Directions in Combinatorial Matrix Theory

BIRS May 7,8 2004

There are 2 discussion/open problem sessions scheduled with 4 leaders. To help them arrange these informal sessions, please send or give your mathematical problems and points for discussion to Steve Kirkland (kirkland@math.uregina.ca). In the past, sessions similar to these have been an excellent feature of WCLAM meetings.

Proceedings of the conference will be published in a special volume of ELA, regularly refereed. Thanks to the Editorial Board of ELA. The special editors for this volume are: Steve Kirkland, Bryan Shader and Pauline van den Driessche. Please send your manuscript to one of these editors as soon as possible. It will be very good to have a timely collection of papers together with the theme of combinatorial matrix analysis.

We thank ILAS, BIRS, PIMS for their support, and Ariana Clapton (UVIC) for superb help with putting together the program.

Organising Committee:

- Shaun Fallat (University of Regina)
- Hadi Kharaghani (University of Lethbridge)
- Steve Kirkland (University of Regina)
- Bryan Shader (University of Wyoming)
- Michael Tsatsomeros (Washington State University)
- Pauline van den Driessche (University of Victoria).

Agenda

Friday, May 7th

9:00	C.R. Johnson	“Patterns of Commutativity”
9:30	W. So	“Integral Regular Graphs with Structures”
10:00	K. Vander Meulen	“On Biclique Partitions and Related Matrices”
10:30		**BREAK**
11:00	M. Fiedler	“Matrices and Graphs in Euclidean Geometry”
11:30		ILAS Sponsored Lecturer
12:00		**LUNCH**
12:30		**LUNCH**
13:00		**LUNCH**
13:30	D. Olesky	“Minimal Spectrally Arbitrary Sign Patterns”
14:00	J. McDonald	“The Peripheral Spectrum of a Nonnegative Matrix”
14:30	Z. Li	“On Minimum Ranks of Sign Pattern Matrices”
15:00	T. Britz	“Matrix Inversion and Digraphs: The One Factor Case”
15:30		**BREAK**
16:00	R. Craigen	“Developments in Orthogonal Matrices”
16:30	F. Barioli	“On the Minimal Rank and Path Cover Number for Certain Undirected Graphs”
17:00	A. Pothen	“Combinatorial Scientific Computing and Combinatorial Matrix Theory: The Synergy”
17:30	F. Chung	“Laplacians for Directed Graphs and Non-Reversible Markov Chains”
18:00		**DINNER**
19:00		**DINNER**
20:00		Discussion/Open Problems
21:00		- Led by Chi-Kwong Li and Steve Kirkland
22:00		

Saturday, May 8th

9:00	A. Berman	“Symmetric Nonnegative Factorizations”
9:30	J. Stuart	“Ray Nonsingularity and Ray Determinants”
10:00	L. Hogben	“Spectral Graph Theory and the Inverse Eigenvalue Problem of a Graph”
10:30		**BREAK**
11:00	S. Hedayat	“Crossover Designs for Medical and Pharmaceutical Studies”
11:30	R. Loewy	“The Minimum Rank of a Graph”
12:00		**LUNCH**
12:30		**LUNCH**
13:00		**LUNCH**
13:30		Discussion/Open Problems
14:00		- Led by Rob Craigen and Michael Tsatsomeros
14:30		
15:00		
15:30		

On the Minimal Rank and Path Cover Number for Certain Undirected Graphs

Francesco Barioli

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For a given undirected graph G, the minimum rank of G is the smallest possible rank over all real symmetric matrices A whose associated graph is G. The path cover number of G is the minimum number of vertex-disjoint paths occurring as induced subgraphs of G that cover all the vertices of G. For trees, the relationship between minimum rank and path cover number is completely understood. However, for non-trees only sporadic results are known. In this talk we present formulae for the minimum rank and path cover number for graphs obtained from vertex-sums and edge-sums of simpler graphs.

This is joint work with S.M. Fallat and L. Hogben.

Symmetric Nonnegative Factorizations

Abraham Berman

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We will discuss the following questions:

1. Given a symmetric nonnegative matrix A, can it be factored as $A = BB^T$, where B is nonnegative?

2. If such factorizations are possible, what is the smallest possible number of columns of B?

3. Given a symmetric nonnegative integral matrix A, can it be factored as $A = BB^T$, where B is a $(0, 1)$ matrix?

Matrix Inversion and Digraphs: The One Factor Case

Thomas Britz

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University of Victoria, British Columbia, Canada

A nearly reducible matrix is a real square matrix that is irreducible but becomes reducible whenever any nonzero entry is replaced by zero. In terms of digraphs, a real square matrix is nearly reducible if and only if its digraph is minimally strongly connected. Nearly reducible matrices and their digraphs are well-known to have interesting structural properties.

