Reaction-diffusion and Free Boundary Problems

Peter Constantin (University of Chicago, USA),
Francois Hamel (Université Aix-Marseille III, France),
Robert Jerrard (University of Toronto, Canada)
Jean-Michel Roquejoffre (Université Paul Sabatier, Toulouse, France),
Lenya Ryzhik (University of Chicago, USA)

March 18-23, 2006

1 Introduction and overview of the Field

Reaction-diffusion equations, semilinear diffusion equations and free-boundary problems form an important domain of the theory of partial differential equations that is both very rich and challenging mathematically and is intricately related to numerous applications in physical, chemical and biological sciences.

The purpose of this conference was to bring together researchers in various areas of this field as well as applied mathematicians to highlight the recent developments and discuss the open problems that are of interest both from the mathematical perspective and from the point of view of applications. Due to the enormous activity of the field, it was impossible to cover every topic in reaction-diffusion equations. We have, chosen to lay the emphasis on the following items, that we considered as particularly interesting in view of their mathematical richness, and potential applications. The following subject have particularly been focused upon:

- *Singular perturbations, free boundary problems and reaction-diffusion equations.* This topic is a classical one in reaction-diffusion equations - see for instance Fife [15], but has undergone very important developments in the last years, such as the recent progress in the proof of the de Giorgi conjecture, the description of the Ginzburg-Landau vortices dynamics, the regularity theory of free boundary problems and the dynamics of reaction-diffusion systems.

- *Complex propagation phenomena in reaction-diffusion equations*. Although some mathematical milestones in the theory of reaction-diffusion equations date back to the 1930's, they were mainly concerned with homogeneous situations. More realistic heterogeneous reaction-diffusion equations or systems have been handled only relatively recently. Over the recent years, mathematical results have considerably enriched our understanding of these models and their biological applications. A very partial list of examples of areas where considerable recent progress has been made include the propagation phenomena related to the existence and the dynamical properties of travelling fronts in heterogeneous environments.

- *Homogenization, stochastics and dynamics of reaction-diffusion equations*. Homogenization of reaction-diffusion and Hamilton-Jacobi problems in a periodic medium is by now well understood. However, only recently progress has been made in similar issues for random media. Reaction-diffusion equations are closely connected to the large deviation problems for diffusion processes and weak stochastic perturbations of dynamical systems. Recently much progress has been made in asymptotic theories in this area, including the situations when the underlying dynamical system is itself random.

2 Recent Developments and Open Problems

2.1 Singular perturbations, free boundary problems and reaction-diffusion systems

2.1.1 Phase transitions, geometric methods in elliptic equations, and the de Giorgi conjecture

The equilibrium state of a binary alloy may be described by the celebrated Allen-Cahn equation: if $u(x) \in (-1,1)$ denotes the local proportion of each component, we have

$$-\Delta u = \frac{1}{\varepsilon^2} W'(u), \quad x \in \mathbb{R}^N \tag{1}$$

where W is an even potential, having global minima at ± 1. When ε is - at least formally - sent to 0, the limiting solution u of (1) takes the values 1 or -1, the interface between the regions $\{u = 1\}$ and $\{u = -1\}$ being separated by a surface Γ with zero mean curvature. The interface equation is not so hard to derive in a formal fashion: assuming that Γ is smooth, and letting $\phi_0(x)$ be the unique solution of

$$-\phi_0'' = W'(\phi_0), \quad \phi_0(\pm\infty) = \pm 1, \; \phi_0(0) = 0,$$

a plausible ansatz for the solution u_ε of (1) is

$$u_\varepsilon(x) \sim \phi_0(\frac{d(x)}{\varepsilon}), \quad \text{with } d(x) = \mathrm{dist}(x, \Gamma) \text{ (signed distance)}$$

which yields: $\Delta d = 0$ on Γ. This precisely says that the mean curvature of Γ is zero. A mathematically rigorous derivation of that fact is, of course, much more difficult. Modica and Mortola [31] prove the following version of this fact: a sequence of minimizers $(u_\varepsilon)_\varepsilon$ of the functional

$$u \mapsto \int (\frac{1}{2}|\nabla u|^2 - \frac{1}{\varepsilon} W(u))\, dx$$

converges to a difference of characteristic functions of the form $\chi_E - \chi_{\Omega \setminus E}$; moreover the set $\partial E \cap \Omega$ is a minimal hypersurface.

The de Giorgi conjecture states the following:

(i) (Nonexistence part) Given a potential W as above, analytic in its argument, let $u(x)$ satisfy

$$-\Delta u = W'(u), \; x \in \mathbb{R}^N; \qquad \frac{\partial u}{\partial x_N} \geq 0. \tag{2}$$

Then the level sets of u are hyperplanes, at least if $N \leq 8$.

(ii) (Existence part) For $N \geq 9$, there are truly multi-dimensional solutions of (2).

This conjecture was motivated by:

- a theorem of J. Simons [38], asserting that any minimal graph, defined over the whole space $\mathbb{R}^{N-1}$, has to be an affine function,

- a theorem of Bombieri, de Giorgi, Giusti [8] asserting that, for $N - 1 = 2m \geq 8$, the Simons cone $\{\sum_{i=1}^{m} x_i^2 = \sum_{i=m+1}^{2m} x_i^2\}$ is minimal.

The de Giorgi conjecture is also deeply related to the study of the level sets of converging sequences of solutions of (1): the nonexistence part says that these level sets are uniformly Lipschitz - and that an internal layer expansion is justified.

The nonexistence part of the conjecture was recently proved, in full generality, by Savin [34]. Earlier results were proved by Ghoussoub-Gui [21] ($N = 2$), Ambrosio-Cabré [1] ($N = 3$), Ghoussoub-Gui [22] (particular cases of the dimensions 4 and 5).

