linear group in characteristic p; such ideals are characterized by a combinatorial property on the multidegrees of their monomial generators. The interest in studying such ideals comes from the fact that the generic initial ideals are p-Borel fixed in characteristic p. The minimal free resolution of a 0-Borel fixed ideal is known; it is the Eliahou-Kervaire resolution.

Problem 3.9.2. Find the minimal free resolution and a formula for the regularity of a p-Borel fixed ideal if $p > 0$ (Note that the resolution is considered in characteristic 0).

One can consider Problem 3.9.1 for toric ideals as well; however, proofs are usually considerably harder in the toric case.

Problem 3.9.3. Introduce new constructions and ideas on free resolutions of toric ideals.

3.10. Problems from Algebraic Combinatorics. The expository paper [St] discusses several open problems in Algebraic Combinatorics. We focus on three problems related to Hilbert functions. Recall that there exists a polynomial $h = h_0 + h_1 t + h_2 t^2 + \dots$, such that the Hilbert series of S/I is $\dfrac{h(t)}{(1-t)^{\dim(S/I)}}$. The coefficients of this polynomial form the h-vector of S/I.

A sequence $c_0, \dots, c_r$ of real numbers is called unimodal if for some $0 \le s \le r$ we have $c_0 \le \dots \le c_{s-1} \le c_s \ge c_{s+1} \ge \dots \ge c_r$. The sequence is called log-concave if $c_i^2 \ge c_{i-1}c_{i+1}$ for all $1 \le i \le r-1$. A log-concave sequence of positive numbers is unimodal.

Conjecture 3.10.1. (Stanley) If S/I is a Cohen-Macaulay integral domain, then its h-vector is unimodal.

The following problem is in the spirit of the above conjecture:

Problem 3.10.2. Find classes of graded minimal free resolutions such that the sequence of (total or graded) Betti numbers is log-concave or just unimodal.

Problem 3.10.3. (Charney-Davis-Stanley) [RW] Let I be a quadratic monomial ideal such that S/I is Gorenstein with h-vector $(h_0, \dots, h_{2e})$. Is it true that

$$(-1)^e(h_0 - h_1 + h_2 - \dots + h_{2e}) \ge 0?$$

It might be reasonable to generalize the problem to all Koszul (non-monomial) ideals.

Problem 3.10.4. If I is a monomial ideal such that the quotient S/I is Gorenstein with an h-vector $(h_0, \dots, h_d)$, then is it true that $h_0 \le h_1 \le \dots \le h_{\frac{d}{2}}$? It might be reasonable to generalize the problem to non-monomial ideals.

A set A of subspaces in $\mathbf{C}^n$ is called a subspace arrangement.

Problem 3.10.5. Study problems that relate the properties of subspace arrangements and minimal free resolutions.

The above problem sounds vague, but this is a new area of research and there are no specific conjectures yet; rather, the idea is to explore in this direction. Another problem of this type is about the complement $\mathbf{C}^n \setminus A$, whoose topology has been extensively studied in topological combinatorics.

Problem 3.10.6. Relate the cohomology algebra of the complement $\mathbf{C}^n \setminus A$ and Tor-algebras.

We continue with open problems and conjectures on *infinite* free resolutions. [Av] provide nice expository lectures in this area. In view of the examples in [Ei3] showing that the beginning of an infinite free resolution can be unstructured, it is natural to focus on the asymptotic properties of the resolutions. In the rest of this section, we assume that $R = S/I$ and $I \subseteq (x_1, \dots, x_n)^2$.

3.11. Growth of the Betti numbers. It is known that the growth of the Betti numbers in a minimal free resolution over a quotient ring is at most exponential. The following problems are widely open, cf. [Av].

Problem 3.11.1. (Avramov) What types of growth can the sequence of Betti numbers have (so far, only polynomial and exponential growth have been observed)?

Problem 3.11.2. (Avramov) Is it true that the Betti numbers of every finitely generated graded R-module are eventually non-decreasing?

The following special cases of 3.11.2 are of interest:

Problem 3.11.3. (Ramras) Is it true that if the Betti numbers are bounded, then they are eventually constant?

Problem 3.11.4. Does there exist a periodic module (that is, a module isomorphic to some of its syzygies) with non-constant Betti numbers?

3.12. Rate and the degrees of the entries in the differential maps. In [EH] the following problem is stated in the case when R is a complete intersection, but the authors remark that this assumption might be unnecessary.

Problem 3.12.1. (Eisenbud-Huneke) [EH] Let $\mathbf{G}$ be a graded minimal free resolution over R. Do there exist a number p and bases of the free modules in $\mathbf{G}$, such that for all $i \geq 0$ we have that each entry in the matrix of the differential d_i has degree less than p.

Let M be a finitely generated R-module. If its regularity is infinite, then a meaningful numerical invariant, introduced by Backelin, is

$$\mathrm{rate}_R(M) = \sup\left\{ \frac{p_i - 1}{i-1} \,\middle|\, i \geq 2 \right\}, \quad \text{where } p_i \;=\; \max\{\; j \mid b^R_{i,j}(M) \neq 0 \text{ or } j = i\},$$

called the rate of M. It is known that $\mathrm{rate}_R(M) < \infty$.

Problem 3.12.2. Find classes of rings over which you can give a sharp bound on the rate of k.

