Weierstraß-Institut für Angewandte Analysis und Stochastik

BIRS workshop

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work in progress, joint with

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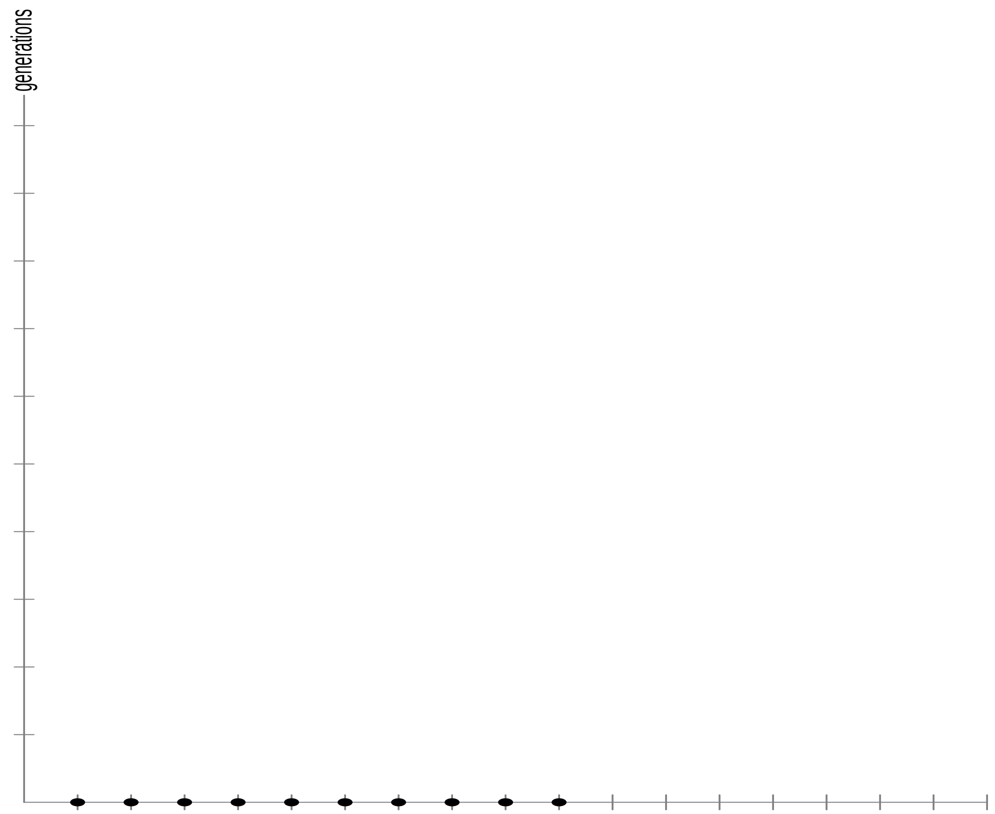
Continuous-mass stable branching and its resampling counterpart

Leibniz Gemeinschaft

Free branching: Galton-Watson processes

- ▷ individuals have a random number of offspring, independently, according to a fixed probability distribution
- ▷ the totality of offspring forms the next generation
- ▷ da capo ...

Free branching: Galton-Watson forests



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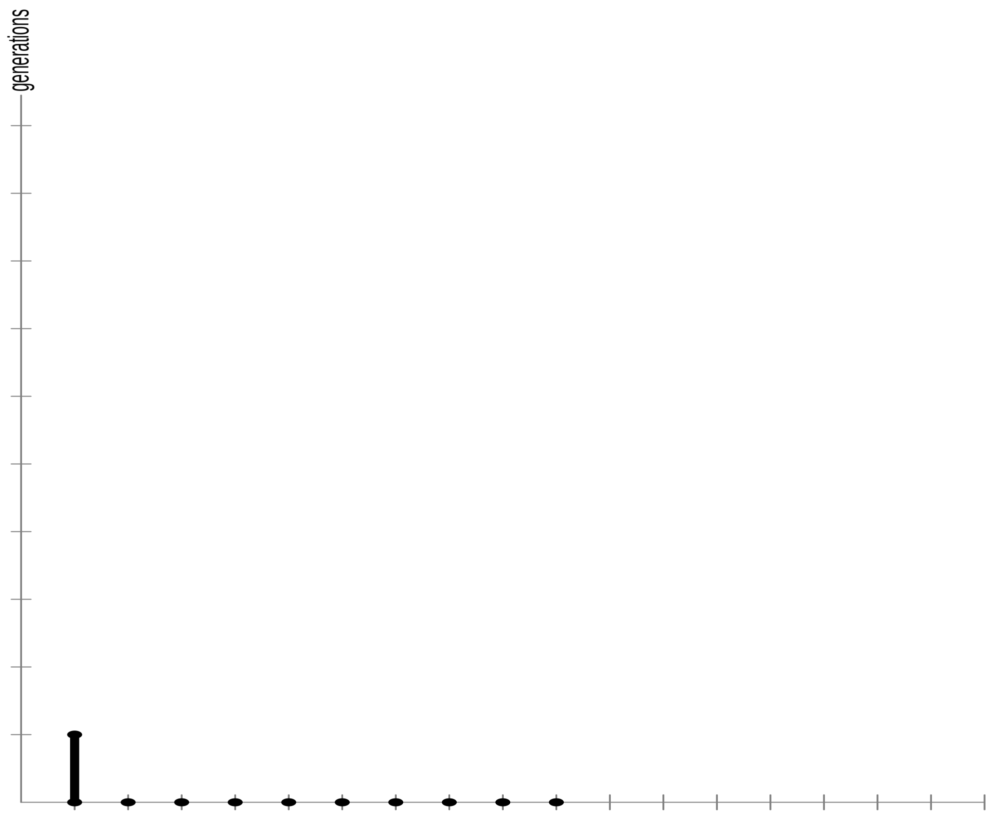

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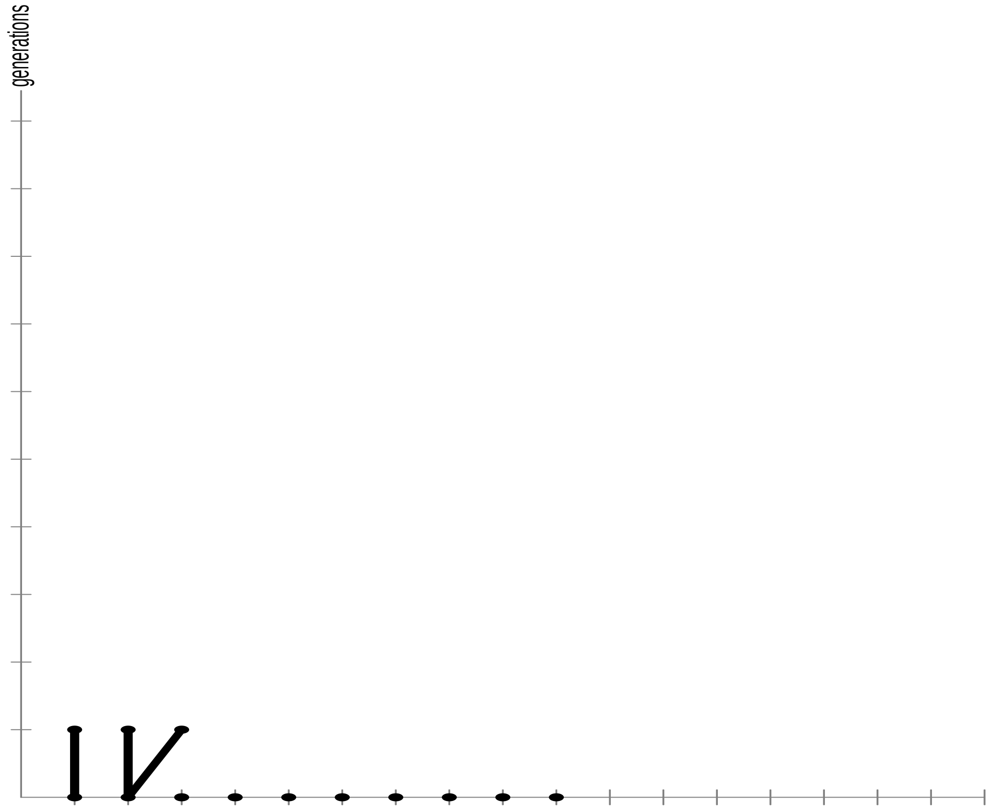

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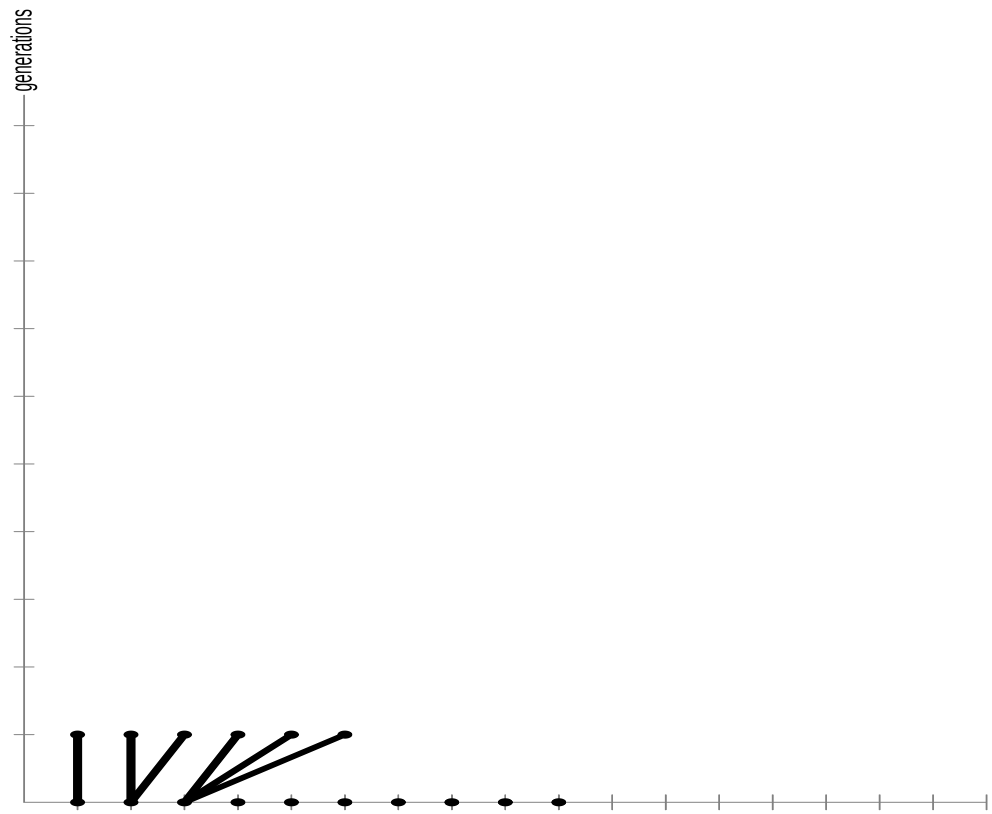

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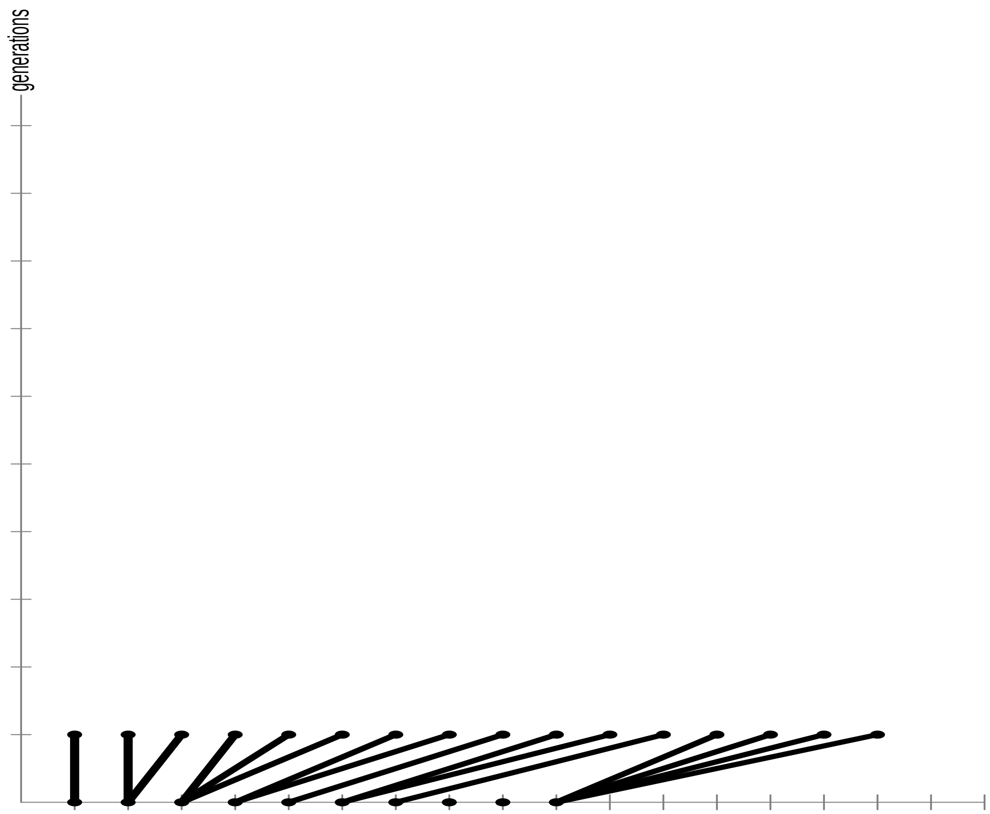

