Calabi-Yau Varieties and Mirror Symmetry

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This is a report for the five day workshop at the Banff International Research Station (BIRS) on "Calabi–Yau Varieties and Mirror Symmetry" held from December 6 to 11, 2003. There were 38 participants. The workshop was a huge success. This workshop was the second in the series, following the first one held at the Fields Institute in the summer of 2001. We have witnessed enormous progress since the last one, and we are hoping to have the third workshop in two years time elsewhere.

We are planning to publish the Proceedings of this workshop. In fact, the contract for the Proceedings has already been signed with International Press/American Mathematical Society. The editors will be James Lewis, Shing-Tung Yau and Noriko Yui. All papers will be refereed rigorously. The Proceedings is open to all participants, not only to the presenters of talks. The deadline for submitting papers is tentatively set at the end of August 2004.

Here is a sample response from participants.

Dear Noriko,
First let me congratulate you on the success of the conference, it was really unique and beautiful. Even John McKay told me that it was probably one of the best conferences he's ever attended, especially for the variety of the themes that all come together (and I think the food also was a factor).
Best regards,
Abdellah Sebbar

Dear Noriko,
As promised, some CY-problems as TeX-file. Best, and thanks again for organizing such a nice meeting,
Eckart Viehweg

Dear Noriko,
yes, it was a nice workshop. I will try to send the problem list as soon as possible.

Best, Duco van Straten

Dear Noriko,

The BIRS workshop definitely was very enjoyable and also very useful. I enjoyed talking to you and a number of other participants, in particular Jan and Helena, and also Eckart Viehweg, Klaus Hulek and Duco van Straten. Duco has great stamina even when the time is close to midnight!

I'll write up the problems over the weekend and send it to you early next week.

Thanks again for the invitation to such a nice workshop!

Best,

Rolf Schimmrigk

Dear Noriko,

Thanks a lot for your email. The workshop was really a fun. I will send you my problems very soon. Merry Christmas and Happy New Year!!!

Best

Andrey Todorov

Dear Noriko Yui,

Thank you again for the wonderful workshop. As I told you, for the first family, $x + 1/x + y + 1/y + z + 1/z - k$ and $k = 18, 30, 102, 198$, you get a tau quadratic satisfying respectively $6x^2 + 5 = 0$, $6x^2 + 7 = 0$, $6x^2 + 13 = 0$, $6x^2 + 17 = 0$ defining number fields with class number 4.

Best wishes and Happy New Year.

Marie Jose BERTIN

The main themes

1. Arithmetic of Calabi–Yau varieties and mirror symmetry: Arithmetic of elliptic curves, $K3$ surfaces, Calabi–Yau threefolds, and higher dimensional Calabi–Yau varieties in connection with mirror symmetry. These will include the following topics and problems: Interpretation of mirror symmetry phenomena of Calabi–Yau varieties in terms of zeta-functions and L-series of the varieties in question, the modularity conjectures for Calabi–Yau varieties, the conjectures of Birch and Swinnerton-Dyer for elliptic curves and abelian varieties, the conjectures of Beilinson–Bloch on special values of L-series and algebraic cycles. Calabi–Yau varieties of CM (complex multiplication) type and their possible connections to rational conformal field theories.

2. Algebraic cycles, (classical and p-adic) Hodge theory, K-theory, Quantum cohomology theory for Calabi–Yau varieties. Of particular interest here are the regulators of [higher] algebraic cycles, and some evidence that the Calabi–Yau varieties provide the "most interesting" examples of regulator calculations.

3. Moduli theory for Calabi–Yau manifolds. Moduli of abelian varieties, $K3$ surfaces and Calabi–Yau threefolds. These will lead to classification problems of Calabi–Yau varieties, e.g., computations of period maps and period domains for $K3$ surfaces, classification of rigid Calabi–Yau threefolds.

4. Mirror symmetry for Calabi–Yau varieties and modular forms. Characterization of mirror maps in connection with mirror moonshine phenomenon. Rigorous definition of D-branes, and geometry behind D-branes. Borcherds product formula and mirror symmetry. Modular forms in mirror symmetry.

