The topology of hyperkähler quotients

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1 Overview of the Field

The purpose of this workshop was to address a question concerning the computation of topological invariants of hyperkähler quotients. Our approach is based on the successful and well-developed similar theory for the case of *symplectic* quotients, so we begin with a brief account of that theory.

Symplectic geometry is the mathematical framework of classical mechanics. A symplectic manifold is a manifold equipped with a *symplectic form*, i.e. a non-degenerate closed differential 2-form, which is the geometric data needed to translate a Hamiltonian function on the system to the dynamics of the system. Examples of symplectic manifolds are any 2-dimensional surface equipped with its area form, cotangent bundles T^*M, toric varieties, and flag manifolds. A symplectic manifold is Kähler if there is also a complex structure compatible with the symplectic form; when there are *three* Kähler structures on M, with associated compatible complex structures interacting like the quaternions, then M is *hyperkähler*. Many hyperkähler manifolds appear naturally in physics and representation theory. Examples from physics are $T^*\mathbb{P}^1$ with the Eguchi-Hansen metric, K3 surfaces, and moduli spaces of Higgs bundles over a Riemann surface [7]; examples arising in representation theory are quiver varieties, as studied by Nakajima [14].

In the theory of hyperkähler or symplectic quotients, we are primarily concerned with a situation in which there is a symmetry of the system, as encoded by the action of a compact Lie group G. Symplectic manifolds with an action of a Lie group G and a corresponding moment map, which is a suitably compatible collection of Hamiltonian functions, are called Hamiltonian G-spaces. For a hyperkähler manifold M, we require that there be a moment map $\mu_i : M \to \mathfrak{g}^*, i = 1, 2, 3$, for *each* of the three Kähler structures. Given a symplectic Hamiltonian G-space, the symplectic quotient is defined as $M//G := \mu^{-1}(0)/G$. The reduced space inherits a symplectic structure from M. In the hyperkähler case, we take the hyperkähler quotient $M////G$ to be the quotient by G of the intersection of the zero-level sets of all three moment maps; this is again hyperkähler.

Hyperkähler quotients, and hyperkähler geometry in general, has recently attracted much attention due to its relationship between many other fields of mathematics. The topological invariants of hyperkähler manifolds, such as rational cohomology or integral K-theory, are often quite interesting. For example, the K-theory of quiver varieties give geometric realizations of representations of certain algebras associated to quivers. There are also close connections between the cohomology of hyperkähler analogues of toric varieties and the combinatorial theory of hyperplane arrangements.

We now give an overview of our approach towards the computation of the topology (more specifically, the rational cohomology ring) of hyperkähler quotients. There is a "meta-principle" for computing such invariants of Hamiltonian quotients of various types, which we call here the *Kirwan method*. Let M be a Hamiltonian G-space of some type (symplectic or hyperkähler, and M_G the appropriate Hamiltonian quotient of M by G. Then the Kirwan method consists of the following three steps:

"The Kirwan method":

1. **"Meta-Theorem" (Kirwan surjectivity):** For M and M_G as above, there is a natural ring homomorphism
$$\kappa : H^*_G(M;\mathbb{Q}) \to H^*(M_G;\mathbb{Q})$$
which is *surjective*. In particular, in order to compute $H^*(M_G;\mathbb{Q})$, it suffices to compute $H^*_G(M;\mathbb{Q})$ and $\ker(\kappa)$.

2. Compute $H^*_G(M;\mathbb{Q})$.

3. Compute $\ker(\kappa)$.

The point of this method is that one can often compute the last two objects, $H^*_G(M;\mathbb{Q})$ and $\ker(\kappa)$, using *equivariant* techniques which are unavailable on the quotient. In the symplectic case, this "Kirwan method" has been well-developed; in particular, Step (1) in this case was proven by Kirwan [9], and various explicit solutions of Steps (2) and (3) can be found e.g. in [8, 17, 3, 4]. Thus, our research program is to develop the Kirwan method for hyperkähler quotients. The focus of our BIRS workshop was in the proof of Step (1) for this hyperkähler case.