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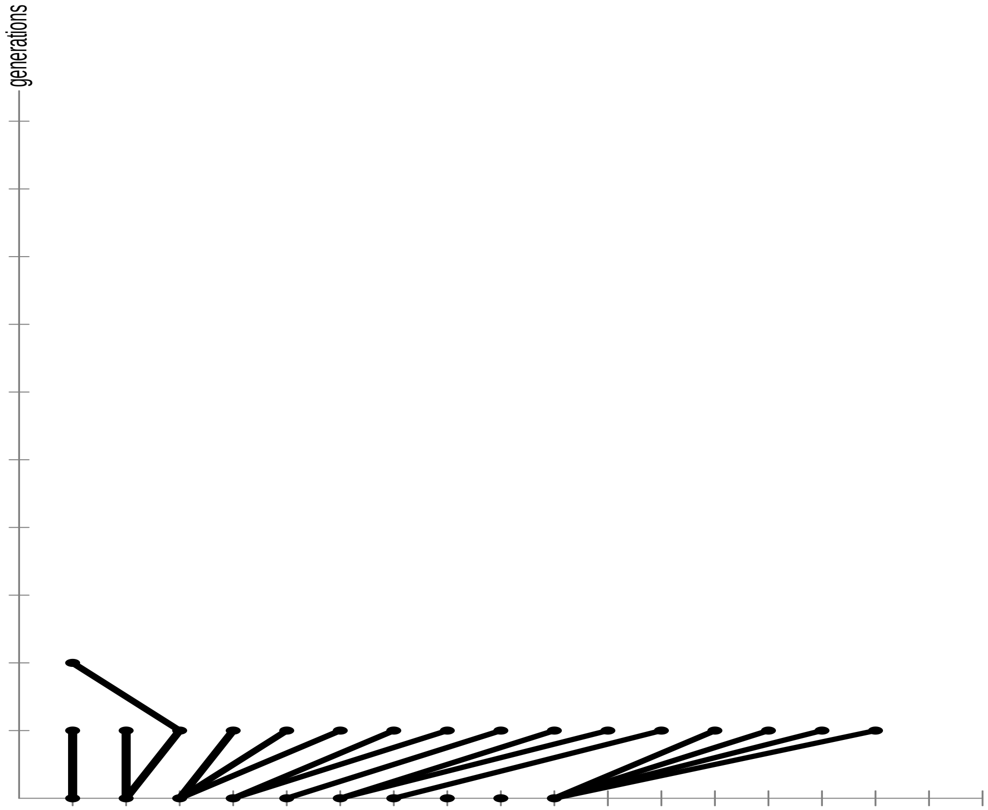

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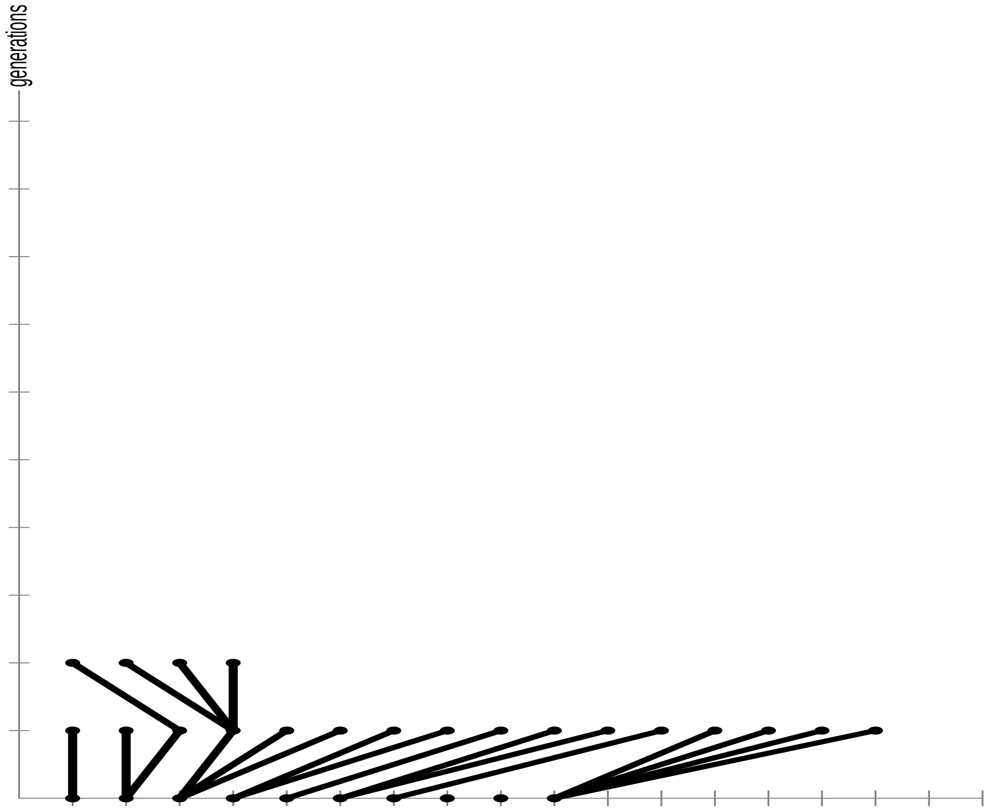

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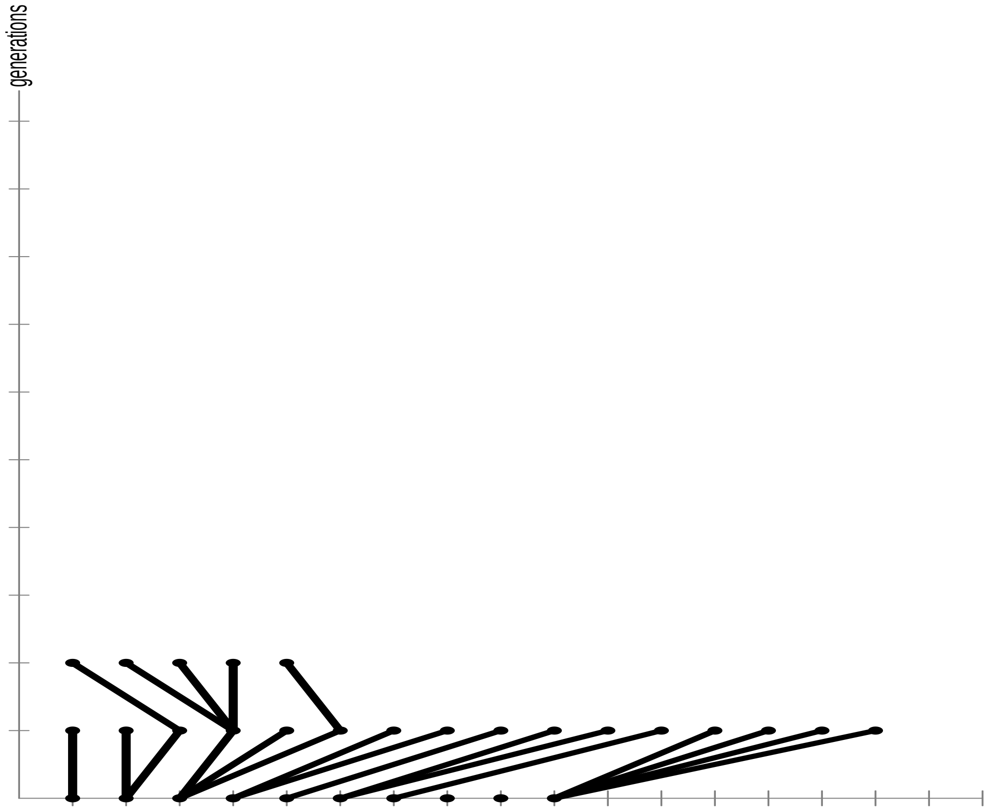

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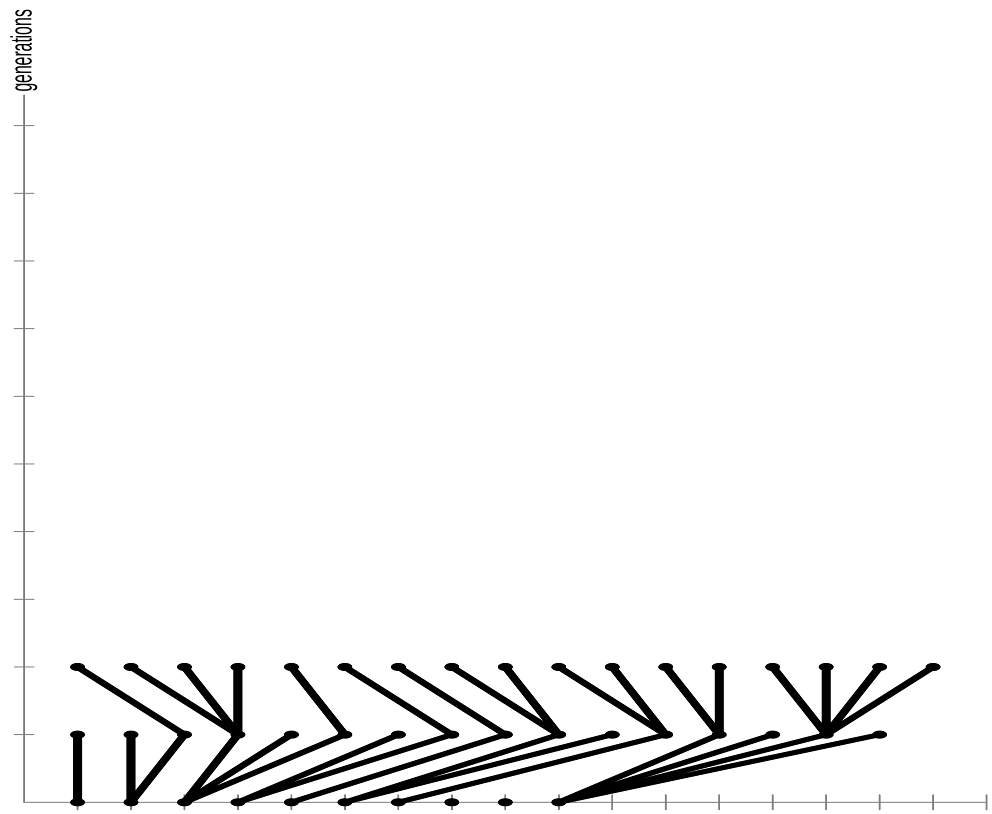