3.13. Koszul rings. Koszul rings play an important role in Commutative Algebra, Algebraic Geometry, and other fields. The ring R is called Koszul if the following equivalent conditions hold:

- if $i \neq j$ then the graded Betti number $b^R_{i,j}(k)$ of k over R vanishes
- the entries in the matrices of the differential (in the minimal free resolution of k) are linear forms.

Problem 3.13.1. Find classes of rings, which are Koszul.

The following is the most interesting currently open conjecture on Koszul toric rings:

Problem 3.13.2. (Bøgvad) Is the toric ring of a smooth projectively normal toric variety Koszul?

In particular, it is not known:

Problem 3.13.3. (Bøgvad) Is the toric ideal of a smooth projectively normal toric variety generated by quadrics?

3.14. Generic Poincarè series. This is a new area of research on infinite free resolutions. Avramov has recently raised the problem to search for meaningful conjectures and ideas on what generic behavior means for Poincarè series. There are several directions in which one might think. One of them is to vary the generators of I:

Problem 3.14.1. Let I be generated by generic forms $f_1, \ldots, f_r$ of degrees $a_1, \ldots, a_r$. What can be said about the Poincarè series of k over R?

For example,

Problem 3.14.2. Is it true that R is Koszul if I is generated by generic quadrics?

And more generally:

Problem 3.14.3. Is the Poincarè series of k over R determined by the Hilbert series of R if I is generated by generic forms?

Another possibility is to fix the generators of I, but vary the module that we are resolving:

Problem 3.14.4. Let R be fixed (we probably want I to be generated by generic forms here, or to have another assumption on I). What can be said about the Poincarè series of the generic finitely generated modules over R?

Problem 3.14.5. Yet another possibility is to fix some parameters or properties, and then search for generic behavior. An example of this kind is that if you take take 7 generic quadrics in 4 variables and a module presented by a (2×2)-matrix, then its Betti numbers are constant and equal to 2.

3.15. Complete intersections and exterior algebras. There has been a lot of exciting progress on the structure of minimal free resolutions over complete intersections and over exterior algebras. Although we list only one specific conjecture, we feel that these areas are very fruitful and important, and that it will be of high interest to continue studying such resolutions.

Conjecture 3.15.1. (Eisenbud) [Ei3] Let $\mathbf{G}$ be the minimal free resolution of a finitely generated module over a complete intersection. The Eisenbud operators on $\mathbf{G}$ can be chosen so that they asymptotically commute.

3.16. Infinite resolutions over quotient rings by monomial and toric ideals. Let $\mathbf{G}$ be the minimal free resolution of a monomial ideal over R. Assume that I is either a monomial or a toric ideal. It will be interesting to explore what can be said about the structure of $\mathbf{G}$. For example,

Problem 3.16.1. Develop the theory of infinite cellular resolutions.

Problem 3.16.2. Find how to compute the Betti numbers of $\mathbf{G}$ using simplicial complexes.

Problem 3.16.3. Construct $\mathbf{G}$ in the case it is resolving a lex ideal.

There are a number of other exciting topics on Hilbert functions and syzygies, which we can't discuss because of space limitations. A good way to get a feel of the recent research in this area is to browse the web pages of the participants in the workshop and of other colleagues.

4. Summary of current research by some of the participants in the Banff workshop

Research on Hilbert functions and Syzygies has made substantial progress in the last few years. In this section we provide all the abstracts that were sent to us by participants in the Banff Workshop. They give an excellent overview of some of the current research in the area. The abstracts are ordered alphabetically by the last name of the author.

4.1. Polyhedral and Fansy Divisors (by Klaus Altmann).
Altmann reports results obtained together with Hausen, Hein, and Süss. According to them, varieties X with torus action can be described by divisors $\mathcal{S}$ on their Chow quotients Y. However, this requires the use of rather strange coefficients for $\mathcal{S}$. These coefficients have to be polyhedra if X is affine, and they are polyhedral subdivisions of some vector space in the general case. All coefficients have the same asymptotic behavior – visible as the so-called tail cone or fan of $\mathcal{S}$. This gives $\mathcal{S}$ its name: It is called the "polyhedral" or "fansy" divisor of X on Y. This language comprises that of toric varieties as well as the theory of (good) $\mathbf{C}^*$ actions. They prove the following theorem: Normal, quasiprojective T-varieties X equipped with an open, affine, equivariant covering can be described by fansy divisors.

As an example, Altmann describes the Grassmannian Grass$(2, n)$ as a fansy divisor $\mathcal{S} = \sum_B \mathcal{S}_B \otimes D_B$ on the moduli space $\bar{M}_{0,n}$ of stable rational curves with n marked points. Here, D_B stands for the prime divisor consisting of the two-component curves with a point distribution according to the partition B. The polyhedral subdivisions $\mathcal{S}_B$ look like their common tail fan – only that the origin has been replaced by a line segment whose direction depends on B.

4.2. Affine monoids and Hilbert functions (by Winfried Bruns).
Bruns, Li and Römer investigated seminormal monoids M and their algebras $K[M]$ over a field K and found cohomological criteria for seminormality and various results related to the Cohen-Macaulay property. While there is no direct relationship between seminormality and the Cohen-Macaulay property, various conditions on M imply it or yield at least good lower bound for the depth. All results are accompanied by examples that show their optimality.