2.1.2 Free boundaries in reaction-diffusion equations, and their qualitative properties

A typical instance of the free boundary problems on which the conference focused is the following class of parabolic equations

$$T_t - \Delta T = \frac{1}{\varepsilon^2}(1-T)\exp(\frac{T-1}{\varepsilon}) := (1-T)f_\varepsilon(T), \;\; x \in \mathbb{R}^N. \tag{3}$$

Such an equation is a - fairly good, and still widely employed for qualitative predictions - model for the propagation of a flame in a combustible mixture; the function $T(t,x)$ represents the temperature of the mixture and the right-hand side accounts for the rate at which the chemical reaction proceeds. The parameter ε is the inverse of the - fortunately large - reduced activation energy. As one may realize, the reaction term $f_\varepsilon(T)$ is concentrated at the value $T = 1$, which is here the normalized burnt gas temperature. When ε is sent to 0, the problem can be shown - at least in a formal fashion - to tend to the more singular one:

$$T_t - \Delta T = \delta_{T=1}. \tag{4}$$

The space is here separated into two regions: $\{T < 1\}$ and $\{T = 1\}$, and the normal derivative of the temperature - provided it exists! - undergoes a jump of size 1 at the boundary $\partial\{T = 1\}$. Deriving (4) formally is not so difficult: it is a classical internal layer analysis; doing it in a mathematically rigorous fashion is once again a hard problem.

Important progress has been made in the treatment of free boundary problems by Caffarelli and his collaborators, especially in the understanding of their regularity. The methods range from potential theory and harmonic analysis to geometric measure theory; see for instance the series [9] - regularity of elliptic FBP's, [2] - regularity for the Stefan problem, [10] - monotonicity formulae implying uniform estimates for problems of the type (3); see also [11] where a lot of these ideas are exposed. This wide body of methods and ideas have been applied - to many other types of problems, such as homogenization of free boundary problems, singular perturbations - the proof of the de Giorgi conjecture by Savin is inspired by the ideas of Caffarelli *et al.* -, fully nonlinear reaction-diffusion equations...

2.1.3 The dynamics of reaction-diffusion systems

Reaction-diffusion systems may exhibit complex dynamics, and important hints in their description are provided by singular perturbations. Examples of complex dynamics may already be found by the following slight generalization of equation (3): assume that the chemical reaction follows the single-step scheme $A \to B$, and assume that the reactant A does not diffuse in the same fashion as the temperature: a new parameter - the Lewis number, denoted by Le - enters into play. Let $Y(t,x)$ denote the mass fraction of the reactant; equation (3) becomes

$$\left\{ \begin{array}{rcl} T_t - \Delta T &=& Y f_\varepsilon(T) \\ Y_t - \dfrac{\Delta Y}{Le} &=& -Y f_\varepsilon(T) \end{array} \right. \tag{5}$$

This system has 1D travelling wave solutions, see [7]. A famous computation of Sivashinsky [39] indicates that, as Le gets ε-far from 1, the wave destabilizes into multi-dimensional patterns $(Le < 1)$ or into pulsating waves $(Le > 1)$; this was proved in a rigorous way in [23]. Of interest if the behavior of the flame front - here, the set $\{T - 1 \sim \varepsilon\}$ near the critical parameter; if the front is described by a graph $\{y = \Phi(t,x)\}$, an evolution equation is once again provided by Sivashinsky [39] in the form of the celebrated Kuramoto-Sivashinsky equation:

$$\Phi_t + \Delta^2\Phi + \Delta\Phi + \frac{1}{2}|\nabla\Phi|^2 = 0. \tag{6}$$

A lot has already been said on (6); due to its universal character - it arises in a lot of interface problems - the subject is still extremely active. Its rigorous derivation from (5) seems to be a challenging open problem. Depending on the geometry considered and the values of the Lewis number, the flame front may satisfy extremely diverse types of evolution equations; see for instance [27] for a version of (5) with $Le < 1$.

A singular perturbation may also occur in a reaction-diffusion system under the form of a small diffusion; a generic presentation for the system would be

$$\left\{ \begin{array}{rcl} u_t - \Delta u &=& f(u,v) \\ v_t - \varepsilon\Delta v &=& g(u,v) \end{array} \right. \tag{7}$$

Singular perturbation results for ordinary differential equations date back to the early 60's; however a seminal work of C. Jones, unifying all these results in the framework of geometric theory of dynamical systems, has fostered a large body of works investigating complex wave patterns for (7). Stability of travelling waves is an important topic that has been addressed to in the workshop; some important problems of the moment include

- Complex flame models - such as flames in two-phase flows;
- biological models - such as the Gray-Scott or Gierer-Meinhardt model; see the talk of A. Doelman below;
- detonation models. This last topic is particularly challenging: such models include the whole set of gas dynamics equations, plus an equation for the chemistry. The stability of detonation waves is a complex problem, and the introduction of a reduced model for fast waves in porous media, by Gordon-Kagan-Sivashinsky [24] seems to be quite promising.

2.2 Complex propagation phenomena in reaction-diffusion equations

Reaction-diffusion equations appear in many different areas of physics and of the life sciences. They are commonly used to describe phase transitions in various contexts in physics and in chemistry. In combustion theory, for instance, these equations arise in models of flame propagation. Equations of this kind play a central role in modeling biological invasions in various situations (population dynamics, physiology, wound healing, tumor growth, etc, see the classical books of Murray [32] and Shigesada and Kawasaki [37]).

The existence of traveling wave like solutions is an essential feature of this class of equations that is relevant for all the models mentioned above. It is strongly related to propagation phenomena that are particularly important and again a common feature in these areas.

As a mathematical subject, the study of reaction-diffusion equations, traveling waves and propagation properties is very active now. Even though, it was first introduced in the homogeneous framework in the late 1930s (see [16, 26]), there has been a profusion of works since the 1970s with results that have profoundly enriched our understanding of these equations. It is only relatively recently that researchers have been able to address propagation and traveling fronts in heterogeneous environments and to take into account other phenomena, such as transport, interaction with environment, singular behavior etc. The recent years have indeed seen much progress on these questions.