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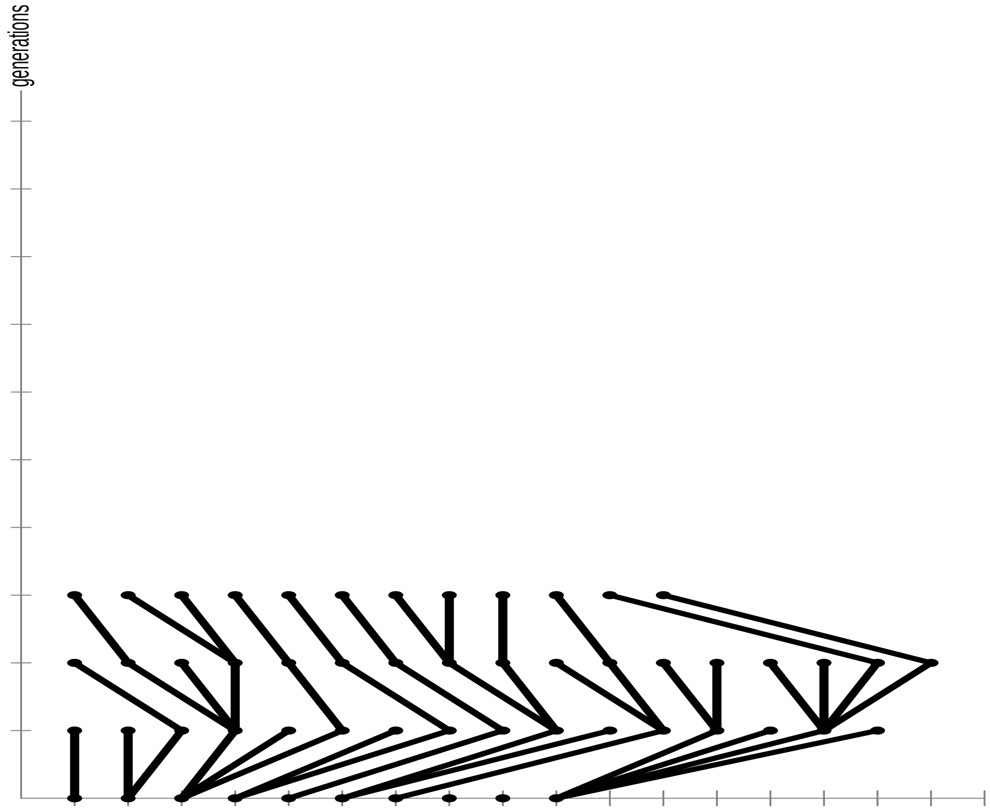

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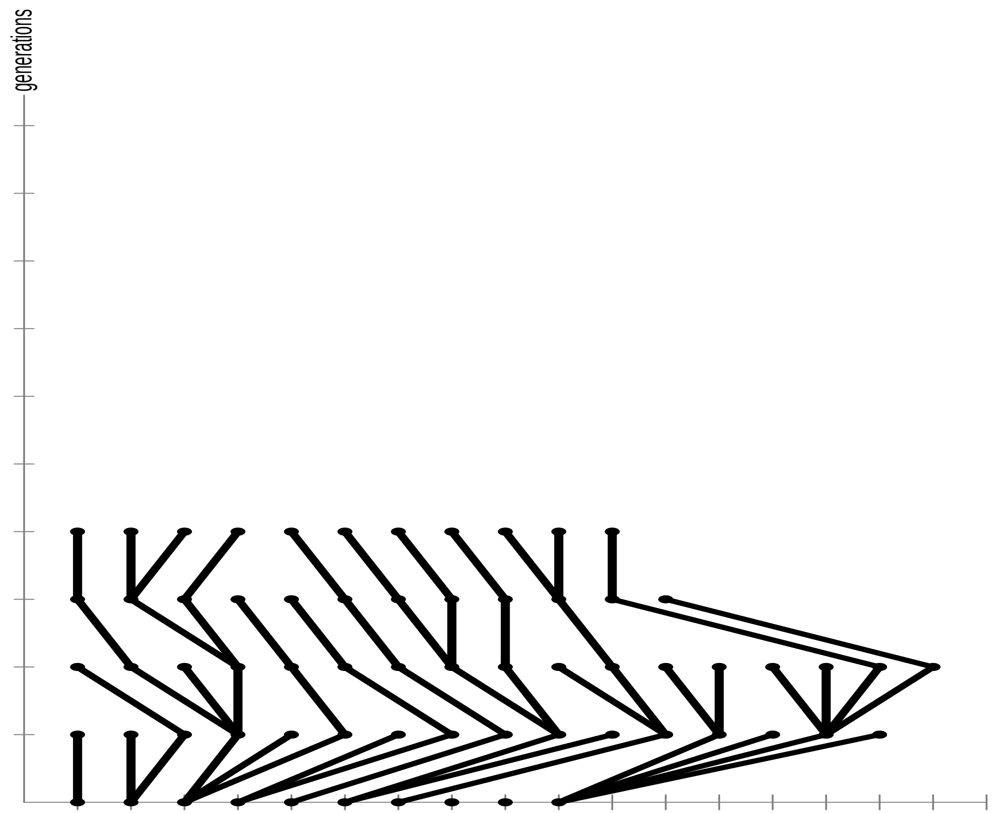

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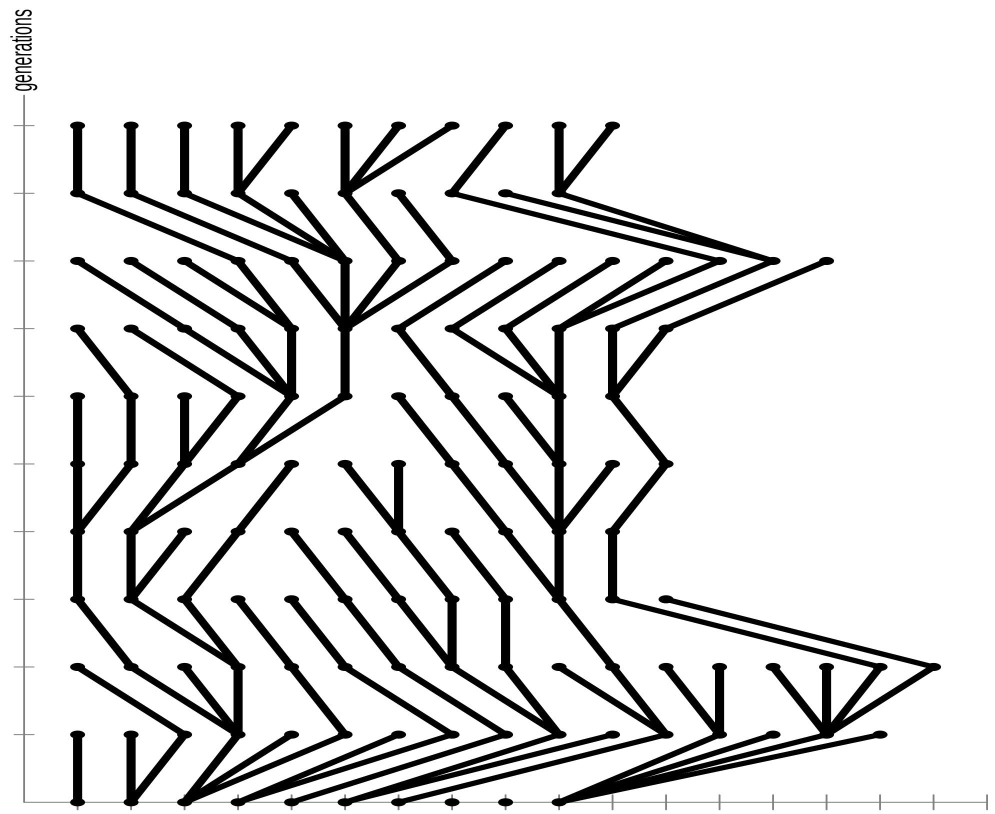

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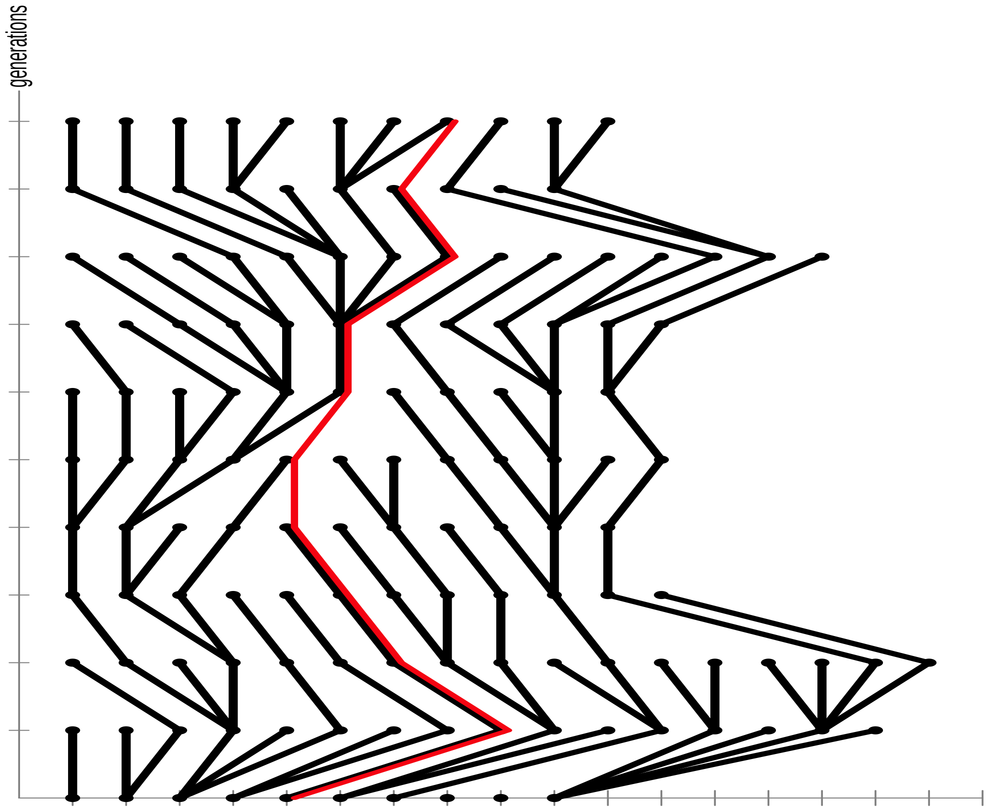
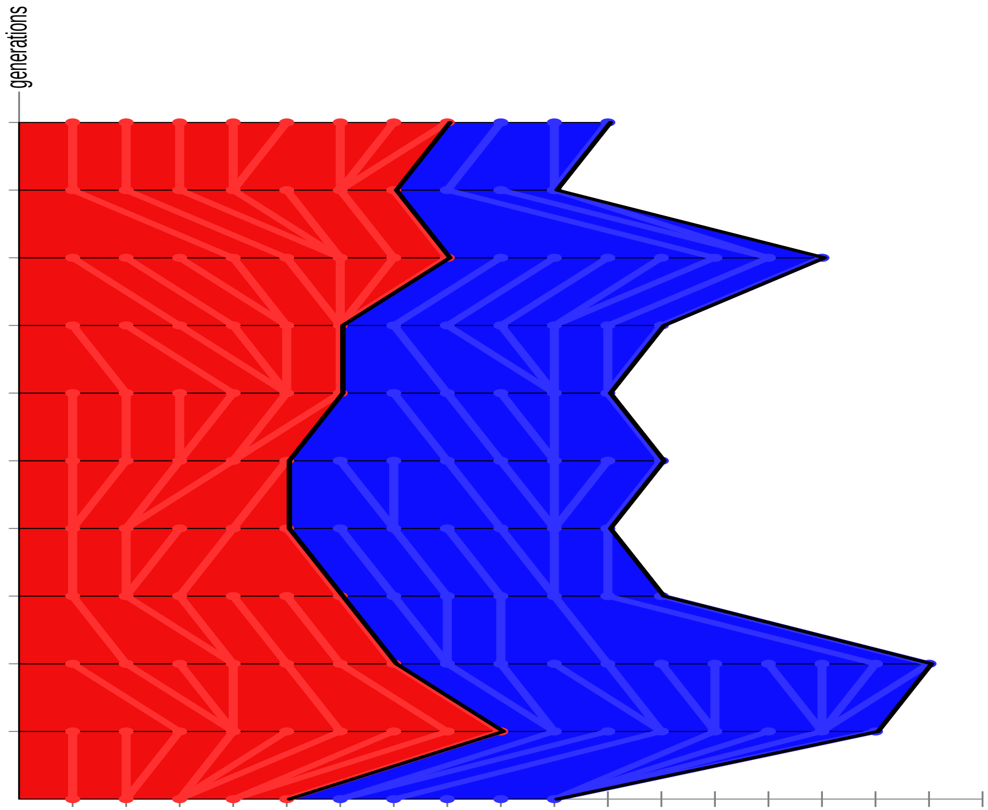


Order the individuals (artificially), let

$$\xi_t(n) := \#\text{ descendants of founders } 1, \ldots, n \text{ alive in generation } t\ .$$

For fixed n, $\xi_t(n),\ t = 0, 1, 2, \ldots$ is a Galton-Watson process with $\xi_0(n) = n$.