Background

A Calabi–Yau variety of dimension d is a complex manifold with trivial canonical bundle and vanishing Hodge numbers $h^{i,0}$ for $0 < i < d$. For instance, a dimension 1 Calabi–Yau variety is an elliptic curve, a dimension 2 Calabi–Yau variety is a K3 surface, and a dimension 3 one is a Calabi–Yau threefold.

(A) One of the most significant developments in the last decade in Theoretical Physics (High Energy) is, arguably, *string theory* and *mirror symmetry*. String theory proposes a model for the physical world which purports its fundamental constituents as 1-dimensional mathematical objects "strings" rather than 0-dimensional objects "points". Mirror symmetry is a conjecture in string theory that certain "mirror pairs" of Calabi–Yau manifolds give rise to isomorphic physical theories. Calabi–Yau manifolds appear in the theory because in passing from the 10-dimensional space time to a physically realistic description in four dimension, string theory requires that the additional 6-dimensional space is to be a Calabi–Yau manifold.

Though the idea of mirror symmetry has originated in physics, in recent years, the field of mirror symmetry has exploded onto the mathematical scene. It has inspired many new developments in algebraic geometry, toric geometry, Riemann surfaces theory, infinite dimensional Lie algebras, among others. For instance, the mirror symmetry has been used to tackle the problem of counting number of rational curves on Calabi–Yau threefolds.

In the course of mirror symmetry, it has become more and more apparent that Calabi–Yau varieties enjoy tremendously rich arithmetic properties. For instance, arithmetic objects such as: modular forms, and modular functions of one and more variables, algebraic cycles, L-functions of Calabi–Yau varieties, have popped up onto the scene. Also special classes of Calabi–Yau manifolds, e.g., of Fermat type, or their deformations, offer promising testing grounds for physical predictions as well as rigorous mathematical analysis and computations.

(B) One of the most significant developments in the last decade in Arithmetic Geometry and Number Theory is the proof of the Taniyama-Shimura-Weil conjecture on the so-called *modularity of elliptic curves defined over the field* $\mathbf{Q}$ by A. Wiles and his disciples. Wiles' idea is to exploit 2–dimensional Galois representations arising from elliptic curves and modular curves, and establish their equivalence. His method ought to be applied to explore arithmetic of Calabi–Yau threefolds. In particular, rigid Calabi–Yau threefolds defined over the field of rational numbers are equipped with 2–dimensional Galois representations which are conjecturally equivalent to modular forms of one variable of weight 4 on some congruence subgroups of $\mathrm{PSL}(2, \mathbf{Z})$. This might be regarded as concrete realizations of the conjecture of Fontaine and Mazur that every odd irreducible 2–dimensional Galois representations arising from geometry should be modular. Recently the modularity of odd mod 7 2–dimensional Galois representations has been established. It is one of our aims to understand this result in the modularity conjecture for rigid Calabi–Yau threefolds over $\mathbf{Q}$. For not necessarily *rigid* Calabi–Yau threefolds over $\mathbf{Q}$, the Langlands Program predicts that there should be some automorphic forms attached to them. We plan to test the so-called *modularity conjectures* for Calabi–Yau varieties over $\mathbf{Q}$ or more generally over number fields, first trying to understand them for some special classes of Calabi–Yau threefolds, e.g., those mentioned in (A) and more generally for motives arising from Calabi–Yau threefolds.

The determination of zeta-functions and L–series of Calabi–Yau will undoubtedly be one of the central themes in this endeavour. Recent works of Candelas, de la Ossa and Villegas have brought in "semi-periods" and the GKZ hypergeometric systems in the determination of zeta-functions of one-parameter deformations of Calabi–Yau threefolds. One of our goals is to understand their findings and their consequences on physics, arithmetic and geometry, and also the work of J. Stienstra on zeta-functions of ordinary Calabi–Yau manifolds via Dwork theory presented at the pilot workshop in July 2001 ought to be analyzed with vigor.