H. Berestycki (EHESS) gave two lectures on recent advances in this area. He first reported on several papers with F. Hamel and N. Nadirashvili [3, 4, 5] on existence and qualitative properties of pulsating traveling fronts in periodic media, for reaction-diffusion-advection equations of the type

$$u_t - \text{div}(A\nabla u) + q \cdot \nabla u = f(x,u), \quad x \in \Omega, \tag{8}$$

when $A(x)$, $q(x)$ and $f(x,u)$ have the same periodicity in the x-variables as the domain Ω itself. The influence of different phenomena involved – such as transport, diffusion, reaction, geometry of the domain – on the speeds of propagation were discussed. For instance, several well-known facts can be proved rigorously: the perforations slow down the propagation, whereas stirring always speeds up the fronts.

Another key notion involved here is the asymptotic speed of spreading in domains which have no periodicity. The spreading speed in a given direction is defined as the speed of the leading edge of the solution of the Cauchy problem at large times. For the solutions of the equation

$$u_t = \Delta u + f(u) \tag{9}$$

in general domains Ω with sub-linear nonlinearities f of the Fisher-KPP type ($0 < f(s) \leq f'(0)s$ for $s \in (0,1)$ with $f(0) = f(1) = 0$), the spreading speeds may depend in general on the domain and on the initial condition, even if the solution is initially compactly supported. Even for this homogeneous equation, very interesting new phenomena appear, due to the complex geometry of the domain. For instance, in very narrow domains, the spreading speed may be infinite.

More complex dynamical behaviors may also occur. Roughly speaking, even for simple models (9) and even in dimension 1, when the nonlinearity f is of the combustion type ($f = 0$ on $[0, \theta]$, $f > 0$ on $(\theta, 1)$ and $f(1) = 0$ with $0 < \theta < 1$) or of the bistable type ($f < 0$ on $(0, \theta)$, $f > 0$ on $(\theta, 1)$ and $f(0) = f(\theta) = f(1) =$

0), then propagation may occur or fail according to the size of the initial condition. For instance, for bistable nonlinearities with positive mass over $[0,1]$, A. Zlatos [40] recently proved that, when the initial condition at time 0 is the characteristic function of an interval, then there is a critical positive interval size below which the solution will eventually converge to 0 uniformly in $x \in \mathbb{R}$, and above which it will converge to 1 locally, and actually develop into two expanding fronts. For the critical interval size, the solution eventually converges to the unstable non-trivial ground state. Even if the results are not as precise when the equation involves heterogeneous coefficients and in particular a non-constant flow, propagation/quenching issues were addressed recently and special attention has been put on the role played by the profile of the underlying flow (see P. Constantin, A. Kiselev, L. Ryzhik, A. Zlatos [12, 25]).

Further generalizations of the notion of traveling front or wave in general heterogeneous frameworks were recently introduced for general systems of partial differential equations. These new definitions are based on uniform limits far away, with respect to the geodesic distance inside the domain, from some hypersurfaces. These notions extend the previous known cases of periodic or almost-periodic environments. General situations like the propagation in curved tubes, exterior domains, etc can now be considered. The determination of the shape of the leading edge of the fronts and the stability of these new fronts are some of the main goals of future work.

The question of propagation in media which are locally perturbed is an open problem which is one of the most important cases for the applications. Indeed, the same issues of propagation can be asked when the medium is homogeneous (or even periodic) outside a localized zone and the definition of generalized waves is also adapted to this situation. The archetype is the equation (8), where the coefficients A, q and f, or the domain Ω, are homogeneous or periodic outside a compact set. This is the case of a tube which has a local stricture. What are the necessary and sufficient conditions to have propagation ?

Another very interesting open problem is to describe the propagation of generalized fronts in media for which some diffusion or reaction coefficients are monotone in the direction of propagation, or more generally when the characteristics of the medium are different far ahead and far behind the front. These questions may depend strongly on the nonlinearity, propagation may fail for bistable nonlinearities whereas, everything else being unchanged, propagation may occur for monostable equations. These problems have concrete applications in combustion or in biological models for instance.

Biological invasions are indeed one of the most common examples of propagation phenomena and it seems fair to say that these are the most widely used equations in ecological and biological modeling (epidemics, epizootics and tumor growths can also be modelled by reaction-diffusion equations). Much progress has been made in the recent years about the mathematical analysis of such models. It helps to have a better understanding of the concrete applications and to be able to make reasonable predictions. For instance, for ecological models of the type

$$u_t = \mathrm{div}(A(x)\nabla u) + (\mu(x) - \nu(x)u)u \tag{10}$$

in periodic fragmented environments, light was recently shed on how a spatially diverse environment affects biological invasions or species survival in this context. A less fragmented medium, which means that the favourable and unfavourable regions are more aggregated, is better for species persistence (see [6]).

More complex models can also be used in the applications. As an example, aggregation phenomena for bacteria can be modelled by systems of equations which involve chemotactic terms, meaning that some species tend to diffuse in the direction of positive concentration gradient of a chemical agent (see [32]). In other contexts, nonlocal models can be used to model long-range dispersion and new versions of the maximum principle, which is one of the most powerful tools in reaction-diffusion equations, were recently established.

In mathematical terms, from a dynamical point of view, front propagation can be thought of as the invasion of a more unstable or less stable state by a more stable or less unstable state. Even if most models do not have a variational structure and no Lyapounov functional is available in general, the study of the spectral properties of the linearized equations around the limiting states is crucial. Another important point is to determine the set of all possible limiting states. For instance, for equations as simple as (10), the existence of a stationary positive state is not obvious. Indeed, since the equations are set in unbounded domains, to allow propagation, the lack of compactness creates additional complications. Recent progress was made on these questions, for equations more general than (10), and new qualitative and Liouville classification results were obtained.