Free branching & neutral types



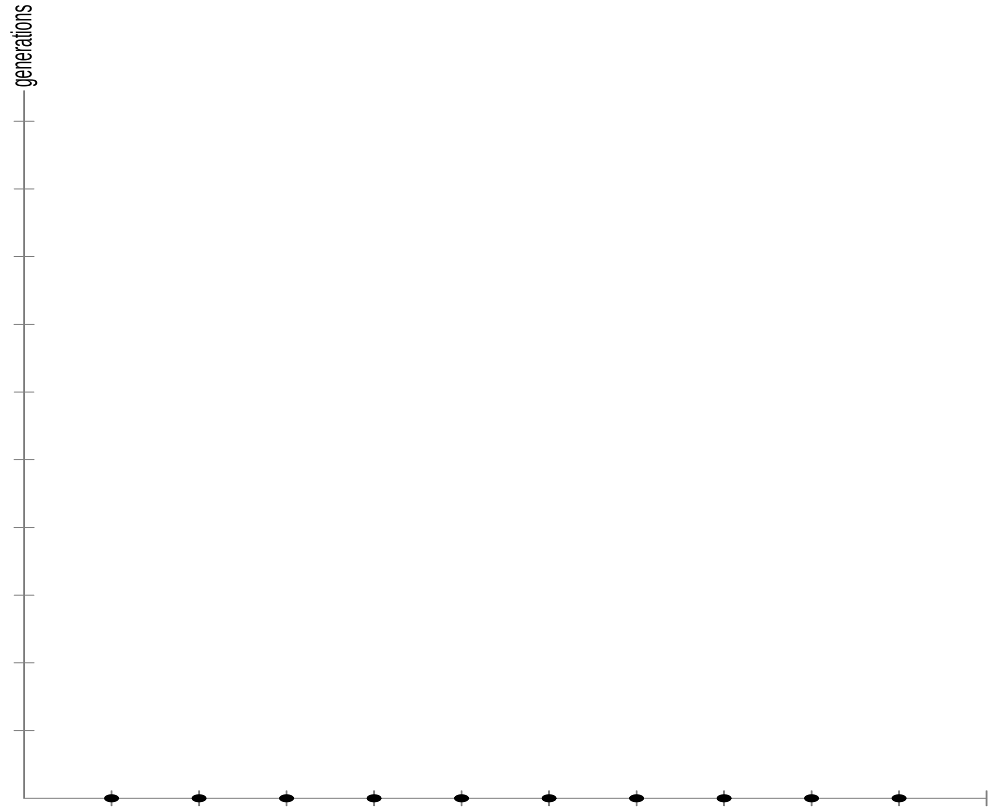
Example: there are 2 neutral types, red and blue.

Fixed population size: Canning's model

Describes a genealogy:

- ▷ fixed population size N
- ▷ individuals have a random number of offspring,
  the offspring numbers $(\nu_1, \ldots, \nu_N)$ are exchangeable with $\sum_{i=1}^{N} \nu_i = N$
- ▷ the totality of offspring forms the next generation
- ▷ da capo ...

Fixed population size: Canning's model



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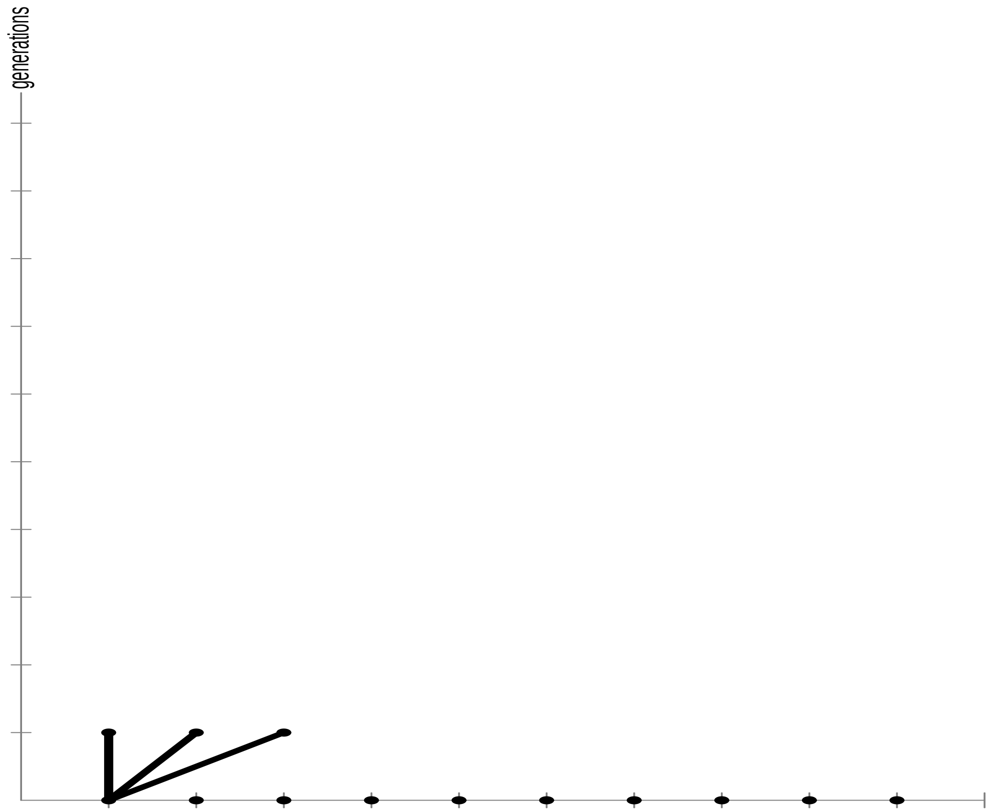

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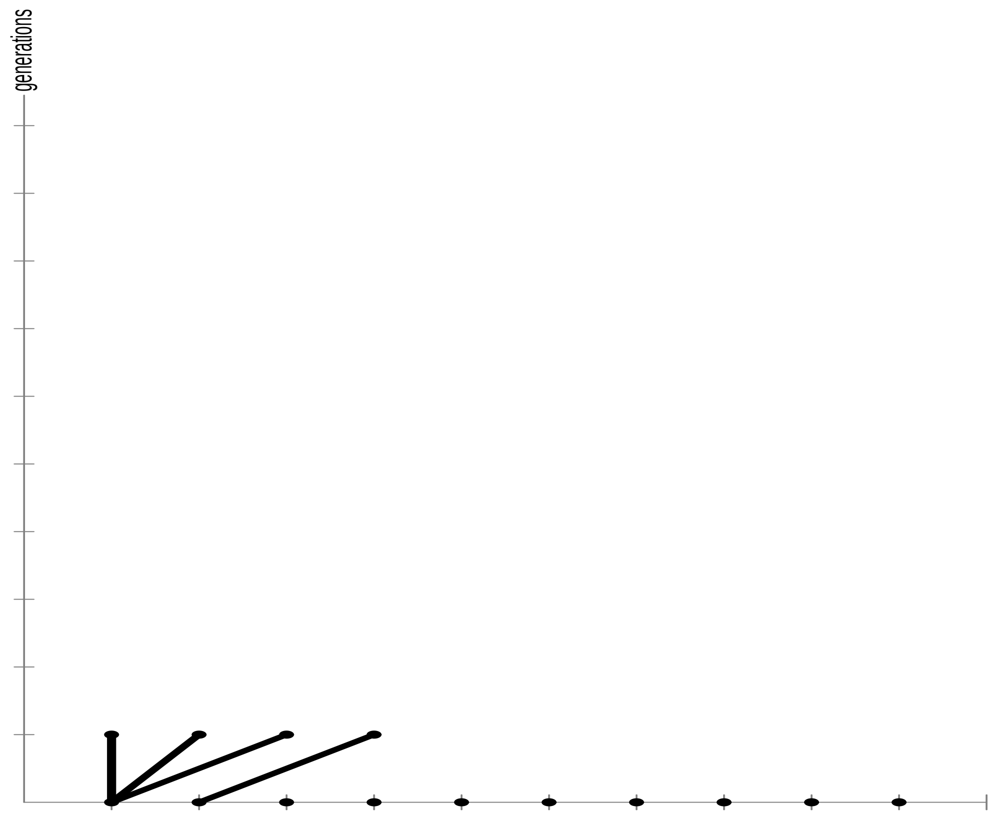

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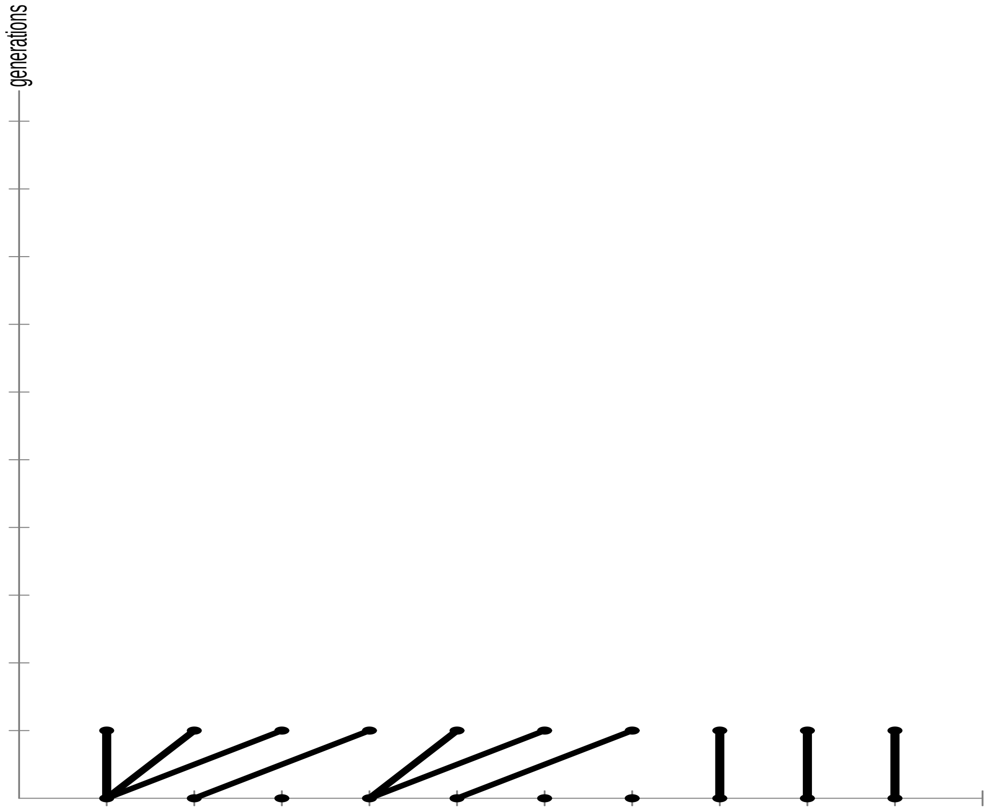

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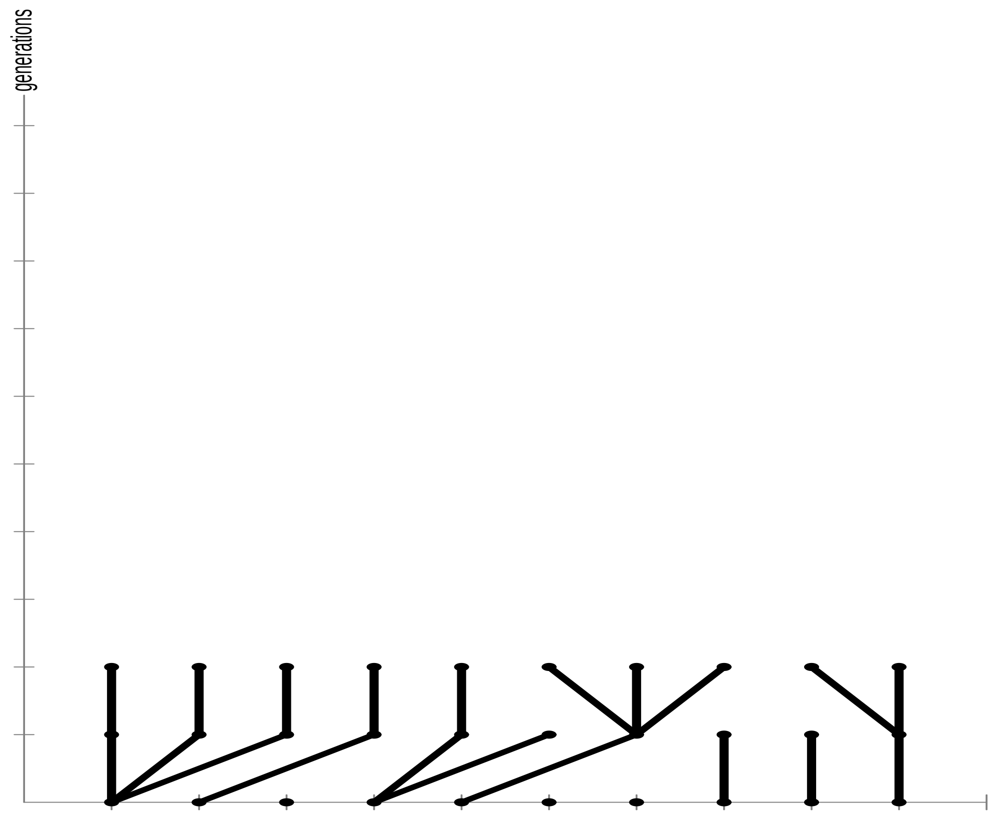

