# Maximal functions in non-commutative analysis

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In recent years we have seen a very deep connection between the recent theory of operator spaces and quantum probability. This leads to fascinating new ideas and much work to be done. The two organizers are partially involved in this new development.

In our first project we discussed the application of these ideas to noncommutative ergodic theory and maximal functions. One of the main results there is the noncommutative maximal ergodic inequality, which is the noncommutative version of the classical Dunford-Schwarcz maximal ergodic inequality. This solves an old (maybe a main) open problem in noncommutative ergodic theory. This maximal inequality is closely related to as well as inspired by the noncommutative Doob maximal inequality for martingales, recently established by one of the organizers. Another main result of this work is a noncommutative version of Stein's maximal ergodic inequality for symmetric positive Markovian semigroups. As an application, we obtain a maximal inequality for the Poisson semigroup of a free group and a version of the almost everywhere radial convergence for this semigroup. This is the free group analogue of the classical results on the Poisson semigroup on the torus. Some work had been done before the meeting at Banff. However, there were many details that required clarification, discussion and improvement. Let us mention, for example, the notion of almost everywhere convergence. There is a vast literature in the noncommutative setting with at least seven different notions. In our final result on convergence of ergodic averages we could provide a functional analytic formulation which implies all reasonable forms of convergence discussed in the literature before. Finally this project (a paper of approximately 50+ pages) is ready for closure.

The second work achieved during our stay deals with the noncommutative Rosenthal inequalities. This is a continuation of our previous work on the noncommutative Burkholder inequality for martingales. We obtained several noncommutative versions of Rosenthal's inequality on independent mean zero random variables. The applications of this family of inequalities ranges from $\ell_p$-norms estimates of the singular values for random matrices, to operator-valued versions in free probability and $L_p$-estimates in classical Araki-Wood factors obtained from quantum mechanics. Some of these estimates were developed for the understanding of operator space analogue of the work of Rosenthal on the structure theory commutative $L_p$ spaces. For example we applied these results to the study of symmetric subspaces of noncommutative $L_p$-spaces. Let us mention one of our applications. Let $M$ be a von Neumann algebra and $L_p(M)$ be the associated noncommutative $L_p$-spaces. Assume $2 < p < \infty$. Let $X \subset L_p(M)$ be a symmetric subspace. Then $X$ is isomorphic either to $\ell_p$ or to $\ell_2$. This is a result in the category of Banach spaces. Its counterpart for operator spaces reads as follows: If $X$ has a completely symmetric basis, then $X$ is completely isomorphic to one of the four spaces: $\ell_p$, $C_p$, $R_p$ and $C_p \cap R_p$, where $C_p$ and $R_p$ are respectively the $p$-column and $p$-row space.

The collaboration in the third project on operator space techniques and quantum probability was motivated by the fast current development in the field. Parallel with the work of Pisier and Shlayhtenko, one organizer started to investigate the connection between operator spaces and free probability in type III von Neumann algebras. A similar theory can also be developed for the classical Araki-Woods factors known from quantum mechanics. We developed the material for a book (or memoirs) which should illustrate these new area of research and techniques. This also required the review the material known to either one of the participants. Indeed, due to the extensive exchange in Banff we avoided huge amount of overlap between the two participants. We believe that we have gained a much better and deeper understanding of the very recent works on the operator space Grothendieck inequalities, the embedding of Pisier's operator Hilbertian space $OH$ into a noncommutative $L_1$ spaces (predual of a von Neumann algebra) by Pisier-Shlyakhtenko and the organizers. We have obtained several applications of these representations. In particular, we proved that the class of completely 1-summing maps on $OH$ coincides with the Orlicz-Schatten class $S_\Phi$, where $\Phi$ is the Orlicz function $\Phi(x) = x^2|\ln x|$ $(x > 0)$. This result is in strong contrast with the corresponding Banach space result which asserts that the class of 1-summing maps on $\ell_2$ is the class of Hilbert-Schmidt operators. Note that the logarithmic factor in $\Phi$ is at the origin of the same logarithmic factor in the operator space little Grothendieck inequality and the projection constant of $OH_n$.

# References


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