Bruns and Römer proved unimodality for the h-vectors of the Ehrhart functions of lattice polytopes with regular unimodular triangulation and a deformation theorem for Gorenstein normal affine monoid domains. This generalizes a theorem of Athanisiadis that in turn proved a conjecture of Stanley. It seems that the theorem obtained is optimal with respect to the methods available. If the g-theorem could be generalized to simplicial spheres, then the condition on regularity can be dropped, but giving up the existence of a unimodular triangulation would require a completely new approach.

Bruns and Ichim proved a theorem on the coefficients of Hilbert quasipolynomials of graded modules M over positively graded K-algebras R: let I be the ideal in R generated by all homogeneous elements of degree coprime to the period of the Hilbert quasipolynomial $q(n)$ of M. Then all coefficients of the powers n^k, $k \geq \dim R/I$, are constant. The theorem was motivated by a theorem of Ehrhart, McMullen and Stanley on Ehrhart quasipolynomials of rational polytopes.

4.3. Infinite syzygies over small rings (by Lars Winther Christensen and Oana Veliche).
A finitely generated module over a local ring is called *infinite syzygy* if it is a cokernel of a differential in a complex with no homology, of finite free modules over the ring. Such a complex is called *acyclic*. Over Gorenstein rings every maximal Cohen-Macaulay module is an infinite syzygy and all acyclic complexes are *totally acyclic*, i.e. the Hom dual complex against the ring is also acyclic. There exist plenty examples of totally acyclic complexes even for non-Gorenstein rings. Of a special interest is an example of V. Gasharov and I. Peeva (1990); this is a totally acyclic complex over a local ring with radical cube zero. On the other hand there exists only one example known of an acyclic complex that is not totally acyclic. This was found by D. Jorgensen and L. Sega (2006), also over a local ring with radical cube zero, having as a source of inspiration the example of V. Gasharov and I. Peeva.

In a joint work Lars W. Christensen and Oana Veliche describe the minimal acyclic complexes over rings with radical cube zero from two points of view: One is the growth of the ranks of the free modules in the complex; only two behaviors are possible and these are described by the two examples above. The other point is to describe the structure of rings that admit infinite syzygies: If a non-Gorenstein local ring $(R, \mathbf{m}, k)$ with $\mathbf{m}^3 = 0$ admits an infinite syzygy, then the embedding dimension e and the socle dimension r differ by 1, i.e. $e = r + 1$, and the Poincaré series of k is $\frac{1}{(1-t)(1-rt)}$. Moreover, the associated graded ring is a Koszul algebra, and under a mild extra hypothesis also the Bass series of R is a rational function determined by r, namely $\frac{r-t}{1-rt}$. The same work also answers a question posed by D. Jorgensen and L. Sega in 2004 about the behavior of Betti numbers of infinite syzygies.

In ongoing joint work, the authors investigate the syzygies of the canonical module for a Cohen–Macaulay local ring. The motivation for this study is the following open question: If the ring is not Gorenstein, does the sequence of Betti numbers of the canonical module have exponential growth?

4.4. Free divisors, generic initial ideals and linear resolutions (by Aldo Conca).
Aldo Conca is currently working on the four problems listed below:

1) Description of families of free divisors. A free divisor is a polynomial f with the property that the module $D(f)$ of the K-derivations d satisfying $d(f) \in (f)$ is free. One of the main goals is to decide, for a given polynomial f, whether there exists g such that gf is a free divisor.

2) The structure of the generic initial ideal for particular type of ideals as, e.g., stable ideals. It is know how the gin of a "principal" stable ideal looks like. One would like to extend the description to a larger class.

3) Products of ideals of linear forms are known to have linear resolution. Thanks to a recent result of Derksen, we have a description of the Betti numbers for "general" ideals in this class. One would like to get also a description of the maps and also to extend the result to non-general configurations. Another interesting question is whether ideals in this class have linear quotients.

4) Let I be an ideals generated by a vector space V of quadrics. Does I have a Gröbner basis of quadrics? If V has codimension ≤ 2 in the space of quadrics then the answer is positive with, essentially, one exception. The goal is to extend this result to space of quadrics of codiension 3.

4.5. The Eisenbud-Green-Harris Conjecture (by Susan Cooper).
Fix $S = k[x_1, \ldots, x_n]$ with the usual grading, where k is an algebraically closed field. It follows from the work of Macaulay that the Hilbert function of any quotient S/I, where I is a homogenous ideal, is also attained by S/L where L is a *lex-segment ideal*. In the same spirit, it has been conjectured by Eisenbud-Green-Harris that *lex-plus-powers ideals* play the role of the lex-segment ideals when restricting to homogeneous ideals containing a regular sequence of forms in the fixed degrees $a_1 \leq a_2 \leq \cdots \leq a_n$.

The Eisenbud-Green-Harris Conjecture is only known to be true in some special cases. The conjecture has been proven in the cases where L is an almost complete intersection (C. Francisco), where I is a monomial ideal containing $x_1^{a_1}, \ldots, x_n^{a_n}$ (Clements-Lindström), and for arbitrary ideals I containing $x_1^{a_1}, \ldots, x_n^{a_n}$ (Cooper-Roberts). B. Richert has verified the conjecture for $n = 2$, and for $n \leq 5$ with each $a_i = 2$. In addition, Caviglia-Maclagan have proven the conjecture in general under certain restrictions on the degrees a_i. On the geometric side, S. Cooper has proven the conjecture for ideals of finite sets of distinct points in $\mathbf{P}^2$, as well as in $\mathbf{P}^3$ provided that when $a_1 \geq 4$, then $a_3 \geq a_1 + a_2 - 1$. Cooper has also studied the geometric consequences of the conjecture where it is known to be true.