We describe how the inversion of a nonsingular nearly reducible matrix may be characterized in simple terms using the associated digraph. An immediate consequence of this characterization is that a nonsingular nearly reducible matrix is strongly sign-nonsingular. Among other consequences is that the inverse of a nonsingular nearly reducible matrix is itself nearly reducible if and only if it is strongly sign-nonsingular if and only if the matrix and its inverse contain the same number of nonzero entries.

This is joint work with Dale Olesky and Pauline van den Driessche.

Laplacians for Directed Graphs and Non-Reversible Markov Chains

Fan Chung Graham

Department of Mathematics
University of California, San Diego, U.S.A.

We consider Laplacians for directed graphs and the Cheeger inequalities which relate the eigenvalues of the Laplacian to other invariants for directed graphs. These techniques can be used to deal with various problems arising in the study of non-reversible Markov chains.

Developments in Orthogonal Matrices

Robert Craigen

Department of Mathematics
University of Manitoba, Canada

Since Sylvester's groundbreaking paper on Inverse Orthogonal matrices in 1867 these matrices have been studied in many disguises, Hadamard, weighing matrices and their various generalizations in different rings, orthogonal designs, symmetric designs, and so on. These are not the "orthogonal" matrices discussed in most elementary texts on linear algebra, but $n \times n$ multiples of unitary matrices (or their analogues in some appropriate domain), usually normalized so that all row "norms" are equal to n.

These matrices and combinatorial questions regarding their structure and existence have caught the interest of many practical-minded people in coding theory, information theory, communications, optical masking and filtering, range-finding applications and so on. More recently they have arisen in quantum information theory and quantum learning, chaos theory and other emergent fields. Often workers in other fields undertake a comprehensive study of these objects, not realising that a rich literature on them already exists.

I begin by introducing various objects to demonstrate the breadth and richness of the field; I will survey some of the major questions in the field and what is known: existence, structure and classification; and discuss some recent powerful developments. If time permits I will outline a proposal for a uniform theory of orthogonal matrices and a unification of several different fields under this banner.

Matrices and Graphs in Euclidean Geometry

Miroslav Fiedler

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Gram matrices form a natural link between positive semidefinite matrices and systems of vectors in a Euclidean space. Therefore, many properties of these matrices are reflected by geometric relationships. Quite often, the corresponding geometric facts are difficult to find geometrically. This is in particular true about inequalities. An example follows:

Theorem 1 ([4], Thm. 4,1). *Let the vectors $u_1, \dots u_n, v_1, \dots, v_n$ form a pair of biorthogonal bases (i.e., the inner product of u_i and v_j is the Kronecker delta δ_{ij}) in a Euclidean n-space E_n. Then for the lengths,*

$$|u_i||v_i| \geq 1, \quad i = 1, \dots, n,$$

$$2\max_i(|u_i||v_i|-1) \le \sum_i(|u_i||v_i|-1).$$

Conversely, if nonnegative numbers $\alpha_1,\ldots,\alpha_n$, $\beta_1,\ldots,\beta_n$ *satisfy*

$$\alpha_i\beta_i \ge 1, \quad i=1,\ldots,n,$$

$$2\max_i(\alpha_i\beta_i-1) \le \sum_i(\alpha_i\beta_i-1),$$

then there exists in E_n *a pair of biorthogonal bases* u_i, v_j, *such that*

$$|u_i|=\alpha_i, |v_i|=\beta_i, \quad i=1,\ldots,n.$$

In matrix theory, various relatively new directions appeared which did not have analogies in Euclidean geometry, as say, completion problems, combinatorial matrix theory, Hadamard products, and others. Some rare exceptions in completion problems could be, however, mentioned.

As usual, an n-simplex in E_n (a generalization of the triangle in the plane, tetradehron in the three-dimensional space) is determined by its $n+1$ vertices, say, $A_1,\ldots,A_{n+1}$. It has edges A_iA_j, $(n-1)$-dimensional faces ω_i (opposite A_i), interior angles ϕ_{ij} (between ω_i and ω_j), etc. One can ask about necessary and sufficient conditions for the lengths of some set of edges that they are edges of some n-simplex. (For instance, strict triangle inequalities have to hold for every "closed" triple, and a strict "polygonal inequality" for every closed polygon.) Similarly as in matrix theory, graphs play here an important role.