(C) There are a number of intriguing developments in the theory of algebraic cycles in the past 25 years, that not surprisingly, should open the door to an infusion of new techniques in the study of Calabi-Yau manifolds and mirror symmetry. The impact of classical Hodge theory as well as the p-adic Hodge theory, is clearly evident. On the algebraic side, there is the relationship of algebraic K-theory and Chow groups, leading to the Bloch-Quillen-Gersten resolution description of Chow groups. There is also the more recent relationship of Bloch's higher Chows groups and higher K-theory (a higher Riemann-Roch theorem), and a conjectured "arithmetic index theorem". The influence of the work of Bloch and Beilinson on the subject of algebraic cycles is profound. For instance there are the fascinating Bloch-Beilinson conjectures on the existence of a natural filtration on the Chow groups, whose graded pieces can be described in terms of extension data, and their conjectures about injectivity of certain regulators of cycle groups of varieties over number fields. There is also the work of others on how conjecturally this filtration can be explained in terms of kernels of higher regulators and arithmetic Hodge structures. The Calabi-Yau manifolds present an ideal testing ground for some of these conjectures.

In particular, the theory of D-branes ought to be pursued providing rigorous mathematical definition, and its connection to algebraic cycles, etc.

Objectives

The recent progresses mentioned above (A), (B) and (C), based on so many interactions with so many areas of mathematics and physics, have contributed to a considerable degree of inaccessibility to mathematicians and physicists working in their respective fields, not to mention, graduate students. Perhaps one of the greatest obstacles facing mathematicians and physicists is that each camp has its own language. Mathematicians have had difficulties isolating mathematical ideas in physics literatures, and vice versa for physicists. At the pilot workshop of this program at the Fields Institute in July 2001, we have witnessed firsthand how these barriers have started melting away. We hope that our proposed half year program at the Fields Institute would follow up the ground breaking efforts of the pilot workshop to the full fruition.

Geometry around mirror symmetry and string theory has been pursued by many mathematicians (complex geometers, toric geometers, and others), and great progress has been witnessed in understanding geometric aspects of the problem. In fact, recently a number of excellent books and survey articles have been published explaining complex geometric aspects of mirror symmetry on Calabi–Yau threefolds as well as on K3 surfaces.

Further, in the past two decades, a number of people who have studied that part of algebraic geometry dealing with Hodge theory and algebraic cycles, have found applications of their work in Quantum Cohomology, Mirror Symmetry and Calabi-Yau manifolds. One anticipates that these interactions between the various "schools" will blossom in the near future.

Arithmetic aspects on Calabi–Yau varieties and mirror symmetry, however, are yet to be explored vigorously. For instance, Wiles' method may be used to establish the *modularity for rigid Calabi–Yau threefolds defined over the field of rational numbers* a la Fontaine and Mazur. Also, investigation on the intermediate Jacobians of Calabi–Yau threefolds ought to be pursued using, for instance, p-adic Hodge theory, modular symbols. Again the recent paper of Candelas, et. al on the computation of the zeta-functions of Calabi–Yau manifolds over finite fields reveal a surprising connection of mirror symmetry to p-adic L-functions (which are the essential ingredients in Iwasawa theory). Further investigation on p-adic analysis in physics pertinent to mirror symmetry is proposed in this program. The construction of algebraic cycles on Calabi–Yau threefolds (generalizing the method of Bloch), investigation of L-functions of Calabi–Yau threefolds a la the conjectures of Beilinson and Bloch, among others, ought to be pursued with more rigor and intensity. In fact, the pilot workshop in July 2001 was mostly concentrated on arithmetic aspects of Calabi–Yau Varieties and Mirror Symmetry. One of the outcomes of the pilot workshop is that we have begun to understand the mirror symmetry phenomenon for a mirror pair of quintic hypersurfaces in $\mathbf{P}^4$ at the level of local zeta-functions.

Moduli theory of Calabi–Yau manifolds has gained maturity in recent years. For instance, modular functions, McKay–Thompson series, Borcherds product formula, are coming into the center stage. Investigation on moduli theory of Calabi–Yau manifolds will certainly be one of the main themes pursued in this program.

Our goal is to bring together experts working in physics, geometry and arithmetic around Calabi–Yau varieties and mirror symmetry, and to exchange ideas and learn the subjects first-hand, mingling with researchers with different expertise. We expect these interactions to lead to progress in solving open problems in mathematics and physics as well as to pave ways to new developments.