4.6. Minimal number of generators of Gorenstein ideals (by Juan Elias).
Let I be a perfect height h ideal in a regular local ring (R, m), $k = A/m$ is the residue field. The problem of the determination, and in particular to find sharp upper bounds, of the Betti numbers of R-modules started with the celebrated examples of Macaulay, [F.S. Macaulay, The algebraic Theory of modular systems, Cambridge University (1916)]. Elias, Robbiano and Valla give in [J. Elias, L. Robbiano, and G. Valla, Number of generators of ideals, Nagoya Math. J. 123 (1991), 39–76] a sharp upper bound of the number generators of perfect ideals in terms of the multiplicity and height. Valla extended these bound to all Betti numbers of perfect ideals, provided that k is a characteristic zero field, [G. Valla, On the Betti numbers of perfect ideals, Compositio Mathematica 91(3) (1994), 305–32]. Elias give in [J. Elias, Sharp upper bounds for the Betti numbers of Cohen-Macaulay modules, Illinois J. Math. 41 (1997)] characteristic free sharp upper bounds of Betti numbers of perfect modules.

Recently Elias and Valla deal with the determination of the number of elements and the structure of the minimal system of generators of Artin Gorenstein ideals I such that $H_A(2) \leq 2$. They give an explicit minimal system of generators of I and a description of the moduli classes of such a rings under k-algebra isomorphisms. In particular the minimal number of generators of I depends only on the height of I and the Cohen-Macaulay type of A. As a corollary of these results Elias and Valla prove that if $A = R/I$ is a multiplicity $e(A)$ Gorenstein ring such that $e(A) \leq h + 4$ then the minimal number of generators of I is equal to $v(I) = \binom{h+1}{2} - 1$.

4.7. Connecting commutative algebra and combinatorics (by Chris Francisco).
Chris Francisco's current research explores connections between combinatorial objects and monomial ideals, especially using the tools of free resolutions and Alexander duality. The overall goal is to build a dictionary between combinatorial properties of simplicial complexes and algebraic features of the monomial ideals associated to them; on the algebraic side, there is a particular focus on (sequential) Cohen-Macaulayness. Recent work with Van Tuyl demonstrates that the edge ideals of chordal graphs are sequentially Cohen-Macaulay, generalizing an earlier result of Herzog, Hibi, and Zheng on Cohen-Macaulay chordal graphs. Other research with Hà investigates the algebraic effect of adding "whiskers" to a graph, extending work of Villarreal. Both projects use Alexander duality and the notion of a componentwise linear ideal to obtain proofs far different from the work on which they build. In a similar direction, Francisco used a combination of Fröberg's Theorem on graphs and resolutions along with Alexander duality to characterize explicitly which tetrahedral curves are arithmetically Cohen-Macaulay. Francisco and his collaborators are now working on generalizations of their work to higher-dimensional simplicial complexes and are also interested in extensions of Fröberg's result to higher dimensions. These investigations have raised a number of interesting related questions, including methods for computing Betti numbers of monomial ideals and predicting when resolutions of monomial ideals are characteristic-dependent.

4.8. Resolutions of ideals of fat points in $\mathbf{P}^2$ (by Brian Harbourne).
Given distinct points $p_1, \ldots, p_n \in \mathbf{P}^2$: *What are the Hilbert functions h_I and graded Betti numbers* for ideals $I = I(p_1)^{m_1} \cap \cdots \cap I(p_n)^{m_n}$ generated by all forms vanishing to order at least m_i at each p_i? Guardo-Harbourne (to appear, J. Alg.) give a matroid-like answer when $n \leq 6$ (see `6ptsNef-K/6reswebsite.html` at `http://www.math.unl.edu/∼bharbourne1/`). Cases $n = 7$ and $n = 8$ look doable, since Geramita-Harbourne-Migliore give a similar answer for Hilbert functions for $7 \leq n \leq 8$ (see `8ptres/8reswebsite.html` and `FatPointAlgorithms.html`, loc. cit.) and Fitchett-Harbourne-Holay (J. Alg. 2001) do the case of $n \leq 8$ general fat points. To do $n > 8$ arbitrary points looks extremely hard, since the possible Hilbert functions are not known. For general points, there at least is a well tested conjecture for h_I, the SHGH Conjecture (for Segre, Harbourne, Gimigliano and Hirschowitz). Let α be the least j such that $I_j \neq 0$. For simplicity, let $t > \alpha$ and let $c_1C_1 + \cdots + c_rC_r$ be the divisorial part of the base locus of I_t. In this special case the *SHGH Conjecture asserts* the curves C_j are rational and $h_I(t) = \binom{t+2}{2} - \sum_i \binom{m_i+1}{2} + \sum_j \binom{c_j}{2}$. Gimigliano, Harbourne and Idà have now extended this to a conjecture for the graded Betti numbers in degrees past $\alpha + 1$. To state it, note that one can reduce to the case that the base locus of I_{t+1} contains no curves. Assuming this, let $c'_1C_1 + \cdots + c'_rC_r$ be the divisorial part of the base locus of I_{t-1}, let $c''_j = c'_j - c_j$ and let $\pi_j : \bar{C}_j \to C_j$ be the normalization morphism. Then $\pi_j^*(\Omega_{\mathbf{P}^2}(1)|_{C_j}) \cong \mathcal{O}_{\bar{C}_j}(-a_j) \oplus \mathcal{O}_{\bar{C}_j}(-b_j)$ for some integers $a_j \leq b_j$, and *the conjecture is that* there should be g_{t+1} degree $t+1$ homogeneous generators for I in any minimal set of generators, where $g_{t+1} = \sum_j \left(\binom{c''_j - a_j}{2} + \binom{c''_j - b_j}{2}\right)$. To determine the a_j and b_j in terms of the m_i is open, but sometimes it's easy and in any case computing them is much easier than computing g_{t+1} directly.