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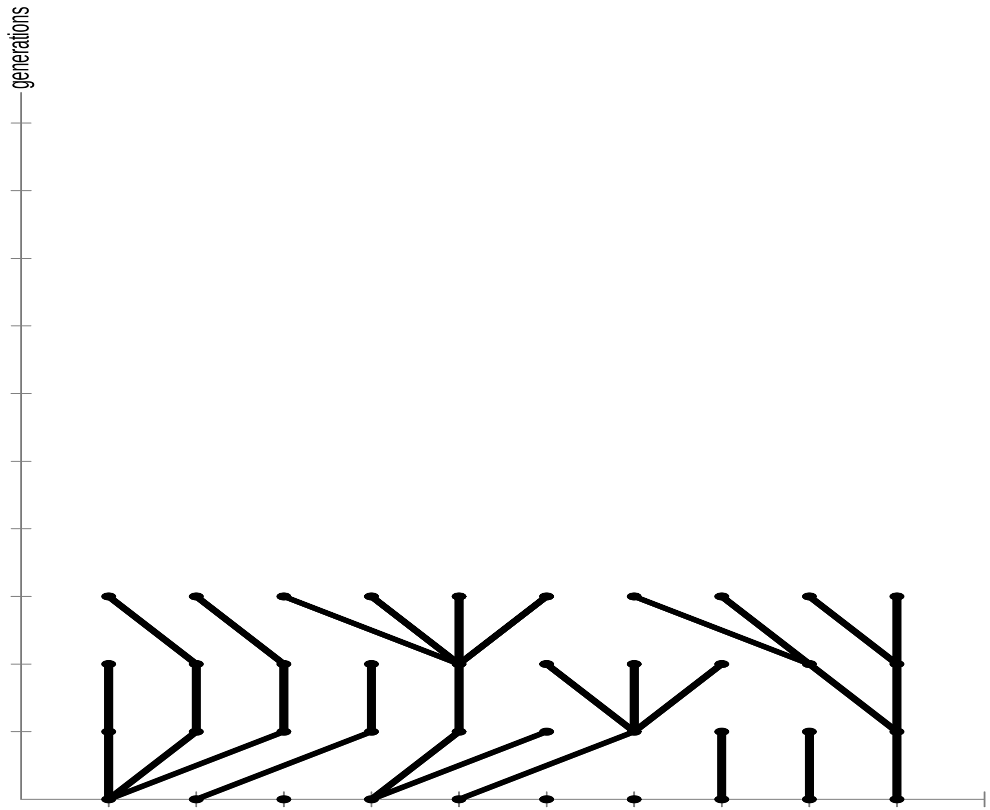

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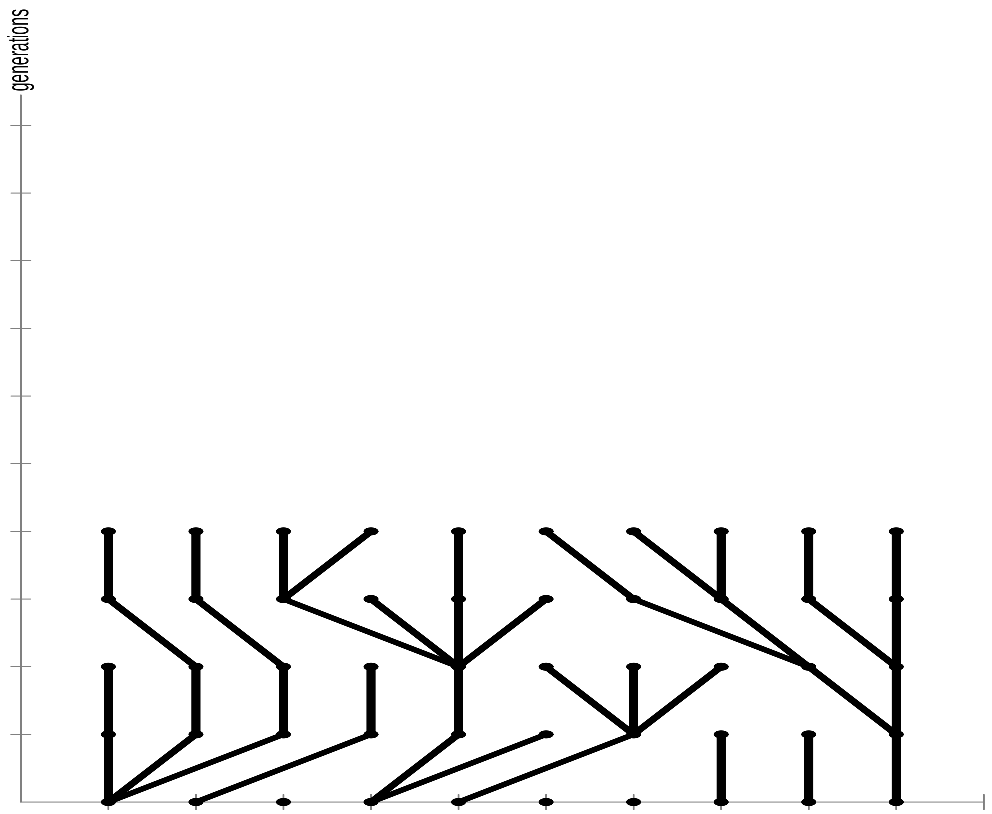

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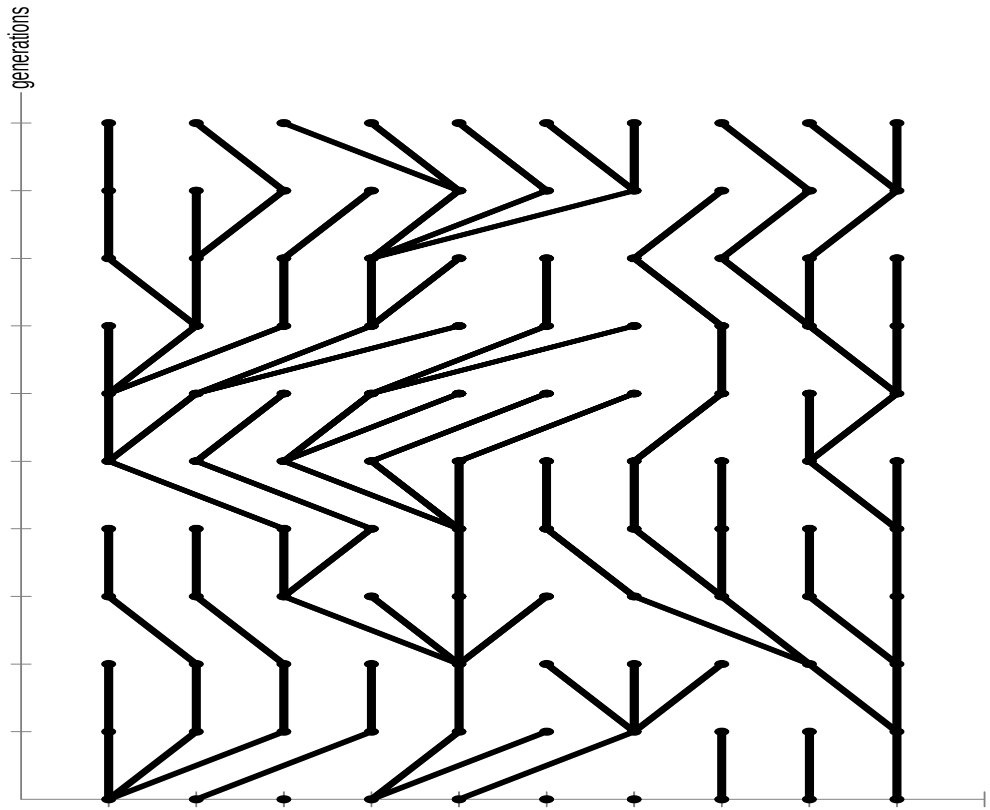
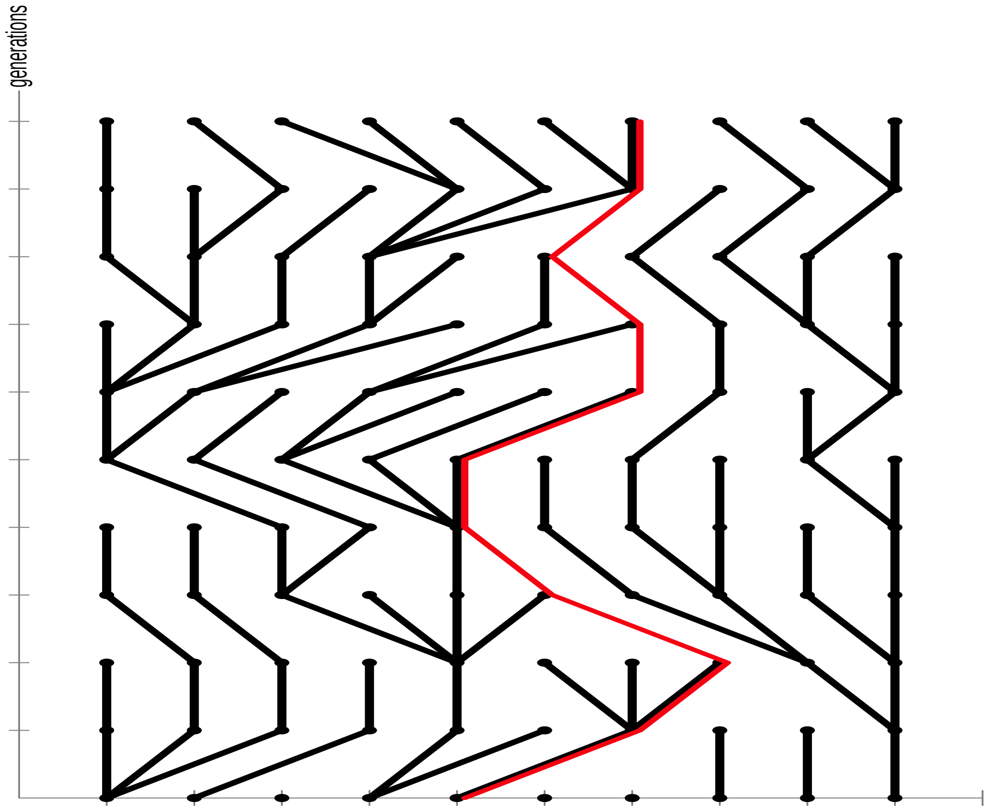
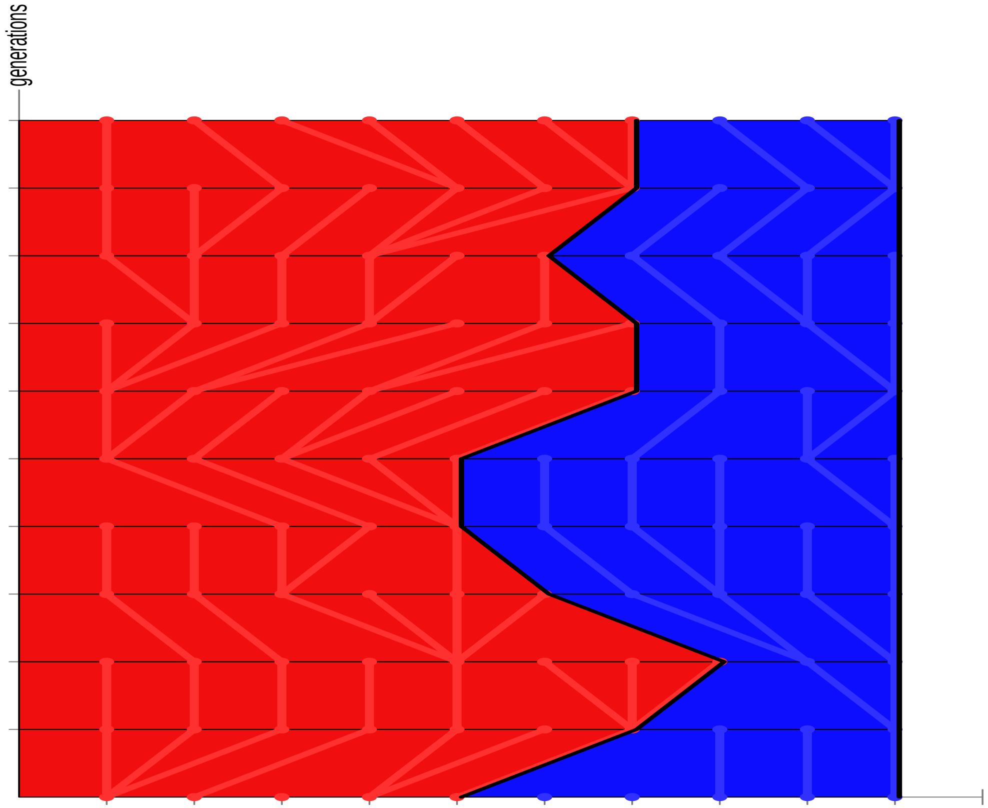