The Participants

This is the participant list of the workshop.

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There were unfortunately several last minutes cancellations. S.-T. Yau had to cancel his participation as the Chinese Premier was visiting the United States in the same period as the workshop, and he had to go to Washington DC to meet the Premier. A. Klemm had to cancel his participation as he was not able to find a replacement for his teaching (he has just moved to University of Wisconsin from Berlin). N. Shepherd-Barron had to cancel his participation due to a family reason.

Abstracts of Talks

Jim Bryan (University of British Columbia)

Topological Quantum Field Theory and Gromov–Witten Invariants of Curves in Calabi–Yau Threefolds

Topological Quantum Field Theory, as formulated by Atiyah, provided a general frame-work for understanding invariants of manifolds. The structure of TQFTs in dimension $1+1$ (i.e. surfaces with boundaries) is completely understood by elementary means yet they can still yield surprising results. For each positive integer d, we define a one-parameter family of $(1+1)$-dimensional TQFTs $Z_d(t)$ which specializes at $t=0$ to the famous Witten-Dijgraaf-Freed-Quinn TQFT for gauge theory with finite gauge group S_d (the d-th symmetric group). Our family of TQFTs completely encodes all the degree d local Gromov-Witten invariants of a curve (of arbitrary genus) in a Calabi-Yau threefold. This provides us with a "structure theorem" for these local invariants (a.k.a. multiple cover formulas). Using these ideas we completely determine the local invariants for $d<7$.

Kentaro Hori (University of Toronto, Physics and Mathematics)

Calabi–Yau Orientifolds

I introduce orientifolds to mathematicians and discuss some applications. Orientifolds are associated with unoriented strings. Real algebraic geometry plays an important role, just as symplectic geometry and algebraic geometry did for oriented strings.

Xi Chen and James D. Lewis (University of Alberta)

Hodge D-Conjecture for $K3$ and Abelian Surfaces

We prove that the real regulator map is surjective on a general K3 or Abelian surface. The heart of the proof involves the use of rational curves on K3 and a degeneration argument. It is closely related to the recent progress on the enumerative geometry on K3. This is joint work with James Lewis.

Chuck Doran (Columbia University)

Integral Structures, Toric Geometry, and Homological Mirror Symmetry

We establish the isomorphisms over $\mathbf{Z}$ of cohomology/K-theory, global monodromy, and invariant symplectic forms predicted by Kontsevich's Homological Mirror Symmetry Conjecture for certain one dimensional families of Calabi-Yau threefolds with $h^{2,1}=1$. These families arise as hypersurfaces or complete intersections in Gorenstein toric Fano varieties, and their mirrors are described by the Batyrev-Borisov construction. Our method involves (1) classifying all rank four integral variations of Hodge structure over $\mathbf{P}^1 \setminus \{0,1,\infty\}$ with maximal unipotent local monodromy about 0 and local monodromy about 1 unipotent of rank 1, and (2) checking, using properties of nef partitions of reflexive polytopes, that the $\mathbf{Z}$-VHS of our Calabi-Yau families match those picked out by the K-theory of their mirrors via the HMS Conjecture. This is joint work with John Morgan.

Andrey Todorov (University of California Santa Cruz)

Higher Dimensional Analogues of Dedekind Eta Function

It is a well known fact that the Kronecker limit formula gives an explicit formula for regularized determinants of flat metrics on elliptic curves. It established the relation between the regularized determinant of the flat metrics on elliptic curves and their discriminants. This relation can be interpreted as follows; There exists a holomorphic section (multivalued) of the dual of the determinant line bundle such that its L^2 norm is equal to the regularized determinant of the Laplacian acting on $(0,1)$ forms. Kronocker limit formula established that the holomorphic section constructed from the determinant line bundle is the Dedekind eta function.