Theorem 2 ([3], Thm. 3,1) *A set of lengths of edges can serve as the set of lengths of edges of an n-simplex if and only if it is the set of lengths of edges of such simplex all interior angles of which opposite to missing edges are right.*

Let us note that such simplex is -up to its position in the space- uniquely determined.

For completeness, let us mention an important -and well known in various modifications- condition for the set of all lengths of edges in an n-simplex:

Theorem 3 ([1]). *The numbers* e_{ij} *can serve as* squares *of the lengths of edges between* A_i *and* A_j, $e_{ii}=0$, *if and only if*

$$\sum_{i,j} e_{ij}x_ix_j < 0, \quad \textit{whenever} \quad \sum_i x_i = 0.$$

Remark. This is equivalent to the condition that the matrix

$$M_0 = \begin{pmatrix} 0 & e^T \\ e & M \end{pmatrix},$$

where e is the column vector of all ones and $M = (e_{ij})$ (called the Menger matrix in [5]) is elliptic, i.e. has one eigenvalue positive and the remaining negative. The inverse of the matrix M_0 is then a $(-\frac{1}{2})$- multiple of the matrix

$$Q_0 = \begin{pmatrix} q_{00} & q_0^T \\ q_0 & Q \end{pmatrix}, \tag{1}$$

where the matrix Q is the Gram matrix of the outer normals in the above mentioned n-simplex (in a certain way normalized in order that the sum of the normals be zero).

As is well known, there is an intimate relationship between the irreducibility of a symmetric matrix and the connectedness of the corresponding (undirected) graph:

Theorem 4. *Let $A = (a_{ik})$ be an $n \times n$ symmetric matrix, $G = (N, E)$ its graph, where $N = \{1, \ldots, n\}$and E the set of unordered pairs (i, k), $i \neq k$, for which $a_{ik} \neq 0$. Then A is irreducible (i.e. for no simultaneous permutation of rows and columns having the form $\begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}$ with non-void A_1 and A_2) if and only if the graph G is connected.*

A real matrix A can always be (uniquely) written as $A = A^+ + A^-$, where A^+ contains the positive entries of A, having zeros elsewhere, and A^- contains all negative entries of A, having zeros elsewhere. Let us call A^+ (respectively, A^-) the *positive* (respectively,*negative) part* of A. We have then:

Theorem 5. *Suppose A is a real symmetric positive semidefinite $n \times n$ matrix with rank $n-1$ and such that $Ae = 0$ (e as above). Then the negative part of A is irreducible. Conversely, if C is a nonpositive irreducible symmetric $n \times n$ matrix with zero diagonal entries, then there exists a real symmetric positive semidefinite matrix A with rank $n-1$, such that its negative part is C.*

This theorem implies the following:

Theorem 6. ([2]) *Let us color each edge A_iA_j of an n-simplex with vertices $A_1, \ldots, A_{n+1}$ by one of the following three colours:*

red, *if the opposite interior angle ϕ_{ij} is* **acute**;

blue, *if the opposite interior angle ϕ_{ij} is* **obtuse**;

white, *if the opposite interior angle ϕ_{ij} is* **right**.

Then, the set of red edges connects all the vertices of the simplex.

Conversely, if we color all edges of an n-simplex by three colors red, blue and white in such a way that the red edges connect all vertices, then there exists such deformation of the simplex that opposite red edges there are acute, opposite blue edges obtuse and opposite white edges right interior angles.

This result shows that every n-simplex has at least n acute interior angles, it also enables to define right n-simplices which have exactly n acute interior angles and all the remaining $\binom{n}{2}$ interior angles right, etc.

Let us conclude by expressing the opinion that the n-dimensional Euclidean geometry is a beautiful field for a matrix theorist or graph theorist for both testing results and finding inspiration.

References

[1] L. M. Blumenthal, Theory and Applications of Distance Geometry. Oxford, Clarendon Press, 1953.

[2] M. Fiedler, Über qualitative Winkeleigenschaften der Simplexe. *Czechoslovak Math. J.* 7(82) (1957), 463 - 478.

[3] M. Fiedler, Über zyklische n-Simplexe und konjugierte Raumvielecke. *CMUC* 2,2 (1961), 3 - 26.

