# $p$-adic variation of motives

Kevin Buzzard (Imperial College),
Robert Coleman (UC Berkeley),
Matthew Emerton (Northwestern Univ.),
Eyal Goren (Univ. McGill)

December 13–18, 2003

## 1 The background.

Langlands' conjectures, made in the 1970s, predict an extraordinary link between automorphic forms (essentially analytic objects) and representations of Galois groups (much more algebraic objects). In fact, strictly speaking, the link conjecturally relates automorphic forms to representations of even bigger groups, whose existence is yet to be established and about which we shall say very little. Langlands also made local conjectures and conjectured that the local and global conjectures should be compatible with one another. The link had already been established for automorphic forms on $\mathrm{GL}_1$ when Langlands made his conjectures—indeed the link in this case was essentially equivalent to the main theorems of local and global class field theory. For automorphic forms on other groups, a lot is known about the local case ([12] for example) and the function field case ([19]), but the conjectures are still wide open in the number field case. One should also add that serious breakthroughs in the mod $p$ version of the local conjectures have been made in [26]. The existence of the link (if it could be proved) has many consequences, for example it would give new ways of building automorphic forms via base change and automorphic induction. In a few cases these constructions have already been made, as a consequence of a lot of work ([1], for example). The local and global compatibility of Langlands' conjectures states basically that the local component at a place $v$ of a global automorphic representation should contain essentially the same data as the restriction of the associated global $\ell$-adic Galois representation to the corresponding local Galois group, at least for $v$ and $\ell$ coprime. In the few cases where the global conjectures are known, these local and global compatibilities have frequently also been established (see [12], for example).

The beginnings of the theory of $p$-adic modular forms also emerged in the 1970s, thanks to work of Serre ([25]) and Katz ([17]). This theme lay dormant for a while afterwards, until it was taken up again by Hida in an important series of papers in the 1980s ([14],[15] and others). Hida developed a theory of $p$-adic families of modular forms, and one consequence of his work was that one could attach Galois representations to certain $p$-adic modular forms.

These two stories, Langlands' conjectures and $p$-adic families, have enjoyed a healthy amount of interplay over the years, but it is only apparently recently that people have started to make deeper observations about a hitherto missing piece of the puzzle—the local and global theory of so-called $p$-adic automorphic forms. Let us make some attempt of describing the missing pieces in some degree of concreteness, at least for the group $\mathrm{GL}_2$. On the automorphic side, if one considers continuous representations of the group $\mathrm{GL}_2(\mathbf{Q}_p)$ acting on a $p$-adic vector space, there are many

many more such representations than the usual "admissible" representations which arise in the classical theory. On the Galois side, if one considers 2-dimensional $p$-adic representations of the absolute group of $\mathbf{Q}_p$ then there are far more continuous representations than potentially semi-stable ones—and specialisations of Hida families at forms of negative weight typically give rise to representations which are not potentially semi-stable. The link between automorphic forms and Galois representations is frequently made through arithmetic geometry, and there are far more $p$-adic modular forms than classical modular forms. These three observations are all well-known, but it has taken a long time before people have begun to understand precisely which Galois representations, which $p$-adic modular forms, and which representations of $\mathrm{GL}_2(\mathbf{Q}_p)$ are the "correct" ones, and what the links should be. Indeed, our understanding is still very limited, and for more general reductive groups it is still in its infancy. Our Banff workshop was organised to bring together researchers in these areas, to report on recent progress in our understanding of "the big picture" and to raise precise open problems in this area.

Let us conclude these general remarks with some observations about the current state of the art as of the end of 2003, for $\mathrm{GL}_2$ and the general case. Amongst the $p$-adic modular forms, it seems that the overconvergent ones are the ones of the most interest. Indeed, in the important paper [8], Coleman shows that finite slope overconvergent forms lie in families just as Hida's ordinary forms did. It is also known that ordinary forms are overconvergent. Unfortunately Coleman's work does not generalise too easily to other groups, and we are still searching for a good definition of an overconvergent automorphic form. Good definitions seem to be known for tori ([6]), groups which are compact at infinity ([7]), and certain unitary groups ([16]).