In the talk we discuss the existence of the analogue of the Dedekind eta function for K3 surfaces and CY manifolds. The construction of the generalized Dedekind eta function is based on the variational formulas for the determinants of the Laplacians of a Calabi-Yau metric acting on functions and forms of type $(0,1)$ on CY manifolds, K3 surfaces and Enriques surfaces. Based on the variational formulas we will establish the existence of a holomorphic section of some power N of the dual of determinant line bundle on the moduli space of odd dimensional CY manifolds whose L^2 norm is the N^{th} power of the regularized determinant of the Laplacian acting on $(0,1)$. This holomorphic section of the determinant line bundle is the analogue of the Dedekind eta function for odd dimensional CY manifolds. In case of even dimensional CY manifold and we will show the existence of a holomorphic section of the relative dualizing sheaf of the moduli space.

In case of K3 surfaces the construction of the Dedekind Eta function is done on the moduli of Kähler-Einstein-Calabi-Yau metrics and then projected to the moduli of polarized algebraic K3 surfaces.

We will discuss also that the L^2 norm on the relative dualizing sheaf is a good metric in the sense of Mumford. This implies that the Weil-Petersson volumes of the moduli spaces of CY manifolds are rational numbers. When M is a CY threefold we will outline how to prove that the regularized determinant of the Laplacian acting on $(0,1)$ forms is bounded and that the section η^N vanishes on the discriminant locus.

Brian Forbes (University of California Los Angeles)

Open String Mirror Maps from Picard–Fuchs Equations on Relative Cohomology

A method for computing the open string mirror map and superpotential, using an extended set of Picard- Fuchs equations, is presented. This is based on techniques used by Lerche, Mayr and Warner. For X a toric hypersurface and Y a hypersurface in X, the mirror map and superpotential are written down explicitly. As an example, the case of K_{p^2} is worked out and shown to agree with the literature.

Eckart Viehweg (University of Essen)

Complex Multiplication, Griffiths–Yukawa Couplings, and Rigidity for Families of Hypersurfaces

Reporting on joint work with Kang Zuo: math.AG/0307398.

Let $M(d,n)$ be the moduli stack of hypersurfaces of degree $d > n$ in the complex projective n-space, and let $M(d,n;1)$ be the sub-stack, parameterizing hypersurfaces obtained as a d-fold cyclic covering of the projective $n-1$-space, ramified over a hypersurface of degree d. Iterating this construction, one obtains $M(d,n;r)$. The substack $M(d,n;1)$ is rigid in $M(d,n)$, although the Griffiths-Yukawa coupling degenerates for $d < 2n$, hence in particular for Calabi-Yau hypersurfaces. On the other hand, for all $d > n$ the sub-stack $M(d,n;2)$ deforms. One can calculate the exact length of the Griffiths-Yukawa coupling over $M(d,n;r)$. As a byproduct one finds a rigid family of quintic hypersurfaces over some 4-dimensional subvariety M of the moduli stack, and a dense set of points in M, where the fibres have complex multiplication.

Yi Zhang (Zhejiang University, China)

Some Results on Families of Calabi–Yau Varieties

The aim of the lecture is to show some new results related to the families of projective Calabi-Yau manifolds.

First, the author introduces concisely the results of rigid problem related to Shafarevich conjecture of Calabi-Yau which are included in the author preprint " The Rigidity of Families of Projective Calabi-Yau manifolds, Math.AG/0308034", i.e. he shows that some important families of Calabi-Yau manifolds are rigid,for examples:

(I) Lefschetz pencils of odd dimensional Calabi-Yau manifolds are rigid;

(II) Strong degenerate families (in some sense, an not need to be CY manifolds) are rigid;

(III) Families of CY manifolds admitting a degeneration with maximal unipotent monodromy must be rigid.

The main methods of the author to attack the problems is that the degenerate theory of variation of Hodge Structure, Yang-Mills theory of Higgs bundle and Deligne-Katz's theory on monodromy,etc.

Initiated by the results of rigidity problems,the author study the important and interesting object in family geometry: Mumford-Tate group. The author wants to understand how Mumford-Tate groups control the families,especially the families of Calabi-Yau threefolds? Are there necessary relations between Mumford-Tate groups and rigidity problems? For example,he shows some relations between the global monodromy group and Mumford-Tate group.