Again: order the individuals (artificially), let

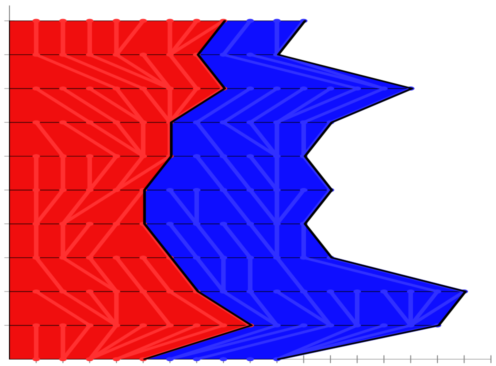
$$\eta_t(n) := \#\text{ descendants of founders } 1, \ldots, n \text{ alive in generation } t\ .$$

Neutral types

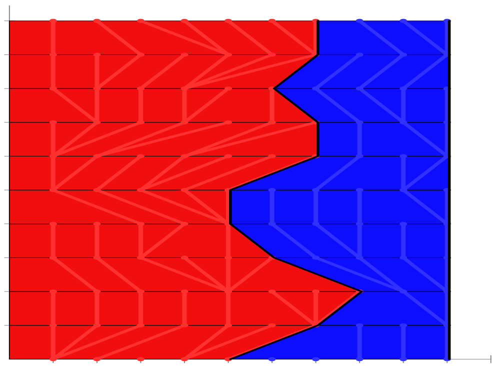


We could again think of neutral types (without mutation), e.g. red and blue.

Relation between



and



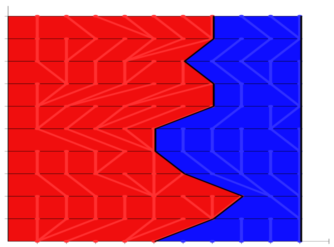
?

Relations?

First answer Condition a Galton-Watson process to have *constant* population size N, obtain a model from Canning's class.

Easy, but is there more?

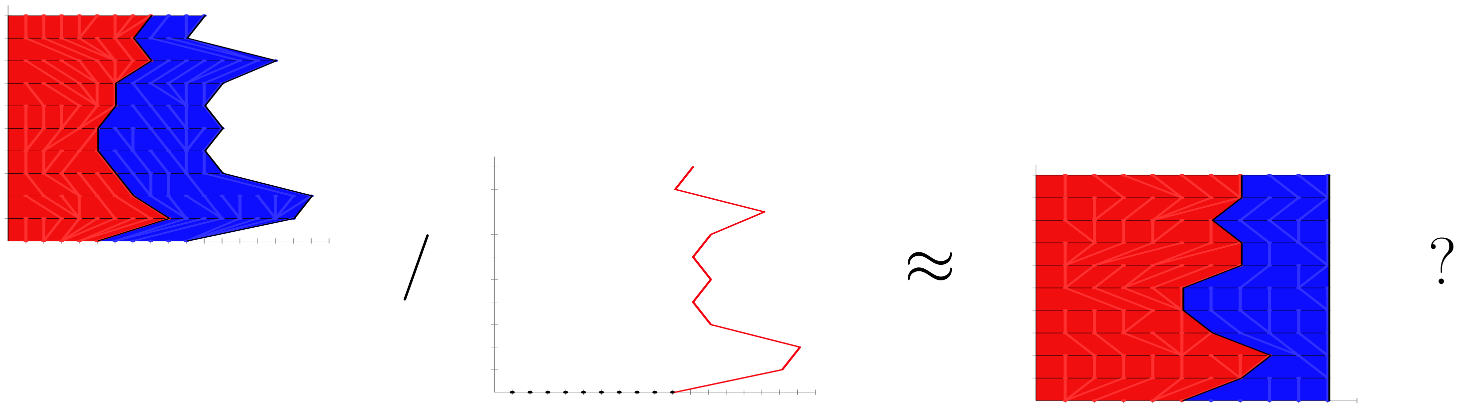
Relations?



In Canning’s model, we think of *relative* frequencies of types.

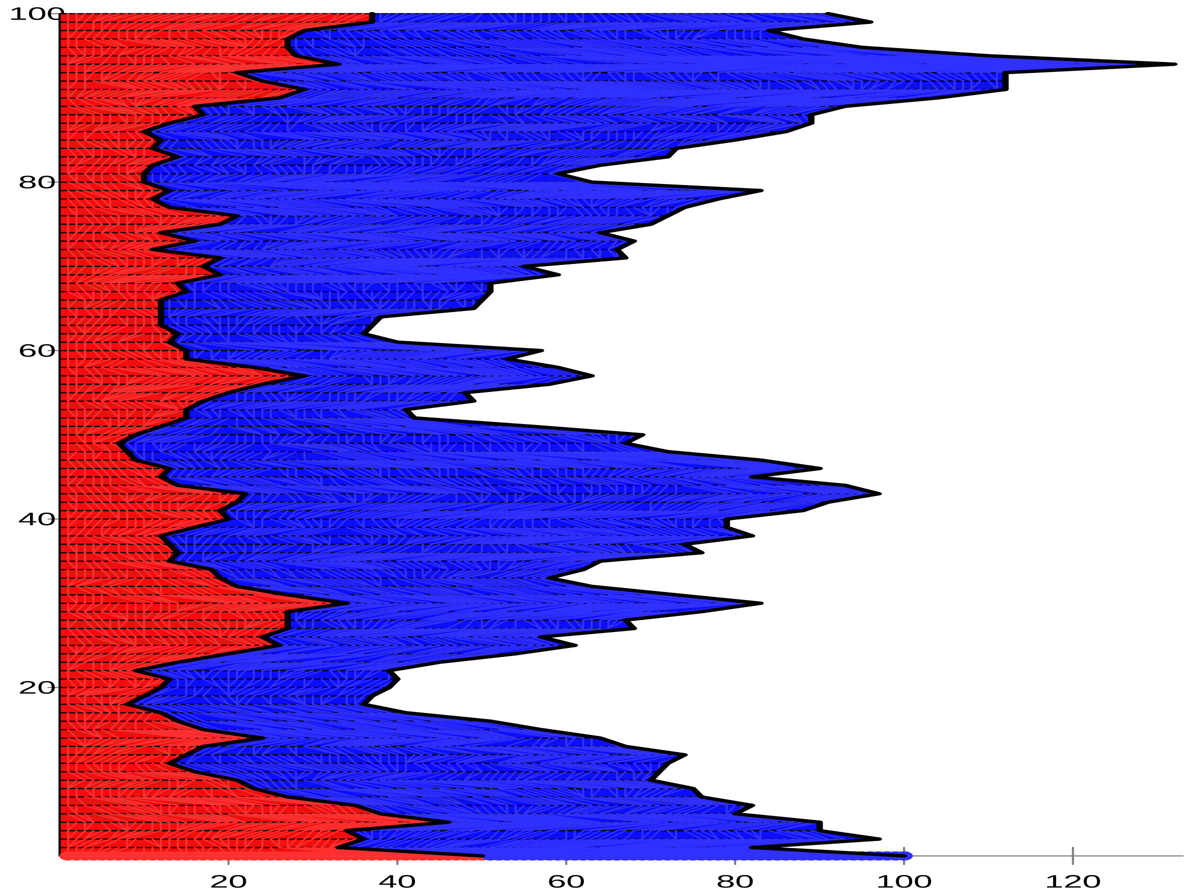
We can do the same in a free branching model by *normalising* with the current total population size (at least before extinction), i.e. considering

$$\tilde{\xi}_t(n) := \frac{\xi_t(n)}{\xi_t(N)}.$$



Relations?

Yes, in a suitable limit of population size $N \to \infty$.



Continuous state branching processes (Jiřina, Silverstein, Lamperti, ...)

$Z^{(N)} = (Z_k^{(N)})_{k=0,1,2,\dots}$, $N \in \mathbb{N}$ a sequence of Galton-Watson processes (possibly with offspring distributions depending on N),

$m_N \to 0$ mass rescaling,

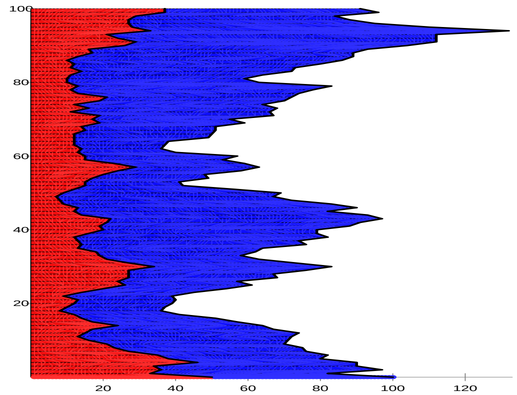
$Z_0^{(N)} = [m_N^{-1}]$

(+ conditions ...).

$$\left(m_N Z_{[Nt]}^{(N)}\right)_{t \geq 0} \Longrightarrow X,$$

where X is a *continuous state branching process*, i.e. an $\mathbb{R}_+$-valued Markov process that enjoys the *branching property*:

$$X, X', X'' \text{ independent copies, } X_0 = 0, X_0' = x', X_0'' = x'', \; x = x' + x''$$
$$\Longrightarrow X \overset{\text{law}}{=} X' + X''.$$



Example Assume the N-th offspring distribution has

▷ mean = $1 + \mu/N + o(1/N)$,

▷ variance = $\sigma^2 + o(1)$

▷ (and, say, uniformly bounded third moment).

$Z_0^{(N)} = N$, then $\left(N^{-1} Z_{[Nt]}^{(N)}\right)_{t \geq 0} \Rightarrow$ Feller's branching diffusion, generator

$$\mathscr{L}_{(2)} f(z) = \frac{1}{2}\sigma^2 z f''(z) + \mu z f'(z).$$

(i.e. $\mathrm{Var}[Z_{t+\Delta t} - Z_t | Z_t] \approx \sigma^2 Z_t \Delta t$, $\mathrm{E}[Z_{t+\Delta t} - Z_t | Z_t] \approx \mu Z_t \Delta t$.)

If the approximating offspring distributions are in the domain of attraction of a *stable law* of index $\alpha \in (0, 2)$, i.e.

$$\mathbb{P}(\text{more than } n \text{ children}) \sim \text{Const.} \times n^{-\alpha}$$

(note: in particular, no variance),
the limit process Z will have discontinuous paths. Generator

$$\mathscr{L}_{(\alpha)} f(z) = c_\alpha z f'(z) + z \int_{(0,\infty)} \{f(z+h) - f(z) - h\mathbf{1}_{(0,1]}(h) f'(z)\} h^{-1-\alpha} dh.$$

Interpretation:
if the present mass is z, a new litter of mass h is produced at rate $z \times h^{-1-\alpha}$.

Stable CSBPes, some properties

Why the name?

- ▷ Lamperti's construction connects them with stable Lévy processes:

$$X_t = Y\big(\int_0^t X_s\, ds\big),$$

where Y is a stable process without negative jumps, stopped upon hitting 0.