2.3 Homogenization, stochastics and dynamics of the reaction-diffusion equations

2.3.1 Reaction-advection-diffusion equations and weak perturbations of dynamical systems

The question of the interplay of a strong advection and weak diffusion is very natural and physically relevant, and the subject has a long history. The passive scalar model

$$\phi_t + u \cdot \nabla \phi = \varepsilon \Delta \phi,$$

is probably one of the most studied PDEs in both mathematical and physical literature. One important direction of research focused on homogenization, where in a certain limit (typically small diffusion) the solution of a passive advection-diffusion equation converges to a solution of an effective diffusion equation. We refer to [29] for more details and references. The corresponding reaction-diffusion models

$$\phi_t + u \cdot \nabla \phi = \varepsilon \Delta \phi + \frac{1}{\varepsilon} f(\phi),$$

and

$$\phi_t + \frac{1}{\varepsilon} u \cdot \nabla \phi = \Delta \phi + f(\phi),$$

have been also extensively studied. Usually, the existence of such a limit requires additional assumptions on the scaling of u (see e.g. [30] for further references). The Freidlin-Wentzell theory [17, 18, 19, 20] studies such problems in $\mathbb{R}^2$ and, for a class of flows, proves the convergence of solutions as the flow strength tends to infinity to solutions of an effective diffusion equation on the Reeb graph of the stream-function. The graph, essentially, is obtained by identifying all points on any streamline. The conditions on the flows for which the procedure can be carried out are given in terms of certain non-degeneracy and growth assumptions on the stream function. Recently this theory has been extended to a class of three-dimensional flows, where the limit problem is formulated on an "open book" rather on a graph. The dynamics is once again described in terms of the slow variables with the fast variations averaged out. Another direction has been taken in [13] – instead of trying to identify a limit problem, the question is what flows are most effective in mixing the solutions as their strength tends to infinity. It turns out that with an appropriate and natural definition of mixing one can provide a sharp classification of such "relaxation-enhancing" flows.

2.3.2 Homogenization of Hamilton-Jacobi and reaction-diffusion equations

Homogenization of the Hamilton-Jacobi equations in a periodic medium has been well understood since the unpublished preprint by Lions, Papanicolaou and Varadhan from the late 1980's. The problem is to homogenize the (possibly second-order) equation

$$\frac{\partial u_\varepsilon}{\partial t} - \frac{\varepsilon}{2} \Delta u_\varepsilon + H\left(t/\varepsilon, x/\varepsilon, \nabla u_\varepsilon, \omega\right) = 0,$$

and find an effective Hamilton-Jacobi problem

$$\frac{\partial u}{\partial t} + \bar{H}(\nabla u) = 0.$$

Here H is a random Hamiltonian and $\bar{H}$ is the deterministic Hamiltonian for the homogenized problem. This problem (as well as a class of related homogenization questions) has been recently studied in a series of papers by P.-L. Lions and P. Souganidis, and independently by E. Kosygina, F. Rezakhanlou and S.R.S. Varadhan. A very interesting and challenging open problem is obtain non-trivial bounds for the homogenized Hamiltonian – this problem remains open even in the periodic case.

3 Presentation Highlights

3.1 Geometric methods for semilinear reaction-diffusion equations

X. Cabré, during his two-hour lecture, presented recent developments on solutions of reaction-diffusion elliptic equations that are related to some classical results in the theory of minimal surfaces. Three results in minimal surfaces theory and their semilinear analogues.

- *Regularity of solutions of elliptic equations in low dimensions.* Inspired by related results for harmonic maps, Cabré discussed semilinear analogues, particularly recent results by Capella and himself on radial solutions of reaction-diffusion equations, including the well-known Gel'fand equation

$$-\Delta u = e^u.$$

  In low space dimensions, they lead to the boundedness or regularity of radial solutions in a ball, and to the instability of radial solutions in the whole space.

- *Flatness of minimal graphs in low dimensions.* This item is related to the de Giorgi conjecture. Cabré explained how bounded solutions in the whole space which are monotone in one variable are always local minimizers of the energy

$$E(u) = \int_{B_R} (\frac{1}{2}|\nabla u|^2 - W(u))\, dx.$$

  This implies that, in low space dimensions, they are necessarily functions of only one Euclidean variable.

- *Saddle solutions.* Guided by this variational approach, Cabré discussed the following generalisation of an earlier result by Schatzman [35]: in $I\!R^{2m}$, equation (2) has a solution whose symmetries are the same as those of the Simons cone; this solution, which is unique up to translations, is called the saddle solution of the Allen-Cahn equation; moreover, if $m = 1$, this solution is unstable. Cabré explained his results results in this direction: instability of the saddle solutions in dimensions $2m = 4$ and 6, relying on a delicate estimate of Modica: if u satisfies $-\Delta u = W'(u)$, W satisfying the standard assumptions, then

$$\frac{1}{2}|\nabla u|^2 \le W(u).$$

  Would these solutions be stable in higher dimensions - as is suggested by the Bombieri-de Giorgi-Giusti analysis, this would lead to a counterexample de Giorgi Conjecture.

O. Savin - who put an end to the non-existence part of the de Giorgi conjecture - discussed viscosity solutions of fully nonlinear elliptic equations

$$F(D^2u, Du, u, x) = 0$$

for which $u \equiv 0$ is a solution. If F is smooth and uniformly elliptic only in a neighborhood of the points $(0, 0, 0, x)$, then u is smooth in the interior if $\|u\|_{L^\infty}$ is sufficiently small. This result - which uses difficult Caffarelli-type estimates on second order derivatives - has applications to the study of the regularity of free boundary problems; in particular it can help to prove regularity when Lipschitz continuity and nondegeneracy of the free boundary are known.

3.2 Free boundary problems and applications

A. Mellet discussed delicate effects in the homogenization of free boundary problems in two cases. First, he considered the scalar thermo-diffusive model for flame propagation

$$T_t - \Delta T = \frac{1}{\varepsilon^2}(1 - T) f(\frac{x}{\delta}, \frac{T - 1}{\varepsilon});$$

the parameter δ accounting for possible heterogeneities in the medium. Hysteresis phenomena occur: passing to the limit in $\varepsilon \to 0$, then $\delta \to 0$ do not yield the same result as taking the limits in the reverse order. Second, he presented a model for the equilibrium of a sticky drop on a rough surface; this amounts to minimising a - nonsmooth - functional of the characteristic function of the drop, with highly oscillating coefficients. There is a homogenisation limit to this problem, namely the drop is almost spherically spherical, and the limiting radius may be computed from data.