2 Recent Developments and Open Problems

Kirwan's proof of Step (1) in the case of symplectic quotients involves showing that the norm-square $\|\mu\|^2$ of the symplectic moment map $\mu : M \to \mathfrak{g}^*$ gives rise to an equivariantly perfect Morse-type stratification of M, which gives surjectivity since the 0-level set of μ is the absolute minimum of the norm-square.[1] We propose to prove an analogue of Kirwan surjectivity in the setting of finite-dimensional hyperkähler quotients using Morse-type methods similar to Kirwan's proof. There are already specific known examples where such a hyperkähler analogue of Kirwan surjectivity result does hold [10, 11]. Moreover, in the specific infinite-dimensional case of the moduli space of Higgs bundles over a Riemann surface, Daskalopoulos, Weitsman, and Wilkin have developed several new Morse-theoretic techniques using the norm-square of the moment map to obtain new Kirwan surjectivity results in rational Borel-equivariant cohomology [2, 18].

In the case of a hyperkähler manifold with the action of a group G which is Hamiltonian with respect to each of the three Kähler structures (a *hyperhamiltonian* group action), there are three moment maps $\mu_i : M \to \mathfrak{g}, i = 1, 2, 3$ (one for each of the Kähler structures).

In an unpublished draft manuscript, Kirwan suggested that one could first use the Morse theory of $\|\mu_2\|^2 + \|\mu_3\|^2 = \|\mu_{\mathbb{C}}\|^2$ (where $\mu_{\mathbb{C}} = \mu_2 + i\mu_3 : M \to \mathfrak{g} \otimes \mathbb{C}$) to construct a map $H^*_G(M) \to H^*_G(\mu_{\mathbb{C}}^{-1}(0))$, and then use the Morse theory of the function $\|\mu_1\|^2$ on the space $\mu_1^{-1}(0)$ to construct a map $H^*_G(\mu_{\mathbb{C}}^{-1}(0)) \to H^*_G(\mu_1^{-1}(0) \cap \mu_{\mathbb{C}}^{-1}(0))$. It would then remain to show that both of these maps are surjective. There are two main technical difficulties in carrying out the second step, firstly that the gradient flow of $\|\mu_1\|^2$ on $\mu_{\mathbb{C}}^{-1}(0)$ might not converge (we need convergence to construct a Morse theory on this space), and secondly that the space $\mu_{\mathbb{C}}^{-1}(0)$ is singular so we would need to provide some extra analysis for the Morse theory to work. Both of these difficulties are new to the hyperkähler situation; the first does not arise in the presence of only *one* moment map (assuming we take the preimage of a regular value), and the second does not arise since in the symplectic or Kähler case one usually assumes that the moment map is proper, so the level set is compact.

Nevertheless, despite these difficulties, Wilkin has made Kirwan's second approach work in the infinite-dimensional case of the moduli spaces of Higgs bundles. In particular, Wilkin (in collaboration with his Ph.D. supervisor Georgios Daskalopoulos and Jonathan Weitsman)

1. proved the gradient flow converges, despite the non-properness of the moment maps [18],

2. showed there exists a Morse-type theory for the norm-square $\|\mu_{\mathbb{R}}\|^2$ on $\mu_{\mathbb{C}}^{-1}(0)$ [2], and

3. showed how to use the singularities in the preimage $\mu_{\mathbb{C}}^{-1}(0)$ to obtain the correct formula for the Poincaré polynomial of the moduli space by developing a theory which can be described as "Morse theory in a stratified sense" [2].

[1] There are technical difficulties arising from the fact that $\|\mu\|^2$ is not, in fact, Morse; this is the technical and important contribution of Kirwan's proof, which has had wide applications.

We intend to follow the approach of Daskaopoulous, Weitsman, and Wilkin, and in particular to prove finite-dimensional analogues of their theorems to obtain a general surjectivity result in the hyperkähler case.

