Differential Invariants and Invariant Differential Equations

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The workshop succeeded in its goal to bring together leading geometers, group theorists, analysts and applied mathematicians to discuss the state of the art in the fields of differential invariants and invariant differential equations. A wide range of stimulating issues and applications was presented in the talks and discussed among the participants. Indeed, it was impressive to see so many participants working together late in the evening or early in the morning in the BIRS lounge. The BIRS facilities provided an ideal setting for this kind of exchange, and it was particularly gratifying to see geometers, analysts and applied mathematicians succeeding in communicating their ideas to each other.

The spectacular recent advances in the analytic study of the mean curvature flow for submanifolds of Euclidean space and of smoothing by the flows associated to some non-linear invariant diffusion equations made the time for this meeting very ripe. Indeed, the connections between integrable soliton equations, Poisson geometry, and geometric motions of curves and surfaces, can be traced back to the remarkable Hasimoto transformation between the equation of motion of a vortex filament and the nonlinear Schrödinger equation. Hierarchies containing both curve shortening flows, vortex filament equations, equations of elastic rods, and thermal grooving have now been established. The past decade has seen a steady increase in the range of theoretical developments, and practical applications in fluid mechanics, elasticity, geometry, computer vision, and soliton theory. Recent work has highlighted the role of differential invariants and invariant evolution equations in understanding and extending this connection to other geometries and other transformation groups. In particular, operators appearing in the invariant Euler-Lagrange complex constructed with the moving frame theory govern the bi-Hamiltonian structure, and hence integrability of these geometrical motions. Moving frames, in their original form as introduced by Elie Cartan, and in their more recent generalization discovered by Fels and Olver provide an extraordinarily powerful approach to these questions. The participants and speakers at the workshop were therefore carefully selected so as to reflect much of the recent activity in these subjects.

A first group of talks was concerned the interplay between group actions, differential geometry, and the profound connections between classical differential geometries — Riemannian, affine and conformal — and integrable (soliton) differential equations. The power of the method of moving frames as a tool for computing differential invariants and invariant differential operators appeared as a recurrent theme in several of these talks.

Tom Ivey lectured on the vortex filament equation for curves in three-dimensional space as a geometric counterpart to the integrable nonlinear Schrödinger equation arising in nonlinear optics. Connections with the Kirchhoff elastic rod and knot theory were presented, with many unresolved issues in the case of higher genus knotted solutions remaining.

Keti Tenenblat presented her recent results on differential systems describing spherical and pseudo-spherical surfaces of constant nonzero Gaussian curvature. Examples include the nonlinear Schrödinger equation and the Landau–Lifschitz equation.

Gloria Marí–Beffa described how to find an invariant moving frame along a curve in a manifold with a conformal structure, and, as a result, a Poisson bracket defined on the space of conformal differential invariants of curves. Certain conformally invariant curve flow induces Hamiltonian and integrable flows on the differential invariants. Jan Sanders further developed the connections between Cartan geometry and integrability in the conformal case, by showing that the structure equations for the flow of the parallel connection of a curve embedded in an n-dimensional conformal manifold leads to integrable bi-Hamiltonian scalar-vector equations.

Stephen Anco discussed applications of frames to nonlinear wave equations and integrable evolution equations. The frame formulation of arclength-preserving curves in Riemannian geometries has been shown to give a geometrical derivation of integrable evolution equations, such as vector mKdV equations, and their associated Hamiltonian operator structures.

Eugene Ferapontov discussed the integrability of (2+1)-dimensional quasilinear systems of hydrodynamic type through decoupling, in infinitely many ways, into a pair of compatible n-component one-dimensional systems in the Riemann invariants. Exact solutions described by these reductions, known as nonlinear interactions of planar simple waves, can be viewed as natural dispersionless analogs of n-gap solutions. As an example of this approach is a complete classification of integrable (2+1)-dimensional systems of conservation laws possessing a convex quadratic entropy.

Robert Bryant presented recent progress in Finsler geometry, a far-reaching generalization of classical Riemannian geometry. Cartan's generalization of Lie's third theorem is applied to study the space of Finsler metrics of constant curvature. The analysis of the geometry of these equations leads to some very interesting Kähler metrics on the complex n-quadric.

Joel Langer showed how Schwarz reflection geometry of analytic curves in the complex plane can be interpreted as an infinite dimensional symmetric space geometry. In the continuous limit, he obtains the geodesic equation, a second order PDE describing conformally invariant evolution of analytic plane curves.