Rolf Schimmrigk (Kennesaw State University)

Complex Multiplication of Calabi–Yau Varieties and String Theory

Abelian varieties with complex multiplication can be identified as the basic cohomological building blocks of certain types of Calabi-Yau manifolds. It is therefore possible to define the notion of complex multiplication for Calabi-Yau spaces via the complex multiplication type of these abelian varieties. The aim of this talk is to show how this symmetry illuminates the exactly solvable conformal field theoretic nature of Calabi-Yau varieties.

4:45pm–5:45pm: Wei-Dong Ruan (University of Illinois at Chicago)

Generalized Special Lagrangian Torus Fibration for Calabi–Yau Manifolds

In light of SYZ conjecture, special Lagrangian torus fibration play important role in mirror symmetry. In this talk, we will discuss new examples of special Lagrangian submanifolds and the construction of global generalized special Lagrangian torus fibration for Calabi-Yau manifolds.

Jan Stienstra (University of Utrecht)

Between Bloch–Beilinson and Seiburg–Witten: Informal subtitle: Six Examples Everybody Thinks (S)He Knows

The same regulator image for elliptic curves in $C * xC*$ turns out to give the relation between Mahler measure, L-function values, modular forms in the articles by Deninger and Rodriguez Villegas and calculations of Gromow-Witten invariants (or instanton numbers) in physics papers by Klemm-Mayr-Vafa and Lerche-Mayr-Warner. The q-parameters in the mathematics and physics papers are inverse functions of each other. This is concretely illustrated for six families of elliptic curves, which have also appeared in many other contexts. There are very interesting connections with recent work of Kenyon, Okounkov, Vafa et al. on dimer models and melting crystals.

Matt Kerr (University of California Los Angeles)

Geometric View of Regulators and Higher Chow Groups

We collect together some techniques and results due to A. Collino, J. Lewis, S. Muller-Stach, S. Saito and others; the main object of study is the indecomposable part of $CH^p(X,n)$ in the cases $3 \geq p \geq n \geq 0$, where X is a Calabi-Yau or Abelian 3-fold or surface. Our aim is to discuss regulator formulas (including those developed in our work), connectivity results, degeneration techniques, and differential equations satisfied by regulator "periods" in families. We will also indicate some interesting open problems.

Shi-shyr Roan (Academia Sinica, Taipei, Taiwan)

Rational Curves in Rigid Calabi–Yau Threefold

We determine all the Kummer-surface-type Calabi-Yau (CY) 3-folds, i.e., those $\widehat{T/G}$ obtained by resolution of a 3-torus-orbifold T/G with only isolated singularities. There are only two such CY spaces: one with $G = Z_3$, and the other with $G = Z_7$. These CY 3-folds $\widehat{T/G}$ are all rigid, hence no complex structure deformation for the varieties. We further investigate problems of rational curves in $\widehat{T/G}$ not contained in exceptional divisors, by considering the counting number d of points in a rational curve C meeting exceptional divisors in a certain manner. We have obtained the constraint on d. With the smallest number d, the complete solution of C in $\widehat{T/G}$ is obtained for both cases. In the case $G = Z_3$, we have derived an effective method of constructing C in $\widehat{T/G}$, and obtained the explicit forms of rational curves for some other d by the method.

Belazs Szendroi (University of Utrecht)

Calabi–Yau Threefolds in Weighted Homogeneous Varieties

Reprting on joint work with Anita Buckley.

Let (X, D) be a Calabi-Yau threefold with quotient singularities, polarized by an ample Q-Cartier divisor. We prove a formula expressing the dimension of the vector space $H^0(X, nD)$ in terms of global numerical invariants of (X, D) and local invariants of D at the quotient singularities of X. Based on this formula, we construct several new families of Calabi-Yau threefolds in weighted homogeneous varieties, generalizations of weighted projective spaces introduced by Corti and Reid. In some cases, we show how to compute Hodge numbers of (smooth Calabi-Yau models of) these threefolds using birational geometry.

Shinobu Hosono (University of Tokyo)

GKZ Hypergeometric Series, Mirror Symmetry, and Singularity Theory

In the last workshop at Fields Institute(July, 2001), I talked about GKZ hypergeometric series taking values in the cohomology group of a Calabi-Yau manifold, and made a conjecture on the period integrals of the mirror Calabi-Yau manifold. In the case of two dimensional toric (non-compact) Calabi-Yau manifolds, I will verify the conjecture by relating the hypergeometric series to the integral solutions of K. Saito's differential equation in singularity theory. I will also present some three dimensional examples, and try to refine the conjecture.