4.9. Algebraic shifting and graded Betti numbers (by Takayuki Hibi).
Let $S = K[x_1, \ldots, x_n]$ denote the polynomial ring in n variables over a field K with each $\deg x_i = 1$. Let Δ be a simplicial complex on $[n] = \{1, \ldots, n\}$ and $I_\Delta \subset S$ the Stanley–Reisner ideal of Δ. We write Δ^s, Δ^e and Δ^c for the symmetric algebraic shifted complex, the exterior algebraic shifted complex and a combinatorial shifted complex, respectively, of Δ. It has been conjectured that, for an arbitrary simplicial complex Δ on $[n]$, one has the following inequalities for the Betti numbers

$$\beta_{ii+j}(I_\Delta) \leq \beta_{ii+j}(I_{\Delta^s}) \leq \beta_{ii+j}(I_{\Delta^e}) \leq \beta_{ii+j}(I_{\Delta^c}) \qquad \text{for all } i \text{ and } j.$$

When the base field is of characteristic 0, the first inequality $\beta_{ii+j}(I_\Delta) \leq \beta_{ii+j}(I_{\Delta^s})$ is proved by

Aramova, Herzog and Hibi. When the base field is infinite, the third inequality $\beta_{ii+j}(I_{\Delta^e}) \leq \beta_{ii+j}(I_{\Delta^c})$ is proved by Murai and Hibi. At present, the second inequality $\beta_{ii+j}(I_{\Delta^s}) \leq \beta_{ii+j}(I_{\Delta^e})$ remains unsolved.

On the other hand, Murai and Hibi prove that the inequalities $\beta_{ii+j}(I_{\Delta}) \leq \beta_{ii+j}(I_{\Delta^c})$ for all i and j, where the base field is arbitrary.

4.10. Linear resolutions of quadratic monomial ideals (by Noam Horwitz).
Noam Horwitz studies the minimal free resolution of a quadratic monomial ideal in the case where the resolution is linear. First, he focuses on the squarefree case, namely that of an edge ideal. He provides an explicit minimal free resolution under the assumption that the graph associated with the edge ideal satisfies specific combinatorial conditions. In addition, he construct a regular cellular structure on the resolution. Then Horwitz extend his results to non squarefree ideals by means of polarization.

4.11. Support of Local Cohomology (by Craig Huneke).
Craig Huneke raised the question of whether the set of associated primes of local cohomology is always finite. This is true for the local cohomology of ideals in regular local rings of equicharacteristic, by work of Huneke and Sharp(is positive characteristic) and Lyubeznik (in equicharacteristic 0). However, it is known to be false in general. The first counterexample was due to A. Singh.

However, Lyubeznik modified the question to ask whether or not the support of arbitrary local cohomology modules are closed? This is equivalent to asking whether there are only finitely many minimal associated primes to local cohomology modules.

There is no counterexample known, but the number of proven cases is very small. Of course, in equicharacteristic regular rings the answer is positive since even the set of all associated primes is finite. In characteristic p, the nth local cohomology of the ring with respect to an n-generated ideal always has finitely many associated primes, an observation of Lyubeznik.

Recent work of Huneke, Katz, and Marley has shown that if the cohomological dimension of an ideal is at most two, then the answer is positive for local cohomology with support in that ideal.

4.12. Class and rank of differential modules (by Srikanth Iyengar).
Twenty years ago Gunnar Carlsson discovered and efficiently exploited similarities between two series of results and conjectures from commutative algebra and algebraic topology. On the topological side they dealt with finite CW complexes admitting free torus actions; on the algebraic one, with finite free complexes with homology of finite length. However, no single statement—let alone common proof—covers even the basic case of modules over polynomial rings.

In recent work, Avramov, Buchweitz, and Iyengar explore the commonality of the earlier results and proved that broad generalizations hold for all commutative algebras over fields. The focus of their work is on a simple construct: a module over an associative ring, equipped with an linear endomorphism of square zero. They call these differential modules. It is a structure that underpins complexes of modules over rings, and also differential graded modules over graded rings. Avramov, Buchweitz, and Iyengar establish lower bounds on the class—a substitute for the length of free complexes—and on the rank of a differential module in terms of invariants of its homology. These results include both Carlsson's theorems on differential graded modules over graded polynomial rings and the New Intersection Theorem for local algebras, due to Hochster, Peskine, P. Roberts, and Szpiro. They also suggest precise statements about matrices over commutative rings, that imply conjectures on free resolutions, due to Buchsbaum, Eisenbud, and Horrocks, and conjectures on the structure of complexes with almost free torus actions, due to Carlsson and Halperin. These conjectures are among the fundamental open questions on both sides of this narrative.