Schneider and Teitelbaum have begun a systematic study of so-called "locally analytic representation theory": the representation theory of $p$-adic lie groups on $p$-adic vector spaces ([21], [22], [23] [24]). In particular, they have introduced several important notions of "admissibility", which cut out various abelian categories of representations. Emerton has introduced a notion of Jacquet module functors in the context of this theory ([9], [10]); passing to the Jacquet module of a representation is the analogue in representation theory of passing to the finite slope part of the space of overconvergent modular forms, in the theory of $p$-adic modular forms.

In the paper [11] Emerton has applied the techniques of locally analytic representation theory to construct many new examples of eigenvarieties, which however do not parameterise families of $p$-adic automorphic forms *per se*, merely $p$-adic analytic families of Hecke eigenvalues. This represents an important breakthrough, although it raises questions as well as answering them—for example, do the eigenvarieties that Emerton constructs coincide with those that are constructed by other, more classical, means? These are important questions that are only just being formulated.

Finally, amongst the representations of $\mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$, Fontaine has singled out the Hodge-Tate and de Rham representations. Here the situation is more delicate. The representations associated to classical modular forms are known to be de Rham. However the Dieudonné module associated to a $p$-adic modular form is only 1-dimensional in general, by recent work of Kisin ([18]). On the other hand, Hodge-Tate representations are probably too general to be of interest. Breuil ([2], [3]) has made some important observations in this area, formulating very precise links between the irreducible admissible locally algebraic $p$-adic representations of $\mathrm{GL}_2(\mathbf{Q}_p)$ (as defined in [20] and [9]) and 2-dimensional de Rham representations, and he has also made some insightful conjectures concerning a mod $p$ version of the theory.

Although the above picture is of a theory that is clearly only in its infancy, the theory has already had some non-trivial applications. Building on work of Wiles, Taylor and his coworkers used Hida families to verify many non-solvable cases of Artin's conjecture for 2-dimensional representations in [5]. Kisin has verified the Fontaine-Mazur conjecture for the Galois representations coming from overconvergent $p$-adic modular forms in [18]. Chenevier's construction of families of automorphic forms in his thesis were used by him and Bellaïche to prove new cases of the Bloch-Kato conjecture. Coleman's theory of the eigencurve explains computational observations of Gouvêa and Mazur, although more precise computations of Buzzard and Gouvêa have thrown up much more precise conjectures that still remain unproven (although see [4] and [13]).

## 2 The talks.

Coleman and Mazur developed the theory of "eigencurves", geometric objects parametrising overconvergent modular eigenforms, and in their paper they raise many questions about the geometry of these objects. Several talks at the conference were about these questions, and the generalisation of the construction to other situations.

Work on explicitly computing regions of eigencurves corresponding to small slope forms has been done by Emerton and Coleman–Stevens–Teitelbaum. The first attempts to work at higher slopes were ideas due to Smithline, and the first concrete results were due to Kilford in his thesis, where he computed the fibre of the 2-adic eigencurve above some explicit points in weight space. Kilford talked about these results at the conference. This work was recently extended by Buzzard and Kilford, who manage to compute the pre-image in the 2-adic eigencurve of an annulus at the boundary of weight space. Buzzard and Calegari, in joint work, have recently managed to deduce from these results that the 2-adic eigencurve is proper over weight space, answering one of the questions raised by Coleman and Mazur in this particular case (Much of the work on this result was in fact done on the plane home after the conference finished.)

Kassaei in his talk illustrated how the Coleman-Mazur ideas could be extended to give $p$-adic families of automorphic forms associated to certain unitary groups, and as a consequence showed how one could deduce Gouvêa-Mazur-like results about the automorphic forms on these unitary groups. Kassaei also explained an important new construction of analytic continuation of overconvergent eigenforms in this case, which enabled him to prove that overconvergent eigenforms of small slope were classical, a generalisation of an old result of Coleman. Kassaei's new proof seems to use much less machinery than Coleman's, involving an "explicit" gluing process, and should certainly have applications in other areas. This is work in progress of Kassaei.