Marie José Bertin (Université Pierre et Marie Curie (Paris 6))

Mahler's Measure and L-Series of $K3$ Hypersurfaces

We express in terms of Eisenstein-Kronecker series the Mahler's measure of two families of polynomials defining $K3$ hypersurfaces. For some of these polynomials we relate their Mahler's measure with the L-series of the corresponding $K3$-surface.

Klaus Hulek (University of Hannover)

Examples of Non-Rigid Modular Calabi–Yau Manifolds

Reporting on joint work with Helena Verrill.

In this talk we want to present some examples of non-rigid Calabi-Yau varieties whose L-series is modular. These examples are constructed by considering nodal Calabi-Yau varieties in the toric variety associated to the A_4 root lattice.

Kenichiro Kimura (University of Tsukuba)

K_1 of a self-product of a curve

Beilinson's conjectures on special values of L-functions predicts the existence of interesting higher Chow cycles on varieties over number fields. I will explain about the attempts to create such cycles mainly in the case of a self- product of a curve.

Keiji Oguiso (University of Tokyo)

Simple Groups, Solvable Groups and $K3$ surfaces

We characterize the following three particular K3 surfaces, among all the complex K3 surfaces, by means of finite group symmetries:

(1) the Fermat quartic K3 surface

$$x_1^4 + x_2^4 + x_3^4 + x_4^4 = 0\,.$$

(2) the Klein-Mukai quartic K3 surface

$$x_1^3 x_2 + x_2^3 x_3 + x_3^3 x_1 + x_4^4 = 0\,,$$

i.e. the cyclic covering of degree 4 of projective plane branched along the Klein quartic curve. This is a joint work with D.-Q. Zhang.

(3) the minimal resolution of

$$s^2(x^3 + y^3 + z^3) - 3(s^2 + t^2)xyz = 0$$

in $\mathbf{P}^1 \times \mathbf{P}^2$, i.e. the minimal resolution of the double cover, branched along two singular fibres, of the (rational) elliptic modular surface with level 3 structure. This is a joint work with J.H. Keum and D.-Q. Zhang.

These three are all singular K3 surfaces in the sense of Shioda. The surface (1) is uniquely characterized as the K3 surface admitting either the solvable finite group action of maximum order or the nilpotent finite group action of maximum order. The surface (2) (resp. (3)) is uniquely characterized as the K3 surface admitting an action of the maximal possible finite extension of the simple group $L_2(7)$ (resp. $L_2(9) \simeq A_6$). In each case, we also show the uniqueness of the groups and their actions.

By a result of Mukai, the finite simple (non-commutaive) groups which can act on some K3 surfaces are only $L_2(7)$, $L_2(9)$ and A_5, the first three groups in ATLAS. Among these three, the first two are the maximal simple groups (with respect to the inclusion of groups) which can acts on K3 surfaces. If possible, I would like to discuss about non-maximal A_5 case, too.

John McKay (Concordia University)

About Everything; Subtitle: Three Sporadic Groups and AFfine Lie Data

I promote two conjectures - one I discovered 25 years ago - and the other just this year. A deep connection exists between affine $E6, E7$, and $E8$ data, and certain Fischer involutions of $F24', B$, and M. The groups of 27 lines on a 3-ic, and 28 bitangents on a 4-ic have a large significant literature but the 120 tritangent planes on a 6-ic curve of genus 4 do not. The fundamental groups of type $E6, E7, E8$ are the Schur multipliers of the corresponding sporadic groups. The second conjecture is the appearance of the class number, 194, of M as a Picard number in "Numerical Oddities" of hep-th/0002012 by Aspinwall, Katz and Morrison.