4.13. The Multiplicity Conjecture (by Manoj Kummini).
Let k be a field and $S = k[x_1, \cdots, x_n]$ be a polynomial ring. Let $I \subseteq S$ be a square-free monomial ideal

of height c; let T_i be the maximum degree of the lcm of any set of i monomial minimal generators of I. Then Herzog-Srinivasan conjecture that (called Taylor bound conjecture) $c!e(S/I) \leq \prod_{i=1}^{c} T_i$. Kummini proves the Taylor bound conjecture of Herzog-Srinivasan for Stanley-Reisner ideals of graphs. He also gives an alternate proof in the case of height 2 ideals, which was earlier proved by Herzog-Srinivasan. For Stanley-Reisner ideals of graphs and of simplicial complexes of dimension 2, a stronger conjecture (due to Herzog-Huneke-Srinivasan) was proved by Novik-Swartz.

4.14. Lexicographic ideals and Hilbert functions (by Jeff Mermin).
Mermin's research has been on Hilbert functions and lex ideals. He has developed the theory of compression (introduced by Macaulay) and used it to produce new proofs of Macaulay's theorem that every Hilbert function of an ideal in the polynomial ring is attained by a lex ideal, and to identify some quotients of the polynomial ring in which Macaulay's result holds. In joint work with Peeva and Stillman, he has described the resolutions of monomial ideals containing the squares of the variables, and proved that the Betti numbers of such ideals are maximized by the lex-plus-squares ideals.This establishes a special case of Graham Evans's Lex-Plus-Powers conjecture.

4.15. New directions for the Multiplicity Conjecture (by Juan Migliore).
This describes a series of projects of J. Migliore and U. Nagel in collaboration with T. Römer. The Multiplicity Conjecture of Herzog, Huneke and Srinivasan gives a very simple lower and upper bound for the multiplicity of a Cohen-Macaulay graded algebra R/I in terms of extremal graded Betti numbers. Stated the right way, they also conjecture that the upper bound holds for the non-Cohen-Macaulay case, although easy examples show that the lower bound does not hold. Recently, extensions of this conjecture have been posed and proved in special cases. Important steps have been contributed by several authors.

In a 2005 paper in Math. Res. Lett., Migliore, Nagel and Römer showed that for codimension two Cohen-Macaulay algebras and codimension three Gorenstein algebras, not only do the original bounds hold, but in fact stronger bounds hold. These stronger bounds are still in terms of the same invariants as the original conjecture. This gave as an immediate corollary that in these two situations, either bound is sharp if and only if the other is sharp, if and only if the algebra has a pure resolution. The first extension of the Multiplicity conjecture is that this latter statement holds in general, and that if R/I is not assumed to be Cohen-Macaulay then the sharpness of either bound implies the Cohen-Macaulayness (and hence the rest of the extended conjecture holds). Only special cases of this extended conjecture are known.

It is natural to ask why the lower bound fails in general, and what modifications can be put into place to fix it. In a paper to appear in Trans. Amer. Math. Soc., Migliore, Nagel and Römer showed that for a non-ACM curve C in $\mathbf{P}^3$, a good lower bound can be given by invoking a certain submodule of the Rao module $M(C)$. No similar extension has been worked out in other non-ACM situations. The authors also gave the first steps toward extending this conjecture to the case of graded modules. It was proven, for example, for finitely generated graded R-modules N of rank 0 and codimension 2, with equality if and only if N is Cohen-Macaulay with pure resolution.

All known special cases involve situations where additional structure is given. The first really general open case is that of codimension three Cohen-Macaulay algebras.

4.16. Problems on Hilbert functions of Gorenstein algebras (by Uwe Nagel).
This is a joint project with J. Migliore and F. Zanello. Hilbert functions of Gorenstein algebras, arising often in disguised form, are of great interest in various areas. They are completely classified if the codimension of the algebra is at most three. The condition is that the h-vector has to be a so-called SI-sequence. In particular, it is unimodal. However, an example of Stanley in 1978 and subsequent work by Bernstein, Boij, Iarrobino, and Laksov showed that in every codimension ≥ 5 there are artinian Gorenstein algebras whose Hilbert function is not even unimodal. The situation in codimension four remains a mystery, but recently

there has been some progress when the initial degree is small. Migliore, Nagel and Zanello have extended a result of Iarrobino and Srinivasan by showing that the Hilbert function of a codimension four Gorenstein algebra is an SI-sequence if the initial degree is at most three.

In Stanley's example non-unimodality occurs in degree two. He went on to conjecture the existence and the value of a limit involving the degree two entry. Stanley and Kleinschmidt gave bounds on the limit assuming that it exists. In a recent preprint Migliore, Nagel, and Zanello establish a lower bound on the degree two entry which, together with the construction of suitable Gorenstein algebras, provides a proof of Stanley's conjecture.

The Hilbert functions of reduced Gorenstein algebras are expected to have much better properties. Evidence for this stems from the famous g-Theorem that characterizes the f-vectors of simplicial polytopes. There is still no algebraic proof of this result. It is conjectured that the g-Theorem extends to all triangulations of spheres. Migliore and Nagel have proposed a more general context for this conjecture by raising the question whether every finite Gorenstein sets of points has the Weak Lefschetz Property.

4.17. Hypersurfaces cutting out a projective variety (by Atsushi Noma).
Let X be a nondegenerate projective variety of degree d and codimension e in a projective space $\mathbf{P}^N$ defined over an algebraically closed field. By considering hypersufaces of degree at most d or $d-e+1$ in $\mathbf{P}^N$ containing X, Noma considers the following two problems and give partial answers: Is the length of the intersection of X and a line L in $\mathbf{P}^N$ at most $d-e+1$ if $L \not\subseteq X$? Is the scheme-theoretic intersection of all hypersurfaces of degree at most $d-e+1$ containing X equal to X? To consider the second problem, Noma studies the locus of points from which X is projected non-birationally.

