Saari's Conjecture

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1 Overview of the Field

Saari's conjecture is notoriously difficult. Donald Saari proposed it in 1970 in the following form [13]: *In the Newtonian n-body problem, if the moment of inertia, $I = \sum_{k=1}^{n} m_k |q_k|^2$, is constant, where $q_1, q_2, \ldots, q_n$ represent the position vectors of the bodies of masses $m_1, \ldots, m_n$, then the corresponding solution is a relative equilibrium*. In other words: Newtonian particle systems of constant moment of inertia rotate like a rigid body.

There have been many attempts to solve this problem. Some of them even led to the publication of incorrect proofs, such as those in [10, 11]. More recently, the interest in this conjecture has grown considerably due to the discovery of the "figure eight" solution (see [2]), which—as numerical arguments show—has an approximately constant moment of inertia but is not a relative equilibrium.

Still, there have been a few successes in the struggle to understand Saari's conjecture. McCord proved that the conjecture is true in the case of three bodies with equal masses [9]. Llibre and Piña provided an alternative proof of this case, but they never published it [6]. Moeckel obtained a computer-assisted proof for the Newtonian three-body problem for any values of the masses [7, 8]. Diacu, Pérez-Chavela, and Santoprete showed that the conjecture is true for all n in the collinear case for any potential that depends only on the mutual distances between point masses [3]. There have also been results, such as [1, 4, 5, 12, 14, 15], that consider the conjecture in contexts different from the Newtonian one.

Even more difficult than Saari's conjecture is its following extension. In the Newtonian n-body case, define the configurational measure of the particle system to be the function $UI^{1/2}$, where U is the Newtonian potential. The extended Saari's conjecture can then be stated as follows: *Every solution of constant configurational measure is homographic*. In particular, if the moment of inertia is constant, then the potential is constant, therefore every homographic solution is a relative equilibrium, so Saari's conjecture follows from its extension.

The extended Saari's conjecture covers new territory. While in the original Saari's conjecture collisions are excluded (because they lead to an unbounded potential, which contradicts the constant moment of inertia assumption) and the motion remains bounded (because the moment of inertia is constant), both collision and unbounded orbits may occur in the extended version of the conjecture.

2 Scientific Progress Made

During the two weeks spent at BIRS, we succeeded to solve the extended Saari's conjecture in some important cases, especially in the case of three bodies. Our results can be outlines in eight theorems, as follows.

Theorem 1 shows that, for homogeneous potentials of order $a < 2$, the extended Saari's conjecture is true for any total-collision solution of the planar or spatial n-body problem. Theorem 2 validates the extended Saari's conjecture for $0 < a < 2$ for any type of collision in the n-body case. Theorem 3 proves the extended conjecture correct in the rectilinear case for $0 < a < 2$. Theorem 4 shows the extended conjecture to be always valid in the collinear case. Theorem 5 proves that, for $0 < a < 2$, the extended conjecture is true in the three-body problem if the solutions stay away from the paths that make them scatter asymptotically towards rectilinear central configurations. Theorem 6 proves the extended conjecture correct in the Newtonian three-body problem with equal masses and non-negative energy. Theorem 7 shows that for any given initial configuration of three bodies, the extended conjecture is valid if the chosen angular momentum is large enough. Finally, Theorem 8 shows that if the angular momentum is chosen first, then the extended conjecture is true if the initial positions are taken close enough to an equilateral triangle of a certain size.

The key tool for obtaining these results is provided by what we call Fujiwara coordinates, which have been earlier developed by one member of this team. These coordinates allowed us to regard the conjecture from a new point of view and thus obtain the results outlined above.

3 Outcome of the Collaboration

The outcome of this "research in teams" collaboration has been very positive. Without the opportunity to meet for two weeks together and work on this problem in the excellent conditions offered by BIRS, we might have not obtained these results. Moreover, this is the beginning of what we hope will be a long and fruitful collaboration.

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