Stevens in his talk explained the status of his work with Ash on generalising the theory of eigencurves to cohomological eigencurves for $\mathrm{GL}_n$. Here a new phenomenon comes into play, that of torsion in cohomology, which makes the construction much more delicate. As a consequence it is still not yet quite a theorem that overconvergent eigenforms for $\mathrm{GL}_n$ lie in analytic families of the expected dimension, and it indeed seems to be the case that this is probably not true in general. On the other hand, he has enough of a theory to, for example, construct the symmetric square of a family of modular forms, by interpolating the symmetric squares of the classical forms in the family.

The representation associated to a $p$-adic modular form was initially constructed via the theory of pseudorepresentations, and hence very little could be said about the local behaviour of such a representation at $p$. Iovita's talk (joint work with Stevens) was on a direct geometric construction of the representations. This construction of course sheds new light on known facts, for example on Kisin's work on the Dieudonné module associated to a $p$-adic modular form, and will no doubt have other applications.

Gouvêa, Buzzard and Stein have all done extensive computations of slopes of classical forms, in an attempt to better understand the general phenomenon of local constancy of slopes of modular forms. Buzzard has made some very precise conjectures about slopes, essentially saying that, for fixed tame level and prime, in many cases one can predict all the slopes of all classical modular forms of all weights. These conjectures were very ad-hoc and based mostly on computer calculations. Herrick has made some much more conceptual observations about these formulae and in his talk he gave some very detailed conjectures which have a lot more structure to them.

The so-called $\mathcal{L}$-invariant has played an important role in the local theory of modular forms and elliptic curves. There have been several different definitions of the $\mathcal{L}$-invariant, due to Coleman, Teitelbaum, Kato–Kurihara–Tsuji, and Fontaine. All definitions are now known to coincide and one crucial piece of work in this area was the theorem of Greenberg and Stevens, who used Hida families to relate the $\mathcal{L}$-invariant to $L$-functions. Hida's talk was about the current state of play in this area.

Moving away from $\mathrm{GL}_2$, Tilouine in his talk gave an update of his progress for the group $\mathrm{GSp}_4$. Much progress has been made on this group in the last decade—work of Weissauer has attached $\ell$-adic representations to eigenforms, and Tilouine and his co-authors,in a series of papers, have developed enough of the theory to generalise work of Wiles and Taylor–Wiles to this situation. Tilouine talked about the ordinary $\Lambda$-adic version of this theorem, which is joint work with Genestier.

Urban in his talk reminded us about the known approaches to proving conjectures of Bloch-Kato type, by interpreting the cohomology groups which arise in the conjectures as groups of extension classes of representations, and relating these representations to automorphic forms. Classically, Mazur and Wiles used classical automorphic forms on $\mathrm{GL}_2$ to deduce the main conjecture for $\mathrm{GL}_1$, and Urban talked about higher-dimensional versions of this construction. Here there are immense technicalities to be overcome but serious progress has been made.

Greenberg in his talk gave us the state of the art about pseudo-null submodules. It is sometimes of great help in Iwasawa theory to know that certain modules which arise naturally in the theory have no non-zero pseudo-null submodules, and Greenberg gave an overview of many cases where this is now known.

Kisin in his talk announced a new and very strong modularity theorem of the form "$\rho$ modular mod $p$ implies $\rho$ modular". Kisin's new insight is how to deal with local deformation conditions in cases which had hitherto been thought intractable. His arguments rely on Breuil's theory of linear algebra associated to finite flat group schemes over DVRs over which $p$ is highly ramified. Kisin's main new idea in this area is a beautiful way of avoiding the technical troubles that Breuil, Conrad, Diamond and Taylor had with their local deformation problems at $p$ and will certainly have other applications to modularity.

Niziol talk about her approach to Fontaine's conjectures via $K$-theory, a method which is now giving totally new proofs of the conjectures.

Finally, Coleman gave a short talk where he mentioned some history and raised some questions about Eisenstein series, series which have played a prominent role in several aspects of the theory, but whose overconvergence properties are still only just becoming known. Some explicit results about overconvergence of 2-adic Eisenstein series are now known, thanks to work of Buzzard and Kilford, but Coleman emphasized that it is important to understand the general case.

One of the most exciting things about the workshop was that many of the talks were on unpublished work. In particular Kassaei's talk on overconvergent forms of small slope being classical and Kisin's talk on deformation rings were all on work that was not even in preprint form at the time.