[4] M. Fiedler, Relations between the diagonal elements of two mutually inverse positive definite matrices. *Czechoslovak Math. J.* 14(89) (1964), 39 - 51.

[5] M. Fiedler, A geometric approach to the Laplacian matrix of a graph. In: *Combinatorial and Graph-Theoretical Problems in Linear Algebra* (R.A. Brualdi, S. Friedland, V. Klee, Editors), Springer, New York 1993, 73-98.

Crossover Designs for Medical and Pharmaceutical Studies

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A typical crossover design for testing v treatments using n subjects in p periods consists of selecting n sequences out of all possible v^p sequences (repetition of sequences are allowed) and assigning one p-sequence of treatments to each subject under the study. Thus at the end of the study each subject provides p observations if there is no drop-outs in the study. Each such design can be represented by a p by n array with columns as p-sequences. Which n sequences should be selected depends on the statistical model for the data and the goal of the study. In this talk we tell the audience what has been done and what is needed to be done. There are many exciting mathematical problems which need solutions.

Spectral Graph Theory and the Inverse Eigenvalue Problem of a Graph

Leslie Hogben

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The pattern of nonzero entries of a real symmetric matrix A can be described by a graph, denoted $G(A)$. If G is a graph and A_G is the adjacency matrix of G, then $G(A_G) = G$, but many other matrices are also described by the graph G. Given a graph G, the problem of characterizing the possible spectra of A such that $G(A) = G$ has been referred to as the Inverse Eigenvalue Problem of a Graph (IEPG). It is a very hard problem and is far from solved, but significant progress has been made recently for trees.

Spectral graph theory is the study of the spectra of certain matrices defined from the graph, including the adjacency matrix. This field is better developed and a number of results have been obtained describing the spectra of adjacency matrices. Clearly any result obtained for IEPG applies to the adjacency matrix of the graph, but (especially in the case of trees) it is possible to extend some results from spectral graph theory to the IEPG.

This talk will survey recent results on IEPG, exploit the connections between these problems, and discuss avenues for future research.

Patterns of Commutativity

Charles Johnson

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College of William and Mary, Williamsburg, U.S.A.

Two n-by-n matrices A and B commute if $AB = BA$. Two n-by-n (sign) patterns commute if they allow a commuting pair of matrices. We give a brief introduction to techniques for dealing with the problem of determining if two (sign) patterns commute and then discuss two recent major results in this area, dealing with commutativity with the full pattern and the all plus sign pattern. Other results will be discussed.

On Minimum Ranks of Sign Pattern Matrices

Zhongshan Li

Department of Mathematics and Statistics
Georgia State University, U.S.A.

A sign pattern (matrix) is a matrix whose entries are from the set $\{+, -, 0\}$. By introducing the notions of weakly dependent sign pattern vectors and strongly independent sign pattern vectors, a new characterization of L-matrices is obtained. A nonnegative sign pattern matrix can also be viewed as a Boolean matrix, by replacing each $+$ entry with 1. For a nonnegative sign pattern, some interesting connections between the minimum rank and the Boolean row (or column) rank are established. Various bounds for the minimum ranks of tree sign patterns are presented. Some other results on the minimum ranks of sign pattern matrices are surveyed and a number of open problems on the minimum ranks of sign patterns are discussed.

The Minimum Rank of a Graph

Raphael Loewy

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Let $G = (V, E)$ be a simple, undirected graph on n vertices, and let $S(G)$ denote the set of all real, symmetric $n \times n$ matrices whose graph is G. Thus, for $A \in S(G)$ $a_{ij} \neq 0$ if and only if $ij \in E$. There is no restriction on the main diagonal entries of A.

The *minimum rank of* G, $mr(G)$, is the minimum rank of all matrices in $S(G)$. It is known that $mr(G) = n - 1$ if and only if G is a path. We consider graphs G with $mr(G) = n - 2$. The analysis of graphs G with $mr(G) = n - 2$ depends on a family of graphs called *2-trees* and their subgraphs.

This talk is based on a joint work with C. R. Johnson and P. Smith.

The Peripheral Spectrum of a Nonnegative Matrix

Judith J. McDonald

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Washington State University, U.S.A.

From the Perron-Frobenius Theorem, we know that the peripheral spectrum of a nonnegative irreducible matrix consists precisely of the spectral radius multiplied by a complete set of roots of unity. Each of the corresponding Jordan blocks is therefore 1×1. In this talk, we will look at necessary and sufficient conditions on the Jordan blocks associated with the peripheral spectrum of a reducible nonnegative matrix.