J.-S. Guo reported on a two-point free boundary problem for a quasilinear parabolic equation, mainly arising in the study of the motion of interface moving with curvature. Global and non-global existence of solutions, was discussed; non-global existence may occur only through a finite-time extinction process - in the case of the mean curvature motion, this amounts to a complete curve shortening. The asymptotic profile at extinction, as well as convergence to a self-simlar profile, were discussed.

The talk of *N. Ghoussoub* concerned the nonlinear elliptic problem

$$-\Delta u = \frac{\lambda f(x)}{(1+u)^2}$$

on a bounded domain Ω of $\mathbb{R}^N$ with Dirichlet boundary conditions. This equation models a simple electrostatic Micro-Electromechanical System (MEMS) device consisting of a thin dielectric elastic membrane with boundary supported at 0 above a rigid ground plate located at -1. When a voltage –represented here by λ– is applied, the membrane deflects towards the ground plate and a snap-through may occur when it exceeds a certain critical value λ^* (pull-in voltage). This creates an instability which greatly affects the design of many devices. The challenge is to estimate λ^* in terms of material properties of the membrane, which can be fabricated with a spatially varying dielectric permittivity profile f. When $\lambda < \lambda^*$ (and when $\lambda = \lambda^*$ in dimension $N \leq 7$), there is at least one steady state, while none is possible for $\lambda > \lambda^*$. More refined properties of steady states –such as regularity, stability, uniqueness, multiplicity, energy estimates and comparison results– are shown to depend on the dimension of the ambient space and on the permittivity profile.

3.3 Asymptotic models of reaction-diffusion systems; application to flame propagation models

Three talks were devoted to various aspects of existence and qualitative properties of waves in reaction-diffusion systems. The talk of *K. Domelevo* reported some results on premixed flames models, where the reactant (i.e. gas fuel) is provided through the vaporisation of liquid fuel droplets. The corresponding simplest mathematical model consists in the usual thermo-diffusive system coupled to the equation for the vaporisation of the droplets. Travelling wave profile exist, and asymptotics with respect to the activation energy reveal new features: if the initial droplet radius is below some explicit threshold, the model is totally similar to the classical thermo-diffusive model. Above the threshold, the combustion is driven by the droplet evaporation. The main result in the talk of *P. Gordon* was a singular perturbation approach to a detonation model in porous media, derived by Sivashinsky; he presented uniqueness results for the speed and wave profile when the thermal diffusion coefficient goes to 0. *M. Haragus* reported on holes in reaction-diffusion systems, i.e.: almost planar interfaces for which the angles of the interface at each point, relative to a fixed planar interface, tend to zero at infinity. She applied dynamical systems ideas - popularised under the name of 'spatial dynamics', to convey the idea that one spatial variable is treated as a time - and showed that, in isotropic reaction-diffusion systems, holes bifurcate from stable planar pulsating fronts.

C.-M. Brauner presented a model of flame front dynamics introduced by Frankel, Gordon and Sivashinsky, more tractable than the classical thermo-diffusive model, and which can yield - by the same process as in the thermo-diffusive model - a single integro-differential equation (Q–S). If the flame front, supposed to evolve in the space $I\!R^2$, is a curve with equation $y = \Phi(t, x)$, then

$$\Phi_t + \frac{\Phi_x^2}{2} - \Phi_{xx} + \alpha(I - \partial_{xx})^{-1}\Phi_{xx} = 0.$$

This asymptotic equation has the same qualitative features as the Kuramoto-Sivashinsky (K–S) one; in particular, it can generate chaotic cellular dynamics. The numerical simulations turn out to be quite convincing.

The modelling of spikes was addressed to in two talks. The talk by *A. Doelman* focussed on how to derive, in a rigorous fashion, an ODE modelling the interaction law between to repelling two-pulse, slowly varying solutions of the a regularized Gierer-Meinhardt system. This system is a heuristic model arising in the description of chemical reactors and biological systems; one of its versions writes

$$\begin{aligned} \varepsilon^2 U_t &= U_{xx} - \varepsilon^2 U + f(U)V^2 \\ V_t &= \varepsilon^2 V_{xx} - V + g(U)V^2 \end{aligned}$$

where $x \in \mathbb{R}, t > 0$, $0 < \varepsilon \ll 1$ is a small parameter, and functions f and g are smooth positive functions. The method employed, based on normalisation group ideas, should be applicable to many other situations.

M. Ward discussed an optimization problem for the fundamental eigenvalue λ_0 of the Laplacian in a planar simply-connected domain that contains N small identically-shaped holes, each of radius $\varepsilon \ll 1$. A Neumann boundary condition is imposed on the outer boundary of the domain and a Dirichlet condition is imposed on the boundary of each of the holes. He presented an asymptotic expansion for λ_0 in terms of certain properties of the Neumann Green's function for the Laplacian. This eigenvalue optimization problem is shown to be closely related to the problem of determining equilibrium vortex configurations in the Ginzburg-Landau theory of superconductivity, and also closely related to the problem of determining equilibrium locations of spikes, to multi-dimensional reaction-diffusion systems.

3.4 Complex propagation phenomena in reaction-diffusion equations

H. Berestycki (EHESS, France) reported first on some results with F. Hamel and N. Nadirashvili on pulsating travelling fronts for reaction-diffusion-advection equations in general periodic framework. The qualitative and quantitative role of the diffusion, advection and reaction terms was explained. Nonlinear propagation phenomena in general unbounded domains of $\mathbb{R}^N$, for reaction-diffusion equations with Kolmogorov-Petrovsky-Piskunov (KPP) type nonlinearities, were then discussed. General domains were considered and various definitions of the spreading speeds at large times for solutions with compactly supported initial data were given. The dependency of the spreading speeds on the geometry of the domain was explained. Some a priori bounds can be obtained for large classes of domains. The case of exterior domains was also explained in detail. H. Berestycki finally reported on very recent works with F. Hamel about further generalizations of the usual notions of waves, fronts and propagation speed in a very general setting. These new notions involve uniform limits, with respect to the geodesic distance, to a family of hypersurfaces which are parametrized by time.