The second group of talks dealt with the more classical topic of symmetries of differential equations, differential invariants and moving frames.

Ian Anderson proposed a general framework for group theoretical origins of superposition principles for nonlinear differential equations. His examples indicate several intriguing open research problems and, in particular, suggest deep relationships between integrable exterior differential systems and group actions on jet spaces.

Michael Eastwood gave an analysis of the higher symmetries of the Yamabe Laplacian, leading to intriguing connection with conformal geometry and the AdS/CFT correspondence.

Boris Doubrov demonstrated how the classical Wilczynski invariants of linear ordinary differential equations can, through linearization, be used to construct differential invariants for classifying nonlinear ordinary differential equations. Applications in the inverse problem of the calculus of variations were presented.

Peter Olver and Juha Pohjanpelto introduced new computational algorithms for infinite-dimensional Lie pseudo-group actions based on the equivariant theory of moving frames and a new, direct construction of invariant Maurer–Cartan forms for infinite-dimensional pseudo-groups. New applications include complete classifications of differential invariants, recurrence relations and syzygies, the invariant bicomplex, resolution of equivalence and symmetry problems, and the calculus of variations.

Harvey Segur posed a question in the study of partial differential equations with multi-symplectic structure, which can have practical value in stability calculations, as in the nonlinear Schrödinger equation in two or more dimensions. He posed the question of why such an extra Hamiltonian structure is sometimes present, and sometimes not. This led to an extensive discussion of the issue with several of the participants.

The third theme was the interaction of symmetry and geometry with numerical and symbolic computational algorithms. This tied in to the rapidly evolving field of geometric numerical integration.

Peter Hydon and Elizabeth Mansfield have developed a new theory of "difference forms" that are designed to play the same role for difference equations, including discretization of continuous systems, that the classical exterior calculus plays for differential equations. They showed how to derive analogues of Stokes' Theorem, de Rham and Çech cohomology, and how to compute nontrivial cohomology groups of rectangular lattices with holes. These constructions have direct interpretations and consequences for various finite difference schemes, including collocation methods. Applications in digital image analysis were indicated.

The key point of Greg Reid's talk on numerical jet geometry was that, in real world applications, one only deals with approximations and so one must rework the calculus of symmetry groups, conservation laws, explicit solutions, etc. taking into account the limited precision of numerical parameters in the system. His goal is to develop a numerical version of algebraic methods such as Gröbner bases, relying recent developments in numeric algebraic geometry and the singular value decomposition. Applications to computer vision and the determination of approximate symmetries of partial differential equations were discussed.

Pavel Winternitz explored the Lie point symmetries of difference schemes involving one dependent and one independent variable. The symmetries act on the equation and on the lattice. If the lattice is suitably adapted, difference schemes have essentially the same symmetries as differential equations obtained as their continuous limits. He showed how symmetries can be used to classify difference schemes, to decrease the order of the scheme, and to obtain exact solutions. Variational symmetries are particularly useful in this context. Consequences for numerical analysis will be discussed, as well as generalizations to multivariable difference schemes.

Thomas Wolf demonstrated his computer packages for determining the existence of higher order symmetries and first integrals for classifying integrable systems. In particular, all quadratic Hamiltonians of Kowalevski type having additional first integral of third or fourth degree were found and quantum analogs of these Hamiltonians given.

Finally, two talks on applications, in quantum mechanics, and in computer vision, rounded out a successful workshop program.

Federico Finkel survey recent results on Sutherland spin models from quantum mechanics. Complete integrability and exact solvability of this model can be established by relating it to a set of differential-difference operators known as Dunkl operators, coming from the theory of orthogonal polynomials.

Kaleem Siddiqi explained the use of flux invariants for shape in image processing. He considers the average outward flux through a Jordan curve of the gradient vector field of the Euclidean distance function to the boundary. In the zero-area limit, this Euclidean-invariant measure serves to distinguish medial points, providing a theoretical justification for its use in a Hamilton–Jacobi skeletonization algorithm. In the case of shrinking circular neighborhoods, the average outward flux measure also reveals the object angle at skeletal points. Hence, formulae for obtaining the boundary curves, their curvatures, and other geometric quantities of interest, can be written in terms of the average outward flux limit values at skeletal points.