3 Progress made at the BIRS workshop and outcomes of the meeting

3.1 The Morse theory of the norm-square of the moment maps

- As in the outline of Wilkin's work in Section 2 above, we first need to prove that the gradient flow of $\|\mu_1\|^2$ converges to a critical point. A result of Lojasiewicz in [12] shows that this problem reduces to proving that the gradient flow remains in a compact set. As a first test case, we proved explicitly that for the case of S^1 acting on $T^*\mathbb{C}^n$, the flow indeed stays in a compact set. We also discussed how Hitchin proved the gradient flow convergence in [7] for the case of Higgs bundles, where he uses the fact that the finite-time gradient flow lies on a $G^{\mathbb{C}}$ orbit to compute estimates along the flow. We can describe the case of quiver varieties via a setup similar to Hitchin's. During the BIRS workshop we computed simple examples of quivers and came up with specfic conjectures of analytic estimates on the gradient flow which would suffice to prove its convergence in the case of quiver varieties.

- We also made progress at the BIRS meeting in understanding the singularites arising in the Morse theory of $\|\mu_1\|^2$. Previous to the meeting, we proved the following theorem about the Morse index of $\|\mu_1\|^2$.

  Theorem. *Let M be a finite-dimensional hyperkähler manifold, and let $f(x) = \|\mu_1(x)\|^2$ on M. At a critical point $x \in \mu_{\mathbb{C}}^{-1}(0) \subset M$ let $N(x) \subset T_xM$ denote the negative eigenspace of f, and let $L(x)$ denote the linearisation of the complex moment map $\mu_{\mathbb{C}}$. Then $N(x) \subseteq L(x)$.*

  This is the first step in relating the Morse index calculations of $\|\mu_1\|^2$ on the smooth manifold M (where Kirwan's results show that the index is well-defined) to the Morse index calculations on the singular space $\mu_{\mathbb{C}}^{-1}(0)$. To carry out the approach of [2] in our case, we need to show that the negative directions at a critical point are *contained within* the space $\mu_{\mathbb{C}}^{-1}(0)$, not just the linearisation of this space. During the BIRS workshop we computed several concrete examples and showed that the negative directions are indeed contained within $\mu_{\mathbb{C}}^{-1}(0)$ in each case. Using these examples, we formulated explicit strategies to prove the more general cases.

- We were also able to prove the following theorem regarding the critical sets of the functional $\|\mu_1\|^2$ on the space $\mu_{\mathbb{C}}^{-1}(0)$ for quiver varieties.

  Theorem. *At a critical point of $\|\mu_1\|^2$ the quiver splits into sub-quivers. In particular, each connected component of the set of non-minimal critical points can be expressed as the product of quiver varieties of simpler quivers.*

  Hence we can inductively build up the critical sets by studying quivers with simpler structures. This is analogous to the well-known setting for the Yang-Mills functional, where a holomorphic bundle splits into sub-bundles at a critical point (see for example [1]). This fact (for Higgs bundles) is used heavily in [2], which leads us to believe that in the case of quiver varieties many of the methods of [2] for Higgs bundles will hold.

3.2 Alternative Approaches

During the BIRS workshop, we also discussed possible alternative approaches to our problem. In particular, we discussed the possibility of first taking the Kähler quotient $N := T^*\mathbb{C}^n//_\alpha G$ with respect to the real moment map $\mu_{\mathbb{R}}$, and then further restricting to $\mu_{\mathbb{C}}^{-1}(0)$, as $T^*\mathbb{C}^n////_{(\alpha,0)}G = N \cap \mu_{\mathbb{C}}^{-1}(0)$. With this method, the Kähler quotient N should be a smooth manifold, and we expect that its relation to the hyperkähler quotient $M////G$ can be obtained using Morse theory for the norm square of the complex moment map $\|\mu_{\mathbb{C}}\|^2$.