J. Coville (CMM-Universidad de Chile, Chile) presented some work devoted to the maximum principles holding for some nonlocal diffusion operators and its applications to obtain qualitative behaviors of solutions of some nonlinear problems with sliding methods. As in the classical case, it can be shown that the nonlocal diffusion satisfies a weak and a strong maximum principle. Uniqueness and monotonicity of solutions of nonlinear equations are therefore expected as in the classical case. J. Coville also presented a optimal condition to have a strong maximum for operator $Mu := J \star u - u$.

S. Luckhaus (University of Leipzig, Germany) reported on joint work with L. Triolo [28], and with A. De Masi and E. Presutti [14], about a hierarchy of scalings in a population model for tumor growth. Interacting particle systems modeling the competition of healthy and malignant cells were considered and lateral contact inhibition and difference of mobility were taken into account in a lattice model. A two scale hydrodynamic limit was derived. On longer time scales the solutions are expected to converge to the tumor growth governed by the eikonal equation. This last step in the scaling hierarchy has not yet been shown starting from the original stochastic process.

H. Matano (University of Tokyo, Japan) reported on recent advances in quenching vs. propagation phenomena for bistable-type equations in heterogeneous media. In some domains with non-periodic perforations, propagation may be blocked by stationary solutions.

K.-I. Nakamura (University of Electro-Communications, Japan) talked about front propagation phenomena for a bistable reaction-diffusion equation in an infinite cylinder with periodic boundaries. By using the first 3 terms of asymptotic expansions of the profile and the speed of front solution, he constructed suitable supersolutions and subsolutions to obtain upper and lower bounds for the front speed when the diameter of the cylinder is very small. These bounds enabled him to show that spatial periodicity always slows down the front propagation in bistable diffusive media.

P. Polacik (University of Minnesota, USA) presented a new result on asymptotic symmetry of positive solutions of parabolic equations on nonsmooth bounded domains. A key ingredient in the proof of this result is a theorem on asymptotic positivity of solutions of linear equations with bounded measurable coefficients. Some perspectives on this technical tool were given.

L. Rossi (Universita Roma I, Italy and EHESS, France) discussed on generalized principal eigenavlue of elliptic operators in $\mathbb{R}^N$ and on some applications. He introduced two different generalizations of the principal eigenvalue for linear elliptic operators in the whole space. He discussed how their signs determine the existence and uniqueness of bounded solutions for an associated class of semilinear equations. The two notions do not coincide in general and some inequalities between these eigenvalues in the case of self-adjoint, one-dimensional and limit-periodic operators were derived.

A. Stevens (Max Planck Institute, Leipzig, Germany) discussed on transport equations for cellular alignment and aggregation and their parabolic limits. A widespread phenomenon in moving microorganisms and cells is their ability to orient themselves with respect to each other and in dependence of chemical signals. Kinetic models for this kind of movement were discussed, which take into account a variety of evaluations of the external chemical field and of the neigboring cells. In case of chemotaxis parabolic limit equations can be derived, which relate the microscopic parameters to the macroscopic ones, e.g. the so-called chemotactic sensitivity.

A. Zlatos (University of Wisconsin, Madison, USA) discussed on spreading of reaction in the presence of strong cellular flows with gaps. He considered a reaction-diffusion-advection equation with an ignition-type reaction term and a cellular flow with a periodic array of gaps. He showed that if the initial flame is large enough, it cannot be quenched by such flows, regardless of their strength.

3.5 Homogenization, stochastics and dynamics of the reaction-diffusion equations

M. Freidlin (University of Maryland) presented two lectures on asymptotic problems for stochastic processes and RDE's which covered material from the introductory level to the state of the art of the field. He presented some old and new results concerning averaging and large deviations for stochastic processes. These results allow, in particular, to describe motion of wavefronts for a class of reaction-advection-diffusion equations and systems, as well as to consider some homogenization problems for reaction in incompressible fluid.

A. Kiselev (University of Wisconsin) presented a talk on diffusion and mixing in fluid flow. Enhancement of diffusion by advection is a classical subject that has been extensively studied by both physists and mathematicians. In this work, the authors considered enhancement of diffusive mixing on a compact Riemannian manifold by a fast incompressible flow. The main result is a sharp description of the class of flows that make the deviation of the solution from its average arbitrarily small in an arbitrarily short time, provided that the flow amplitude is large enough. The necessary and sufficient condition on such flows is expressed naturally in terms of the spectral properties of the dynamical system associated with the flow. In particular, they find that weakly mixing flows always enhance the relaxation speed in this sense. The proofs are based on a new general criterion for the decay of the semigroup generated by a dissipative operator of certain form. They employ ideas from quantum dynamics, in particular the RAGE theorem describing evolution of a quantum state belonging to the continuous spectral subspace of the hamiltonian (and related to a theorem of Wiener on Fourier transforms of measures).

E. Kosygina (CUNY) presented her work on homogenization of Hamilton-Jacobi-Bellman equations with respect to time-space shifts in a stationary ergodic medium. Consider a family $\{u_\varepsilon(t,x,\omega)\},\ \varepsilon > 0$, of solutions of the final value problem

$$\frac{\partial u_\varepsilon}{\partial t} + \frac{\varepsilon}{2}\Delta u_\varepsilon + H\left(t/\varepsilon, x/\varepsilon, \nabla u_\varepsilon, \omega\right) = 0, \quad u_\varepsilon(T,x,\omega) = U(x),$$

where the time-space dependence of the Hamiltonian $H(t,x,p,\omega)$ is realized through the shifts in a stationary ergodic random medium. For Hamiltonians, which are convex in p and satisfy certain growth and regularity conditions, she shows the almost sure locally uniform in time and space convergence of $u_\varepsilon(t,x,\omega)$ as $\varepsilon \to 0$ to the solution $u(t,x)$ of a deterministic "effective" equation

$$\frac{\partial u}{\partial t} + \bar{H}(\nabla u) = 0, \quad u(T,x) = U(x).$$

The averaged Hamiltonian $\bar{H}(p)$ is given by a minimax formula. This is a joint work with S.R.S. Varadhan.

