THE p LAPLACIAN EIGENVALUE PROBLEM

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ABSTRACT

Generalized sine functions S_p, $1 < p < \infty$, were studied by Á. Elbert in [1] as Dirichlet eigenfunctions for the one-dimensional p-Laplacian equation

$$-\left([y']^{p-1}\right)' = (p-1)\lambda [y]^{p-1}, \qquad 0 \le x \le \pi_p, \tag{1}$$

where, by definition, $[z]^{p-1} = z|z|^{p-2}$. The number

$$\pi_p := \frac{2\pi}{p \sin(\pi/p)}$$

is the first zero of S_p, which is the eigenfunction, normalized by $S_p'(0) = 1$, corresponding to the minimal eigenvalue $\lambda = 1$ of (1). Elbert also deduced the relation

$$|S_p|^p + |S_p'|^p = 1 \tag{2}$$

and considered the higher order eigenfunctions, which are given by $S_p(nx)$ with corresponding eigenvalues $\lambda = n^p$, $n = 2, 3, \ldots$. When $p = 2$, (1) corresponds to Fourier's equation, $S_2(x) = \sin(x)$, $\pi_2 = \pi$ and (2) is the Pythagorean relation.

In [2], Ôtani examined analogous functions $\sin_p$ with the factor $(p-1)$ on the right side of (1) replaced by 1. Lindqvist, [3], has studied the $\sin_p$ functions in some detail, noting that hyperbolic versions were introduced as early as 1879. We remark that the S_p and $\sin_p$ functions have become standard tools in the analysis of more complicated equations, with various applications. While the S_p and $\sin_p$ functions can easily be transformed into each other, it turns out that (2) must be modified when S_p is replaced by $\sin_p$.

Despite this activity, it seems that analogues for $1 < p \neq 2$ of the standard completeness and expansion theorems for sine functions have not been discussed previously. Define

$$f_n(t) := S_p(n\pi_p t), \qquad n = 1, 2, \ldots.$$

These functions depend on p, and in the case $p = 2$ they become

$$e_n(t) := \sin(n\pi t), \qquad n = 1, 2, \ldots,$$

which are proportional to a standard orthonormal basis of the Hilbert space $L2(0, 1)$.

Recall that $\{f_n\}$ is a Schauder basis of $L2(0, 1)$, if for any $f \in L2(0, 1)$, there exist unique coefficients c_n, depending continuously on f, so that

$$\Big\| \sum_{n=1}^{N} c_n f_n - f \Big\|_2 \to 0$$

as $N \to \infty$.

In this talk, we establish the following

THEOREM *For* $\frac{12}{11} \leq p < \infty$, *the family* $\{f_n\}_{n=1}^{\infty}$ *forms a Schauder basis of* $L2(0,1)$.

Our main device in the proof of this theorem is a linear mapping T of the space $L2(0,1)$, satisfying $Te_n = f_n$, and decomposing into a linear combination of certain isometries.

REFERENCES

- [1] Á. Elbert. "A half-linear second order differential equation". *Coll. Math. Soc. J. Bolyai* 30 (1979) 153 – 179.
- [2] M. Otani. "A Remark on certain nonlinear elliptic equations", *Proc. Fac. Sci. Tokai Univ.* 19 (1984), 23 – 28.
- [3] P. Lindqvist. "Some remarkable sine and cosine functions", *Ricerche di Matematica* 44 (1995), 269 – 290.