In this case, we hope to use the S^1-action rotating the fibers of $T^*\mathbb{C}^n$ and its corresponding moment maps on the Kähler and hyperkähler quotients in order prove that the cohomology of the Kähler quotient surjects onto the cohomology of the hyperkähler quotient. Using Morse theory to build both of the quotients simultaneously, we note that the minimal level sets of the S^1-moment maps are the same in both cases. As we pass each higher critical level, we hope to prove that surjectivity still holds, and in order to show this we have formulated the following conjecture:

Conjecture. *For each connected component C of the S^1-fixed set of $T^*\mathbb{C}^n//G$, the restriction to $\mu_{\mathbb{C}}^{-1}(0)$ induces a surjection in cohomology, $H^*(C) \twoheadrightarrow H^*(C')$, and the two Morse indices agree: $\lambda_C = \lambda_{C'}$.*

We verified that this argument works for hyperpolygon spaces by performing explicit computations based on [11]. In this case, the S^1-fixed sets in both the Kähler and hyperkähler quotients are compact projective spaces, and our above conjecture holds. If this argument works in general, then we can establish the surjectivity from the Kähler quotient to the hyperkähler quotient $H^*\big(T^*\mathbb{C}^n//_\alpha G\big) \twoheadrightarrow H^*\big(T^*\mathbb{C}^n////_{(\alpha,0)}G\big)$. To establish the hyperkähler analogue of Kirwan surjectivity, we must further establish Kirwan surjectivity for the Kähler quotient: $H_G^*(T^*\mathbb{C}^n) \twoheadrightarrow H^*\big(T^*\mathbb{C}^n//_\alpha G\big)$. Since the spaces involved are non-compact, we must prove that the gradient flow converges in order to apply Kirwan's surjectivity arguments.

A further approach which we discussed at BIRS is to do Morse theory using linear combinations of the S^1-moment map and $\|\mu_{\mathbb{C}}\|^2$. Although we need to restrict to the minimum of $\|\mu_{\mathbb{C}}\|^2$, the S^1-moment map much better behaved. We explored the possibility of starting with one such moment map and perturbing it by adding a multiple of the other, hoping to obtain the same quotient without the Morse-Kirwan difficulties.

3.3 Abelianization

Another related topic which we discussed at BIRS is the "abelianization" of hyperkähler quotients. When working with symplectic quotients, one can use the techniques of Tolman-Weitsman [17] and Goldin [3] to compute the cohomology of $M//T$ where T is abelian. For quotients of the form $M//G$ where G is not abelian, we must first abelianize, by restricting from G to a maximal torus T. Working in the hyperkähler case, we studied the following abelianization conjecture of Tamás Hasusel:

Conjecture *(Hausel) Let G be a compact, connected Lie group and T a maximal torus in G. If both of the hyperkähler quotients $T^*\mathbb{C}^n////G$ and $T^*\mathbb{C}^n////T$ are hypercompact, then*

$$H^*(T^*\mathbb{C}^n////G) \cong \frac{H^*(T^*\mathbb{C}^n////T)^W}{\mathrm{Ann}(e_T(\mathfrak{g}/\mathfrak{t})^2)},$$

where $e_T(\mathfrak{g}/\mathfrak{t}) \in H_T^*(\mathrm{pt})$ is the equivariant Euler class of the representation $\mathfrak{g}/\mathfrak{t}$ of T.

In [6], Hausel and Proudfoot prove an S^1-equivariant version of this abelianization theorem based on Martin's proof [13] of the analogous result for symplectic quotients. They use the S^1-equivariance in order to establish integral formulae, which allow them to reproduce Martin's Poincaré duality arguments in the non-compact setting. However, our alternative proof [5] of Martin's theorem does not use Poincaré duality, and we believe that this will allow us to generalize our techniques to the non-compact hyperkähler setting without S^1-equivariance. A close analysis of this conjecture leads us to believe that it is best approached using the algebro-geometric language of holomorphic symplectic quotients, in lieu of hyperkähler quotients.

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