## References

[1] J. Arthur and L. Clozel, *Simple algebras, base change, and the advanced theory of the trace formula*, Annals of Mathematics Studies 120, Princeton University Press.

[2] C. Breuil, Sur quelques représentations modulaire et $p$-adiques de $\mathrm{GL}_2(\mathbf{Q}_p)$ II, *J. Institut Math. Jussieu* **2** (2003), 1–36.

[3] C. Breuil, Invariant $\mathcal{L}$ et série spéciale $p$-adique, preprint (2003).

[4] K. Buzzard and F. Calegari, Slopes of overconvergent 2-adic modular forms, to appear in Compositio Mathematica.

[5] K. Buzzard, M. Dickinson, N. Shepherd-Barron and R. Taylor, On icosahedral Artin representations, *Duke Mathematical Journal* **109** (2001), 283–318.

[6] K. Buzzard, On $p$-adic familes of automorphic forms. In *Modular curves and abelian varieties*, Progress in Mathematics, **224**, 23–44, Berkhäuser Verlag, 2004.

[7] G. Chenevier, Thesis, Univ. Paris VII 2003.

[8] R. Coleman, $p$-adic Banach spaces and families of modular forms, *Inventiones Mathematicae* **127** (1997), 417–479.

[9] M. Emerton, Locally analytic vectors in representations of locally $p$-adic analytic groups, to appear in *Memoirs of the AMS*.

[10] M. Emerton, Jacquet modules for locally analytic representations of reductive groups over $p$-adic local fields, preprint (2004).

[11] M. Emerton, On the interpolation of systems of eigenvalues attached to automorphic Hecke eigenforms, preprint (2004).

[12] M. Harris and R. Taylor, *The geometry and cohomology of some simple Shimura varieties*, Annals of Mathematics Studies 151, Princeton University Press.

[13] G. Herrick, forthcoming Northwestern PhD thesis.

[14] H. Hida, Galois representations into $\mathrm{GL}_2(\mathbf{Z}_p[[X]])$ attached to ordinary cusp forms, *Inventiones Mathematicae* **85** (1986), 545–613.

[15] H. Hida, On $p$-adic Hecke algebras for $\mathrm{GL}_2$ over totally real fields, *Annals of mathematics* **128** (1988), 295–384.

[16] P. Kassaei, $P$-adic modular forms over Shimura curves over totally real fields, *Compositio Mathematica* **140** , 359–395.

[17] N. Katz, $p$-adic properties of modular schemes and modular forms. In *Modular functions of one variable, III*, Lecture Notes in Mathematics Volume 350, pp. 69–170.

[18] M. Kisin, Overconvergent modular forms and the Fontaine-Mazur conjecture. *Inventiones Mathematicae* **153** (2003), 373–454.

[19] L. Lafforgue, Chtoucas de Drinfeld et correspondance de Langlands, *Invent. Math.* **147** (2002), no. 1, 1–241.

[20] D. Prasad, Locally algebraic representations of $p$-adic groups. Appendix to [22], *Represent. Theory* **5** (2001), 125–127.

[21] P. Schneider, J. Teitelbaum, Locally analytic distributions and $p$-adic representation theory, with applications to $\mathrm{GL}_2$, *J. Amer. Math. Soc.* **15** (2001), 443–468.

[22] P. Schneider, J. Teitelbaum, $U(\mathfrak{g})$-finite locally analytic representations, *Represent. Theory* **5** (2001), 111–128.

[23] P. Schneider, J. Teitelbaum, Banach space representations and Iwasawa theory, *Israel J. Math.* **127** (2002), 359–380.

[24] P. Schneider, J. Teitelbaum, Algebras of $p$-adic distributions and admissible representations, *Invent. Math.* **153** (2003), 145–196.

[25] J.-P. Serre, Formes modulaires et fonctions zêta $p$-adiques. In *Modular functions of one variable, III*, Lecture Notes in Mathematics Volume 350, 191–268. Serre Antwerp

[26] M.-F. Vignéras, La conjecture de Langlands locale pour $\mathrm{GL}(n, F)$ modulo $l$ quand $l \neq p, l > n$. *Ann. Sci. cole Norm. Sup. (4)* **34** (2001), no. 6, 789–816.