Minimal Spectrally Arbitrary Sign Patterns

Dale Olesky

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An $n \times n$ sign pattern $\mathcal{A}$ is spectrally arbitrary if, given any self-conjugate spectrum, there exists a matrix realization of $\mathcal{A}$ with that spectrum. If the replacement of any nonzero entry (or entries) of $\mathcal{A}$ by zero gives a sign pattern without this property, then $\mathcal{A}$ is a minimal spectrally arbitrary sign pattern. For $n \geq 3$, several families of $n \times n$ spectrally arbitrary sign patterns are presented, and their minimal spectrally arbitrary subpatterns are identified. These are the first known families of $n \times n$ minimal spectrally arbitrary sign patterns. Furthermore, all such 3×3 sign patterns are determined and it is proved that any irreducible $n \times n$ spectrally arbitrary sign pattern must have at least $2n - 1$ nonzero entries, and it is conjectured that the minimum number of nonzero entries is $2n$.

This is joint work with Thomas Britz, Judi McDonald and Pauline van den Driessche.

Combinatorial Scientific Computing and Combinatorial Matrix Theory: The Synergy

Alex Pothen

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Old Dominion University, Norfolk VA, USA

Researchers who are working on combinatorial problems in scientific computing have recently organized as a research community under the banner of combinatorial scientific computing (CSC). There are significant opportunities for fruitful interaction between the CSC and combinatorial matrix theory communities. I will point out a few of these research areas, and then talk about graph coloring models for computing Jacobian and Hessian matrices when solving nonlinear problems in optimization.

Ten variant graph coloring problems occur in this context depending on the computational scenario. A unified algorithmic approach is developed to solve the variant problems, and new graph coloring formulations and algorithms are proposed for the problems.

This is joint work with Assefaw Gebremedhin (ODU), Fredrik Manne (Bergen, Norway).

Integral Regular Graphs with Structures

Wasin So

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The problem "Which graphs have integral spectra?" appears intractable. By restricting to regular graphs, the problem becomes managable. Indeed, there are finitely many integral connected r-regular graphs, for a given r. However, a complete description of them is difficult due to a lack of structures. In this talk, several classes of integral regular graphs with structures are discussed, and their characterizations are given. The study of integral graphs is beautiful because it brings together graph theory, matrix theory, number theory, group theory, and more.

Ray Nonsingularity and Ray Determinants

Jeffrey Stuart

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It is well known that a sign pattern is nonsingular if and only if the determinant of the sign pattern is unambiguously signed. That is, if A is an $n \times n$ matrix with entries in $\{-1, 0, 1\}$, and if $Q(A)$ is the set of all $n \times n$ real matrices of the form $A \circ X$ where $\circ$ denotes the Hadamard product and where X is entrywise positive, then every matrix in $Q(A)$ is invertible if and only if the determinant of every matrix in $Q(A)$ has the same sign.

We investigate the analogous issue for ray pattern matrices. Let A be an $n \times n$ matrix with entries in $\{z \in \mathbf{C} : |\mathbf{x}| = \mathbf{1}\} \cup \{0\}$. The ray pattern $Q(A)$ is the set of all $n \times n$ complex matrices of the form $A \circ X$ where X is entrywise positive. It is easy to show that if the determinant of every matrix in $Q(A)$ lies on the same open ray, then every matrix in $Q(A)$ is nonsingular. However, unlike the sign pattern case, it is possible that every matrix in a ray pattern is nonsingular but the determinant of the pattern is ambiguous, forming a cone of rays minus the origin. We construct a variety of ray patterns that have unambiguous ray pattern determinants, and we present results on invertible ray patterns that have ambiguous ray pattern determinants.

On Biclique Partitions and Related Matrices

Kevin Vander Meulen

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We review some results on the partitions of the edge set of a graph by bicliques (complete bipartite subgraphs.) If $b(G)$ is the minimum number of bicliques in a partition of the edge set of G, it is well-known that $b(G) \geq h(G)$ where $h(G)$ is maximum of the number of positive and the number of negative eigenvalues of G. Characterizing the inertia of a rank one perturbation of a Hermitian matrix allows us to determine sharp bounds on $b(G)$ for various graphs, especially joins of graphs. Further, we illustrate known and conjectured values of $b(\mu K_n)$ briefly noting some of the connections to Hadamard matrices and other combinatorial structures. Open problems on minimum biclique partitions will be presented.