- ▷ Scaling properties
- ▷ Equation for Laplace transforms

Existence of mean: $\alpha > 1 \Rightarrow \mathbf{E}_x X_t < \infty$,
$\alpha < 1 \Rightarrow \mathbf{E}_x X_t = \infty$ for all $t > 0$.

Extinction/Explosion: $\alpha < 1 \Rightarrow X$ has growing paths, explodes in finite time,
$\alpha > 1 \Rightarrow X$ becomes extinct in finite time.

Z a given CSBP. The corresponding *Dawson-Watanabe superprocess** is a process X with values in the measures on $[0,1]$ such that

for $B \subset [0,1]$: $\quad (X_t(B))_{t\geq 0} \overset{\text{law}}{=} Z$, started from $Z_0 = |B|$,

$X_\cdot(B_1), X_\cdot(B_2), \ldots, X_\cdot(B_n)$ are independent if $B_1, \ldots B_n$ are disjoint.

Interpretation: $X_t(B)$ = mass of descendants alive at time t whose ancestors where $\in B$

**Note: this is a "toy version".*

$(\Gamma_t)_{t \geq 0}$ with values in the equivalence relations on $\{1, 2, \dots, \}$, exchangeable.

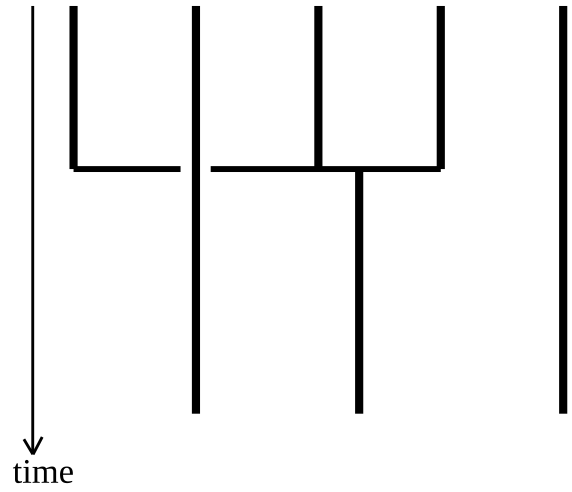
Possible interpretation: $i \sim_{\Gamma_t} j$ iff individuals i and j have a common ancestor at most t time units ago.

For restriction to any $\{1, \dots, N\}$: If there are presently p classes,

at rate $$\beta^{\Lambda}_{p,j} = \int_{[0,1]} x^{j-2}(1-x)^{p-j}\Lambda(dx)$$

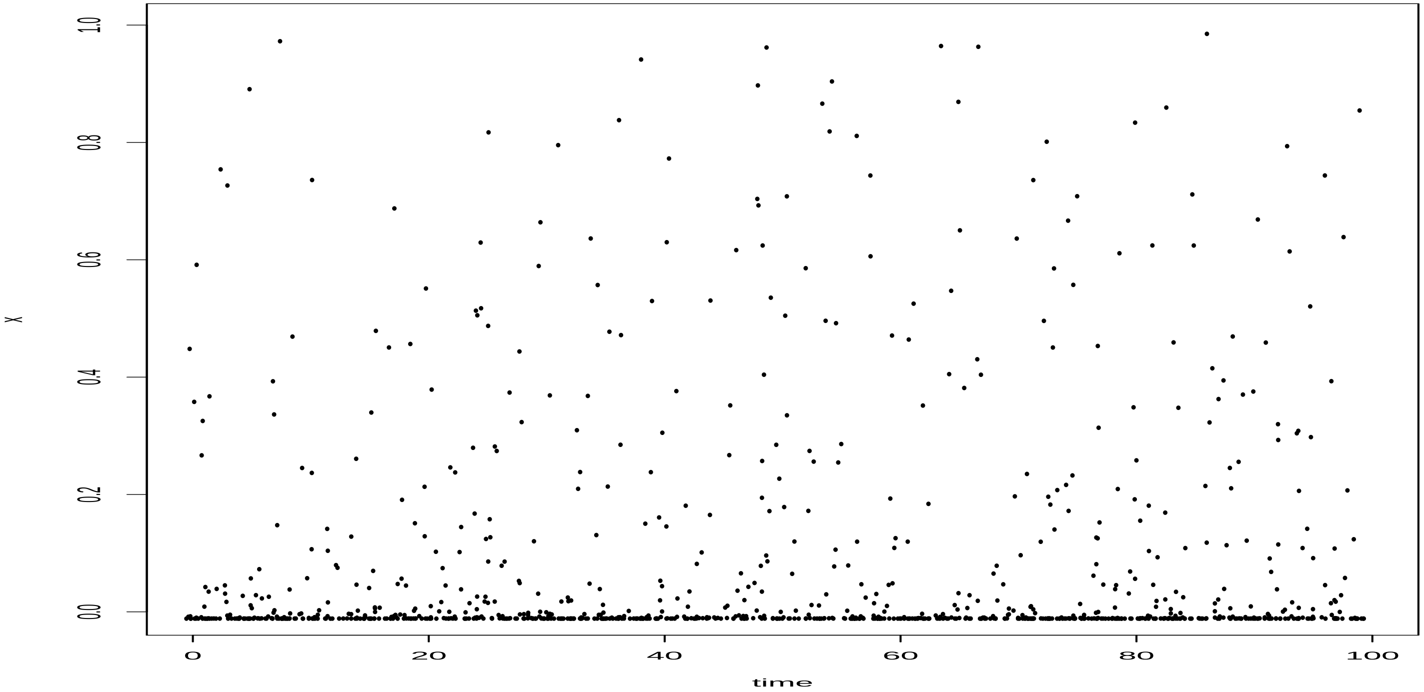
any given j-tuple of classes merges to one ($j \geq 2$).

Λ is a finite measure on $[0, 1]$.



Note: $\Lambda = \delta_0$ corresponds to Kingman's coalescent,
if $\Lambda((0, 1]) > 0$, there will be *multiple mergers*.

Interpretation of “non-Kingman” part:
$\mathcal{N}$ a Poisson point process on $[0,\infty)\times(0,1]$ with intensity measure $dt\otimes x^{-2}\Lambda(dx)$.



For $(t,x)\in\mathcal{N}$: at time t each class throws a coin with success probability x. All the successful classes merge to one.

$(\rho_t)_{t\geq 0}$ a Markov process with values in the probability measures on $[0,1]$.

Interpretation (if ρ_0 = uniform measure):
$\rho_t(B)$ = fraction of mass alive at time t whose ancestors at time 0 where in $B \subset [0,1]$.

On test functions of the form $F(\rho) = \int \dots \int \rho(dx_1)\dots\rho(dx_p) f(x_1,\dots,x_p)$, the generator acts as

$$\mathscr{L}_{FV,\Lambda} F(\rho) = \sum_{\substack{J\subset\{1,\dots,p\},\\ |J|\geq 2}} \int \dots \int \rho(dx_1)\dots\rho(dx_p)\beta^{\Lambda}_{p,|J|}\big(f(\underset{\substack{\uparrow \\ a_i^J = a_{\min J} \text{ if } i\in J}}{a_1^J,\dots,a_p^J}) - f(a_1,\dots,a_p)\big).$$

Interpretation: y-jumps come with intensity $y^{-2}\Lambda(dy)$. At the instant of a y-jump, an individual X is chosen according to the current ρ, and

$$\rho \longrightarrow y\delta_X + (1-y)\rho.$$

Note: alternative form of generator $\mathscr{L}_{FV,\lambda} G(\rho) = \int y^{-2}\Lambda(dy) \int \rho(dx)\big(G(y\delta_x + (1-y)\rho) - G(\rho)\big)$

Duality between coalescent and FV (Bertoin & Le Gall)

For a partition γ of $\{1,\dots,p\}$ with $|\gamma|$ classes and a probability measure ρ on $[0,1]$ consider test functions of the type

$$\Phi_f(\gamma,\rho) := \int \rho(dx_1)\dots\rho(dx_{|\gamma|})\, f\left(\vec{y}(\gamma; x_1,\dots,x_{|\gamma|})\right)$$

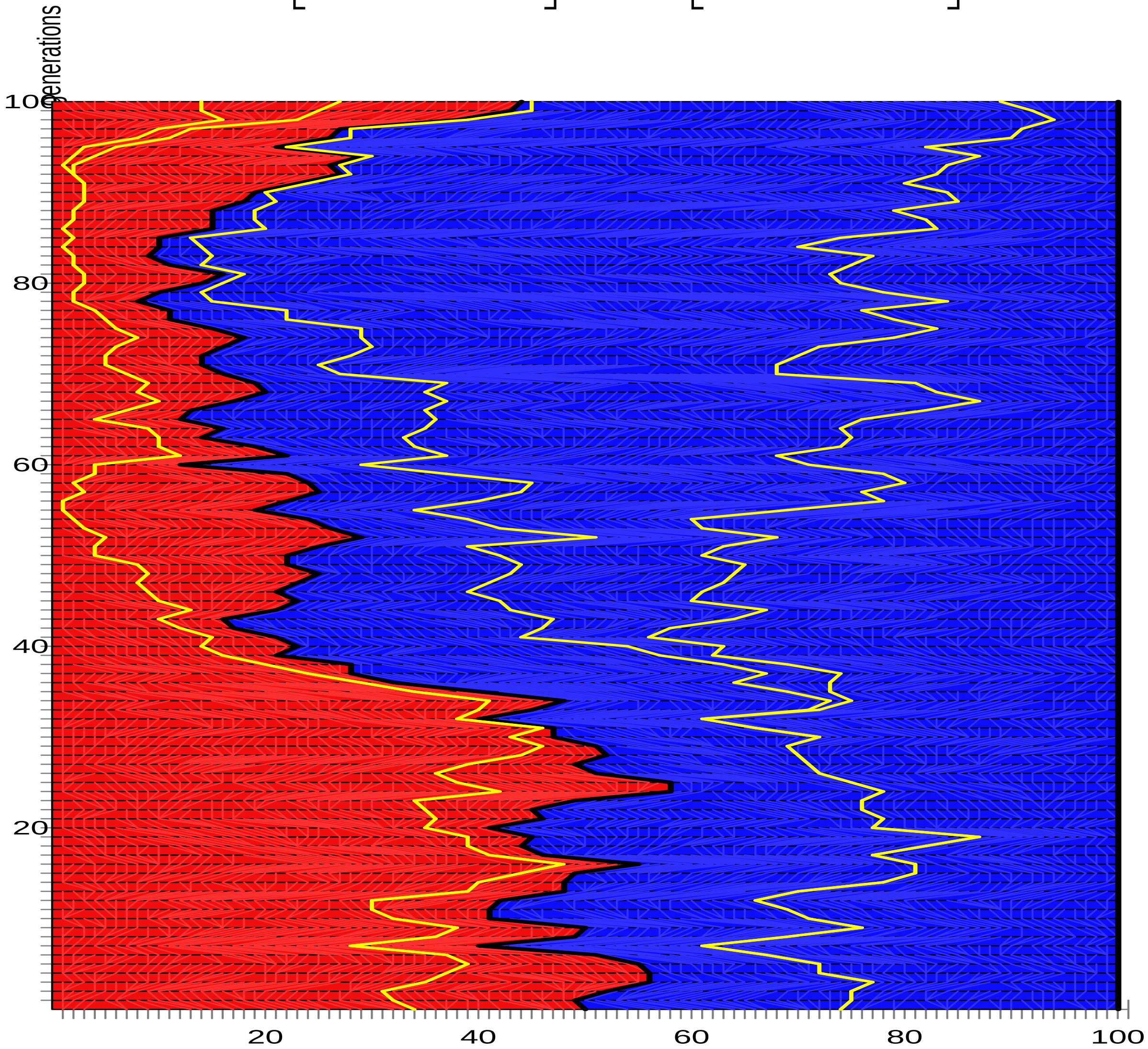
$y_j = x_i$ if $j \in i$-th class of γ

("assign an independent ρ-sample to each class of γ"), where $f:[0,1]^p \to \mathbb{R}$.
Then we have with
$(\Gamma_t^{(p)})$ .... restriction of Λ-coalescent to $\{1,\dots,p\}$,
(ρ_t) .... Λ-FV process

$$\mathbb{E}\left[\Phi_f(\Gamma_0^{(p)},\rho_t)\right] = \mathbb{E}\left[\Phi_f(\Gamma_t^{(p)},\rho_0)\right] \quad \text{for all } t \geq 0.$$

Sample interpretation of the duality

$$\mathbb{E}\left[\Phi_f(\Gamma_0^{(p)}, \rho_t)\right] = \mathbb{E}\left[\Phi_f(\Gamma_t^{(p)}, \rho_0)\right]$$



Let X be a finite variance DW process (on $[0,1]$), i.e. $\alpha = 2$. Then $X([0,a))$ and $X([a,1])$ are independent Feller diffusions.