J. Nolen (University of Texas, Austin) discussed reaction diffusion fronts in temporally inhomogeneous flows. He considered the propagation of fronts that arise from scalar, reaction-advection-diffusion models with the Kolmogorov-Petrovsky-Piskunov (KPP) nonlinearity. For temporally random flows with a shear structure, he established an extension of the well-known variational representation for the front speed, a nonrandom constant. Also, he used this variational representation to analytically bound and numerically compute the speed. The analysis makes use of large deviations estimates for the related diffusion process. The variational principle is expressed in terms of the principal Lyapunov exponent of an auxiliary evolution problem. This is a joint work with J. Xin [33].

A. Novikov (Pennsylvania State University) considered a homogenization approach to large-eddy simulation of incompressible fluids. In the development of large-eddy simulation one makes two primary assumptions. The first is that a turbulent flow can be categorized by a hierarchy of lengthscales. The second assumption states that the small scales have universal properties, characterized by, e.g. a spectral power law. This motivated a number of physical models that attempt to account for the presence of small scales by suitably modifying the corresponding partial differential equations (PDE), the Navier-Stokes equations. Homogenization theory addresses rigorously the issue of modification of PDE in the presence of small scales. The goal of this talk was to apply homogenization methods to LES modeling of fluid flows.

H. Owhadi (Caltech) talked about homogenization of parabolic equations with a continuum of space and time scales. He addressed the issue of homogenization of linear divergence form parabolic operators in situations where no ergodicity and no scale separation in time or space are available. Namely, he considered divergence form linear parabolic operators in $\Omega \subset \mathbb{R}^n$ with $L^\infty(\Omega \times (0,T))$-coefficients. It appears that the inverse operator maps the unit ball of $L^2(\Omega \times (0,T))$ into a space of functions which at small (time and space) scales are close in H^1-norm to a functional space of dimension n. It follows that once one has solved these equations at least n-times it is possible to homogenize them both in space and in time, reducing the number of operations counts necessary to obtain further solutions. In practice they show that under a Cordes type condition that the first order time derivatives and second order space derivatives of the solution of these operators with respect to harmonic coordinates are in L^2 (instead of H^{-1} with Euclidean coordinates). If the medium is time independent then it is sufficient to solve n times the associated elliptic equation in order to homogenize the parabolic equation. (This is a joint work with Lei Zhang.)

J. Quastel (University of Toronto) discussed the effect of noise on KPP traveling fronts. He and co-authors study the effect of small additive Fisher-Wright noise on the speed of traveling fronts in the KPP equation. It had been observed by physicists in the late 90's that the effect is unusually large and Brunet and Derrida have made some very precise conjectures. Quastel described the proofs of some of these. This is joint work with Carl Mueller (Rochester) and Leonid Mytnik (Technion).

M. Soner (Koc University) talked about backward stochastic differential equations and fully nonlinear PDE's. In the early 90's Peng and Pardoux discovered a striking connection between semilinear parabolic PDE's and backward stochastic differential equations (BSDE in short). This connection and the BSDE's have been extensively studied in the last decade and a deep theory of BSDEs have been developed. However, the PDE's that are linked to BSDE's are necessarily semilinear. In joint work with Patrick Cheredito (Princeton) Nizar Touzi (CREST, Paris), Nic Victoir (Oxford), Soner extended the theory of BSDE's by adding an equation for the second order term, which we call 2BSDE in short. Through this extension they are able to show that all fully nonlinear, parabolic equations can be represented via 2BSDE's. He described this theory and possible numerical implications for the fully nonlinear PDE's.

P. Souganidis (University of Texas) presented two lectures on homogenization in random environments and applications to front propagation. In particular, he described recent developments in the theory of homogenization for fully nonlinear first- and second-order pde in stationary ergodic media in his works with L. Caffarelli and P.-L. Lions. He also considered applications to the theory of front propagation in random environments.

3.6 Gizburg-Landau vortices

A. Aftalion (Universite Paris VI) presented her work on vortex lattices in fast rotating Bose Einstein condensates. She described experiments on fast rotating Bose Einstein condensates which display vortex lattices: the lattice is almost triangular with a slight distortion on the edges. The mathematical description can be made with a complex valued wave function minimizing an energy restricted to the lowest Landau level or Fock-Bargmann space. Using some structures associated with this space, she studies the distribution of zeroes of the minimizer.

S. Serfaty (NYU University) gave two lectures on her work on the dynamics of the Ginzburg-Landau vortices. She described the known results on vortex collisions and presented her recent work [36] in this area, extending vortex dynamics past the blow-up time.

4 Outcome of the Meeting

The meeting provded an opportunity for researchers in various sub-areas of the whole domain of elliptic and parabolic partial differential equations to interact with each other. The talks have been devoted to problems ranging from purely mathematical questions such as De Giorgi conjecture to probabilitistic questions, such as stochastic homogenization, and to applied areas including combustion and biology. Nevertheless, the group had a strong core of common interests which held the meeting very coherent and of a high quality. Numerous frutiful discussions have taken place, across the traditional area boundaries. Overall, we believe that the participants found the conference to be very successful and stimulative for their research.

References

[1] L. Ambrosio and X. Cabré, Entire solutions of semilinear elliptic equations in $\mathbf{R}^3$ and a conjecture of De Giorgi, *J. Amer. Math. Soc.* **13** (2000), 725–739.

[2] I. Athanasopoulos, L.A. Caffarelli and S. Salsa, Caloric functions in Lipschitz domains and the regularity of solutions to phase transition problems, *Ann. of Math.* (2) **143** (1996), no. 3, 413–434.

[3] H. Berestycki and F. Hamel, Front propagation in periodic excitable media, *Comm. Pure Appl. Math.* **55** (2002), 949–1032.

[4] H. Berestycki, F. Hamel and N. Nadirashvili, Elliptic eigenvalue problems with large drift and applications to nonlinear propagation phenomena, *Comm. Math. Phys.* **253** (2005), 451–480.

[5] H. Berestycki, F. Hamel and N. Nadirashvili, The speed of propagation for KPP type problems. I - Periodic framework, *J. Eur. Math. Soc.* **7** (2005), 173–213.

[6] H. Berestycki, F. Hamel and L. Roques, Analysis of the periodically fragmented environment model : I - Species persistence, *J. Math. Biology* **51** (2005), 75–113.