4.18. Invariants, syzygies, and combinatorics (by Vic Reiner).
Recently V. Reiner, D. Stanton and D.White, have encountered many instances instances of a pleasing enumerative phenomenon that they call the *cyclic sieving phenomenon*. Unified explanations for why it happens so often are rare, but one explanation uses the following invariant theory set-up.

Assume one is given a finite subgroup W of $GL(V)$ acting on the polynomial algebra $S = \mathrm{Sym}(V^*)$ for which the invariant ring S^W is again a polynomial algebra; typical examples are complex reflection groups W, or $W = GL_n(\mathbf{F}_q)$. Given any subgroup W' of W, for the combinatorial application one needs to understand something about the quotient of Hilbert series $\mathrm{Hilb}(S^{W'}, t)/\mathrm{Hilb}(S^W, t)$. When working over a field k of characteristic zero, $S^{W'}$ is Cohen-Macaulay, and this quotient coincides with the Hilbert series of $S^{W'} \otimes_{S^W} k = \mathrm{Tor}_0^{S^W}(S^{W'}, k)$. Here what one needs is provided by a representation theoretic result of T. Springer. Working over arbitrary fields, one needs a conjectural generalization to $\mathrm{Tor}_*^{S^W}(S^{W'}, k)$. Partial results have by given in joint work with L. Smith, D. Stanton, P. Webb.

4.19. The Lex-Plus-Powers Conjecture (by Ben Richert).
Let $R = k[x_1, \ldots, x_n]$ be a polynomial ring over a field k, H be the Hilbert function of a cyclic R-module, and B_H be the collection of sets of graded Betti numbers of R-modules attaining H. There is a natural partial order on B. In particular, write $\beta^I_{i,j}$ to be the i, j^{th} graded Betti number of R/I (so $\beta^I_{i,j} = \dim_k \mathrm{Tor}^i_R(R/I, k)_j$); then given R/I and R/J with Hilbert function H, one says that $\beta^I \geq \beta^J$, that is, that the graded Betti numbers of R/I are greater than those of R/J, if $\beta^I_{i,j} \geq \beta^J_{i,j}$ for all $i, j \geq 0$. That B_H has a unique largest element, achieved by the lex ideal, was shown by Bigatti, Hulett, and Pardue, while Charalambous and Evans demonstrated that unique smallest elements may not exist. An important problem in commutative algebra is to understand the interior of B_H.

So let $\mathbf{a} = \{a_1, \ldots, a_n\}$ be a sequence of degrees, where $1 < a_1 \leq a_2 \leq \cdots \leq a_n$. We say that an ideal I contains an $\mathbf{a}$-regular sequence, if I contains a regular sequence $f_1, \ldots, f_n$ such that $\deg(f_i) = a_i$. Now, given a Hilbert function H and a sequence of degrees $\mathbf{a}$, let $L(\mathbf{a}, H)$ be the monomial ideal which, if it exists, is minimally generated by $x_1^{a_1}, \ldots, x_n^{a_n}, m_1, \ldots, m_t$ with the property that $y \in R_{\deg m_i}$ and $y >_{\text{lex}} m_i$ implies that $y \in L(\mathbf{a}, H)$. Such ideals are called Lex-Plus-Powers ideals; their graded Betti numbers are conjectured by Charalambous and Evans to give the unique largest element of $B_{H,\mathbf{a}}$, the subset of B_H consisting of the graded Betti numbers of cyclic R-modules whose annihilators contain an $\mathbf{a}$-regular sequence.

Very few cases of the Lex-Plus-Powers (LPP) conjecture are known. Francisco has verified LPP when the Lex-Plus-Powers ideal in question is an almost complete intersection. Richert has shown that the conjecture holds if $n = 2$, or if $n = 3$ and one only considers monomial ideals (this by demonstrating that the whole conjecture is implied in degree $n = 3$ by a seemingly weaker conjecture related to the existence of Lex-Plus-Powers ideals, which is itself known in the monomial case). Most recently Mermin, Peeva, and Stillman have demonstrated that the conjecture is true if one restricts to ideals containing the squares of the variables.

4.20. Arrangements of smooth rational curves in $\mathbf{P}^2$ (by Hal Schenck).
Let $\mathcal{C}$ be a collection of smooth rational plane curves $C_i \subseteq \mathbf{P}^2$, such that $\mathcal{C} = \bigcup_{i=1}^n C_i$ has only ordinary singularities. Schenck and Tohaneanu prove that the addition-deletion operation used in the study of hyperplane arrangements has an extension which works for a large class of arrangements of smooth rational curves, giving an inductive tool for understanding the splitting of $\Omega^1(\mathcal{C})$–the module of logarithmic differential forms with pole along $\mathcal{C}$.

In a recent paper, Yoshinaga gives a freeness criteria for line arrangements in $\mathbf{P}^2$: the multiarrangement obtained by restricting to any line of the arrangement must have the same splitting type. For smooth rational curve arrangements, Schenck-Tohaneanu give an example of a pencil of arrangements, where a special splitting occurs for one fiber. This is related to Ziegler's example that freeness of multiarrangements is not combinatorial; and shows that an analog of Yoshinaga's theorem does not hold in the more genral setting. Schenck-Tohaneanu also show that the analog of Terao's conjecture that splitting of $\Omega^1(\mathcal{A})$ is combinatorially determined is false for arrangements of smooth rational curves with ordinary singularities.