Well known:

$$\left(\frac{X_t([a,1])}{X_t([0,1])}\right)_{t\geq 0} = \left(\frac{X_t([a,1])}{X_t([0,a)) + X_t([a,1])}\right)_{t\geq 0}$$

is a time changed Wright-Fisher diffusion

Main result

Let (X_t) be a stable DW process on $[0,1]$ (with trivial motion) of index $\alpha \in (0,2]$, τ = time of explosion or extinction of X. For $t < \tau$ put

$$R_t(B) := \frac{X_t(B)}{X_t([0,1])}, \quad B \subset [0,1].$$

Consider the additive functional

$$A_t := \int_0^t \frac{1}{(X_s([0,1]))^{\alpha-1}} \, ds, \qquad t < \tau$$

with inverse $\varphi(s) := A^{-1}(s)$.

Theorem *The time changed normalised process*

$$\tilde{R}_s := R_{\varphi(s)}, \quad 0 \le s \le A_{\tau-}$$

is a generalised Fleming-Viot process, dual to the $\mathrm{Beta}(2-\alpha,\alpha)$*-coalescent.*

Note: this yields a “skew product” representation for the DW process:

$$X_t(\cdot) = X_t([0,1]) R_t(\cdot),$$

where the second component is a time-changed generalised FV process.

The time changed normalised process $\tilde{R}_s := R_{\varphi(s)}$ is dual to the $\mathrm{Beta}(2-\alpha,\alpha)$-*coalescent.*

Remarks

- ▷ There is no such skew product representation for general, non-stable DW processes.
- ▷ (Neutral) types & mutation: pass from $[0,1]$ to $[0,1]\times E$, where E is the *type space*, mutation would correspond to a Markov motion on E for the "individual masses".
- ▷ The case $\alpha = 2$ is Perkin's disintegration theorem.
- ▷ The case $\alpha = 1$ corresponds to the well known relation between Neveu's branching process and the Bolthausen-Sznitman coalescent (Bertoin & Le Gall, 2000). Note that there is no time-change in this case.
- ▷ The result suggests a relation between $R^*_{k,n}$, the genealogy connecting a k-sample from generation n in a Galton-Watson process where the individual offspring distribution has tails $\sim x^{-\alpha}$, and that generated by the $\mathrm{Beta}(2-\alpha,\alpha)$-coalescent.

Sketch of proof ($\alpha < 2$)

For a finite measure μ on $[0,1]$ ($\mu \neq 0$) denote $\rho := \mu/\mu([0,1])$.
For test functions of the form

$$F(\mu) = G(\rho) = \int \rho(dx_1) \dots \int \rho(dx_p) \prod_{i=1}^{p} \mathbf{1}_{(a_i,b_i]}(x_i)$$

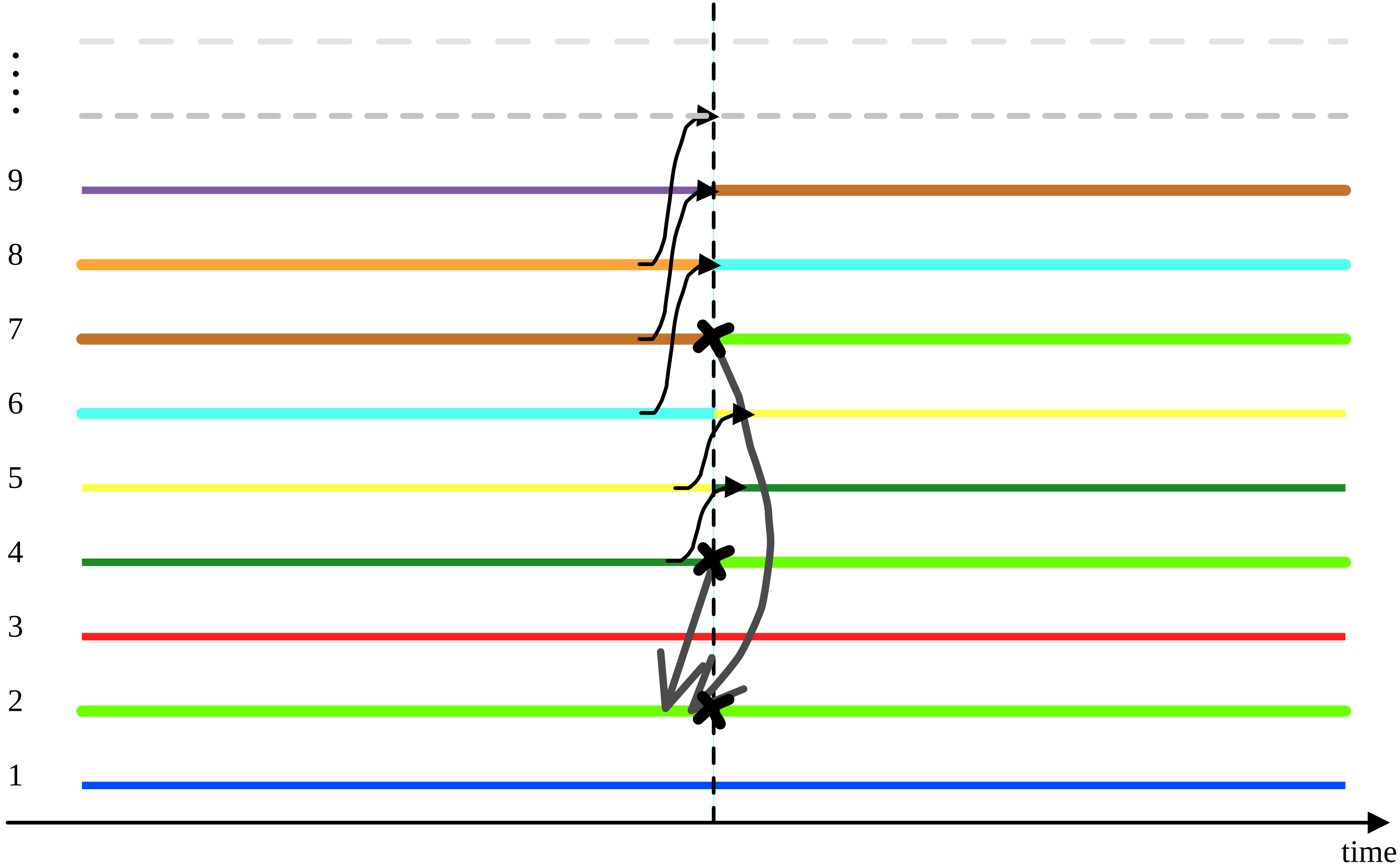
one checks that the generator of the α-stable DW process acts as

$$\begin{aligned} \mathscr{L}_{DW,\alpha} F(\mu) &= \int \mu(dx) \int_{(0,\infty)} dh\, h^{-1-\alpha} \left(G\big(\frac{\mu + h\delta_x}{\mu([0,1]) + h}\big) - G(\rho) \right) \\ &= C_\alpha\, \mu([0,1])^{1-\alpha} \int \rho(dx) \int_{[0,1]} dr\, r^{2-\alpha-1} (1-r)^{\alpha-1} \times \\ &\qquad \frac{1}{r^2} \big(G(r\delta_x + (1-r)\rho) - G(\rho) \big) \end{aligned}$$

remove via time-change

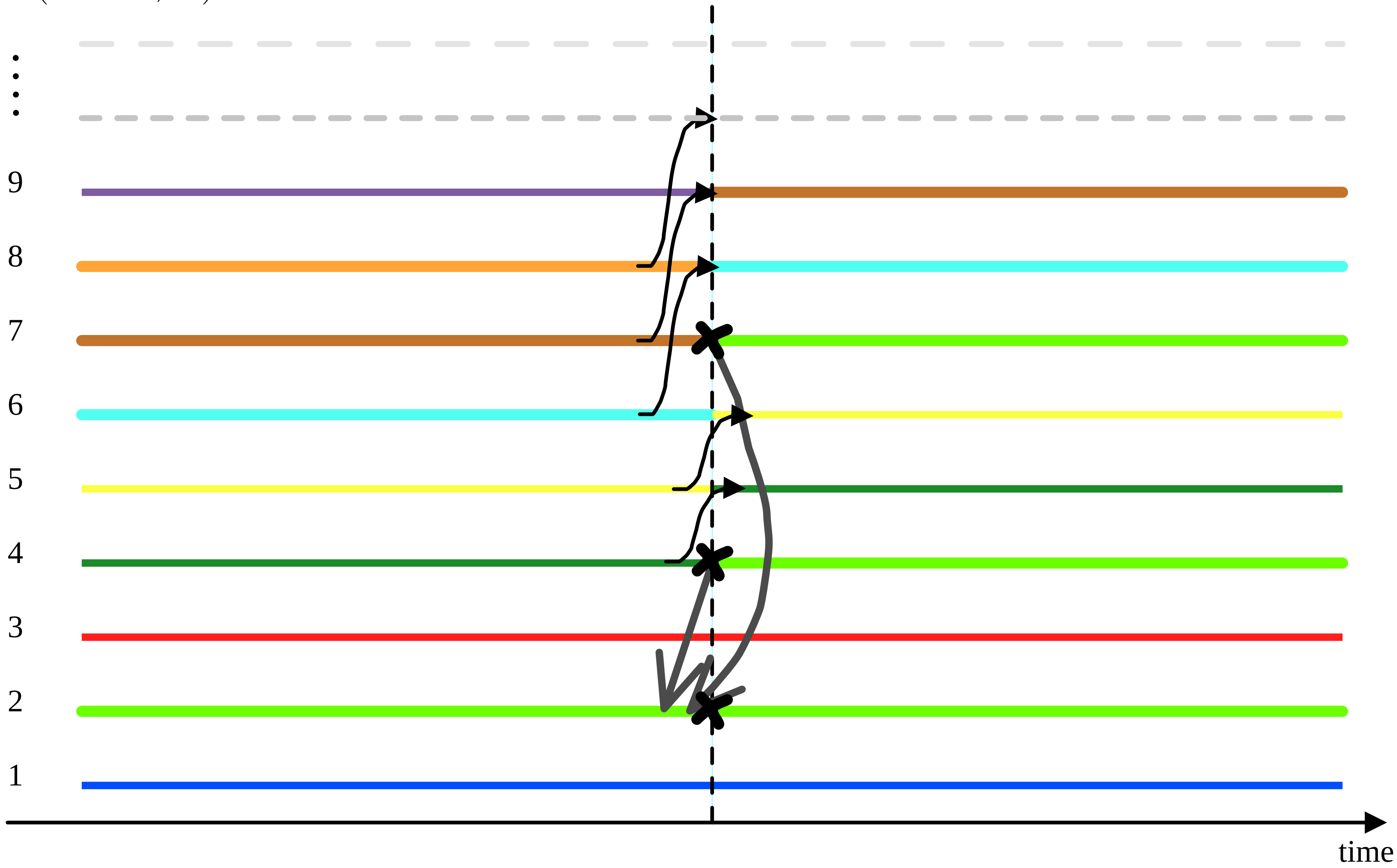
The "modified look down" point of view (Donnelly & Kurtz, 1999)

(Z_t) a stable CSBP, $\alpha < 2$. At the jump times of Z, each level throws a coin with success probability $\Delta Z_t/(\Delta Z_t + Z_{t-})$. All "successful" levels participate in this look down event, and look down to the smallest successful level.



The "modified look down" point of view (Donnelly & Kurtz, 1999)

This is a clever (alternative) way of embedding a genealogy into a CSBP. Moreover, run "backwards" and appropriately time-changed, it yields a $\mathrm{Beta}(2-\alpha,\alpha)$-coalescent.



Conclusion?

- ▷ Free branching and resampling models can generate similar (neutral) genealogies.

- ▷ When will one see multiple mergers in a genealogy?

  – Under a selective sweep,

  – a bottleneck,

  – in a neutral, not exogenously controlled scenario?

- ▷ Result suggests $\mathrm{Beta}(2-\alpha,\alpha)$-coalescents as an interesting subclass of all Λ-coalescents.