[7] H. Berestycki, B. Nicolaenko and B. Scheurer, Travelling wave solutions to combustion models and their singular limits, *SIAM J. Math. Anal.* **16** (1985), 1207–1242.

[8] E. Bombieri, E. De Giorgi and E. Giusti, Minimal cones and the Bernstein problem, *Invent. Math.* **7** (1969), 243–268.

[9] L.A. Caffarelli, A Harnack inequality approach to the regularity of free boundaries. I. Lipschitz free boundaries are $C^{1,\alpha}$, *Rev. Mat. Iberoamericana* **3** (1987), 139–162; II. Flat free boundaries are Lipschitz, *Comm. Pure Appl. Math.* **42** (1989), 55–78; III. Existence theory, compactness, and dependence on X, *Ann. Scuola Norm. Sup. Pisa Cl. Sci.* **15** (1988), 583–602.

[10] L.A. Caffarelli and C.E. Kenig, Gradient estimates for variable coefficient parabolic equations and singular perturbation problems, *Amer. J. Math.* **120** (1998), 391–439.

[11] L.A. Caffarelli and S. Salsa, *A geometric approach to free boundary problems.* Graduate Studies in Mathematics, 68. American Mathematical Society, Providence, RI, 2005.

[12] P. Constantin, A. Kiselev and L. Ryzhik, Quenching of flames by fluid advection, *Comm. Pure Appl. Math.* **54** (2001), 1320–1342.

[13] P. Constantin, A. Kiselev, L. Ryzhik and A. Zlatos, Mixing a fluid flow, Preprint, 2005.

[14] A. De Masi, S. Luckhaus and E. Presutti, Two scales hydrodynamic limit for a model of malignant tumor cells. to appear in Ann. IHP, Prob. and Stat.

[15] P.C. Fife, *Dynamics of internal layers and diffusive interfaces*, CBMS-NSF Regional Conference, Series in Applied Mathematics 53, 1988.

[16] R.A. Fisher, The advance of advantageous genes, *Ann. Eugenics* **7** (1937), 335–369.

[17] M. Freidlin, *Functional Integration and Partial Differential Equations*, Princeton University Press, Princeton 1985.

[18] M. Freidlin, Reaction-diffusion in incompressible fluid: asymptotic problems, *J. Diff. Eq.* **179** (2002), 44–96.

[19] M. Freidlin and A. Wentzell, *Random perturbations of Hamiltonian systems,* Memoir AMS, 1994.

[20] M. Freidlin and A. Wentzell, *Random Perturbations of Dynamical Systems*, 2nd ed., Springer-Verlag, New York/Berlin, 1998.

[21] N. Ghoussoub and C. Gui, On a conjecture of De Giorgi and some related problems, *Math. Ann.* **311** (1998), 481–491.

[22] N. Ghoussoub and C. Gui, On De Giorgi's conjecture in dimensions 4 and 5, *Ann. of Math.* **157** (2003), 313–334.

[23] L. Glangetas and J.-M. Roquejoffre, Bifurcations of travelling waves in the thermo-diffusive model for flame propagation, *Arch. Rational Mech. Anal.* **134** (1996), 341–402.

[24] P.V. Gordon, L.S. Kagan and G.I. Sivashinsky, Fast subsonic combustion as a free-interface problem, *Interfaces Free Bound.* **5** (2003), 47–62.

[25] A. Kiselev and A. Zlatos, Quenching of combustion by shear flows, *Duke Math. J.* **132** (2006), 49–72.

[26] A.N. Kolmogorov, I.G. Petrovsky and N.S. Piskunov, Étude de l'équation de la diffusion avec croissance de la quantité de matière et son application à un problème biologique, *Bulletin Université d'État à Moscou, Série internationale A* **1** (1937), 1–26.

[27] C. Lederman, J.-M. Roquejoffre and N. Wolanski, Mathematical justification of a nonlinear integrodifferential equation for the propagation of spherical flames, *Ann. Mat. Pura Appl.* **183** (2004), 173–239.

[28] S. Luckhaus and L. Triolo, The continuum reaction-diffusion limit of a stochastic cellular growth model. *A*tti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Rend. Lincei (9) Mat. Appl. **15** (2004), 215–223.

[29] A. Majda and P. Kramer, Simplified models for turbulent diffusion: theory, numerical modelling, and physical phenomena, *Phys. Rep.* **314** (1999), 237–574.

[30] A. Majda and P. Souganidis, Large-scale front dynamics for turbulent reaction-diffusion equations with separated velocity scales, *Nonlinearity* **7** (1994), 1–30.

[31] L. Modica and S. Mortola, Un esempio di Γ^--convergenza, *Boll. Un. Mat. Ital. B* **14** (1977), 285–299.

[32] J.D. Murray, *Mathematical biology*, Springer-Verlag, 2003.

[33] J. Nolen and J. Xin, Existence of KPP type fronts in space-time periodic shear flows and a study of minimal speeds based on variational principle. *D*iscrete Contin. Dyn. Syst. **13** (2005), 1217–1234.

[34] O. Savin, Phase transitions: regularity of flat level sets, Preprint.

[35] M. Schatzman, On the stability of the saddle solution of Allen-Cahn's equation, *Proc. Roy. Soc. Edinburgh Sect. A* **125** (1995), 1241–1275.

[36] S. Serfaty, Vortex collisions and energy-dissipation rates in the Ginzburg-Landau heat flow, parts I and II, to appear.

[37] N. Shigesada, K. Kawasaki, *Biological invasions: theory and practice,* Oxford Series in Ecology and Evolution, Oxford, Oxford UP, 1997.

[38] J. Simons, Minimal varieties in riemannian manifolds, *Ann. of Math.* **88** (1968), 62–105.

[39] G. I. Sivashinsky, Instabilities, pattern formation and turbulence in flames, *Ann. Rev. Fluid Mech.* **15** (1983), 179–199.

[40] A. Zlatos, Sharp transition between extinction and propagation of reaction, *J. Amer. Math. Soc.* **19** (2006), 251–263.