4.21. Toric Varieties as Fine Moduli (by Greg Smith).
We'll prove that every projective toric variety is the fine moduli space for stable representations of an appropriate bound quiver. To accomplish this, we study the quiver Q with relations R corresponding to the finite dimensional algebra $End(\bigoplus L_i)$ where $\mathcal{L} = (O_X, L_1, ..., L_r)$ is a list of line bundles on a projective toric variety X. The quiver Q defines a unimodular, projective toric variety, called the multilinear series $\mathcal{L}$, and a map $X \longrightarrow \mathcal{L}$. We provide necessary and sufficient conditions for the induced map to be a closed embedding. As a consequence, we obtain a new geometric quotient construction for projective toric varieties. Under slightly strong hypotheses on $\mathcal{L}$, the closed embedding identifies X with the fine moduli space of stable representations for the bound quiver (Q, R).

4.22. Resolutions of monomial ideals via edge ideals (by Adam Van Tuyl).
Adam Van Tuyl's recent research has focused on monomial ideals from the point of view that the generators correspond to the edges of a graph or hypergraph. If $G = (V, E)$ is a *simple graph* (no loops or multiple edges) on the vertex set $V = \{x_1, \ldots, x_n\}$ and edge set E, then associated to G is the monomial ideal $\mathcal{I}(G) = (\{x_i x_j \mid \{x_i, x_j\} \in E\})$, which is called the *edge ideal* of G. Understanding the properties of the ideal $\mathcal{I}(G)$ and its relation to the graph G has been the focus of many authors (e.g., Francisco, Herzog, Katzman, Sturmfels, Zheng, among others). Adam Van Tuyl's work on edge ideals has focused on the question:

Question. *How do the invariants in the minimal resolution of $\mathcal{I}G)$ relate to the combinatorial data of the simple graph G?*

With M. Roth (`math.AC/0411181`), he showed that in many cases the graded Betti numbers in the *linear strand* of the resolution of $\mathcal{I}(G)$ can be computed directly from information from G. In two papers with T. Hà (`math.AC/0503203, math.AC/0606539`), Adam Van Tuyl used the notion of a splittable ideal due to Eliahou-Kervaire to study the graded Betti numbers of $\mathcal{I}(G)$. The splitting technique enabled them to recover and extend many known results about the resolutions of edge ideals. For example, T. Hà and Adam Van Tuyl showed that the graded Betti numbers of edge ideals of chordal graphs can be computed recursively. As well, they generalized many results about edge ideals to the hypergraph case.

In a current project with R. Villarreal, Adam Van Tuyl is looking for a classification of *sequentially Cohen-Macaulay* bipartite graphs. Such a classification would extend Herzog and Hibi's classification of Cohen-Macaulay bipartite graphs. This work complements Adam Van Tuyl's work with C. Francisco (`math.AC/0511022`) which showed that the edge ideal of all chordal graphs are sequentially Cohen-Macaulay.

Adam Van Tuyl's other interests include the Hilbert functions and resolutions of points in multiprojective spaces, componentwise linear ideals, multigraded regularity, and fat points in special position.

4.23. Combinatorial Commutative Algebra (by Mauricio Velasco).
Velasco has studied the structure of free resolutions of monomial ideals and its relationship with combinatorial topology, aiming towards a classification of minimal free resolutions. In joint work with Irena Peeva he has provided a description of all simplest minimal monomial free resolutions, showing that these are in bijection with simplicial complexes (deriving sharp universal lower bounds for total betti numbers of all monomial ideals). Velasco has also been interested in the extent to which minimal monomial free resolutions can be described via the notion of cellular resolution introduced by Bayer, Peeva and Sturmfels. His work has shown that this paradigm is not sufficient: he has constructed the first family of minimal free resolutions not supported by CW complexes.

More recently his work has focused in studying the total coordinate rings (Cox rings) of non toric varieties. In joint work with Mike Stillman and Damiano Testa he has shown a conjecture of V. Batyrev and O. Popov which characterizes the Cox rings Del Pezzo surfaces as quotients of a polynomial ring by an ideal generated by quadrics.

4.24. Spherical Initial Ideals (by Volkmar Welker).
In recent years there have been many instances in which it has been shown that there are initial ideals $in_{\preceq}(I)$ for homogeneous ideals I in the polynomial ring for which S/I is a Gorenstein domain such that:

(i) $in_{\preceq}(I)$ is the Stanley-Reisner ideal of the join of an m-simplex and a Gorenstein* simplicial complex, simplicial sphere or boundary complex of a simplicial polytope.

(ii) m is the absolute value of the a-invariant of S/I.

The first case, Gorenstein Hibi-rings, was discovered by Reiner and Welker. Athanasiadis generalized their construction to a wider class of rings and then Bruns and Römer showed that it is true for all Gorenstein Ehrhart-rings of integral polytopes which have a regular unimodular triangulation.

Then Jonsson and Welker exhibited the same phenomenon for ideals of Pfaffians of degree r of a generic skew-symmetric n by n matrix. Work of Soll and Welker provides a conjectures initial ideal for ideals of minors of degree r of a generic n by n matrix.

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