Slava Archava (McMaster University)

Hodge Cycles of Milnor Fibers

In this talk I will describe our joint project with Hossein Movasati in which we attempt to study the space of Hodge cycles on a Milnor fiber of a non-composite polynomial (and more generally on an affine hypersurface complement) using the characterization of Hodge cycles by vanishing of appropriate periods. To carry out this program we need a description (as explicit as possible) of the mixed Hodge structure on the cohomology of the variety under investigation and an explicit basis for its homology. In the case of a quasi-homogeneous polynomial we use Steenbrink's description of the Hodge filtration on the cohomology of the Milnor fiber as order of the pole filtration, generalizing classical results of Griffiths for the cohomology of a smooth hypersurface in projective space.

Albrecht Klemm had to cancel his participation at last minute. He was scheduled to speak with the following title.

Albrecht Klemm (University of Wisconsin, Physics)

The Topological Vertex

Authors: Mina Aganagic, Albrecht Klemm, Marcos Marino, Cumrun Vafa
We construct a cubic field theory which provides all genus amplitudes of the topological A-model for all non-compact Calabi-Yau toric threefolds. The topology of a given Feynman diagram encodes the topology of a fixed Calabi-Yau, with Schwinger parameters playing the role of Kahler classes of Calabi-Yau. We interpret this result as an operatorial computation of the amplitudes in the B-model mirror which is the Kodaira-Spencer quantum theory. The only degree of freedom of this theory is an unconventional chiral scalar on a Riemann surface. In this setup we identify the B-branes on the mirror Riemann surface as fermions related to the chiral boson by bosonization.

Problems

During the problem session and informal discussions, a number of problems were proposed, which are collected here for the sake of future discussions.

Problem 1. (Eckart Viehweg, Kang Zuo)
Let M_h be the moduli scheme of polarized Calabi-Yau n-folds. Does there exists a compact curve C and a non trivial morphism $\varphi : C \to M_h$, which is induced by a smooth family $f : X \to C$, such that C is a rigid Shimura curve, controlling the VHS $R^n f_* \mathbf{Q}_X$.

By taking the quotient of a special family of Abelian surfaces by the involution one finds examples for $n = 2$, i.e. for $K3$-surfaces. In "Families over curves with a strictly maximal Higgs field" we show that there are no such families, for n odd, and that in general, the VHS $R^n f_* \mathbf{Q}_X$ is isometric to one, build up by the weight one VHS of the family of Abelian varieties, parameterized by C. So in a vague way, this problem is related to the question, whether such an isometry has some geometric meaning.

Problem 2. (Eckart Viehweg, Kang Zuo) Find ball quotients and Shimura varieties in the moduli scheme of polarized Calabi-Yau manifolds.

Problem 3. (Eckart Viehweg, Kang Zuo) Let $g : Z \to S \times S'$ be a smooth family of Calabi Yau n-folds. In "Complex multiplication, Griffiths-Yukawa couplings, and rigidity for families of hypersurfaces" we have shown, that there exist polarized complex variation of Hodge structures $\mathbf{V}$ and $\mathbf{V}'$ on S and S', respectively, and a Hodge isometry $R^n g_* \mathbf{C}_Z \cong pr_1^* \mathbf{V} \otimes pr_2^* \mathbf{V}'$. Does such a decomposition exist for $R^n g_* K_Z$ where K is a number field, or perhaps even over $K = \mathbf{Q}$? And, does the existence of such a decomposition has any geometric interpretation?

Problem 4. (James D. Lewis) Let X be an algebraic $K3$ surface, and consider the transcendental regulator

$$\Phi : \mathrm{CH}^2(X, 1) \to \frac{H^{2,0}(X)^\vee}{H_2(X, \mathbf{Z})},$$

where $\mathrm{CH}^2(X, 1)$ is Bloch's higher Chow group. Consider the subgroup V of classes in $\mathrm{CH}^2(X, 1)$ obtained in the following way. Let $C \subset X$ be an irreducible nodal rational curve, $\sigma : \tilde{C} \xrightarrow{\approx} C$ its normalization, with points $P,\ Q \in \tilde{C}$, for which $\sigma(P) = \sigma(Q)$ is a nodal point on C. On $\tilde{C}$ there is a rational function $\tilde{f}$ for which $\mathrm{div}_{\tilde{C}}(\tilde{f}) = P - Q$. Since $\mathbf{C}(\tilde{C}) = \mathbf{C}(C)$, $\tilde{f}$ corresponds to a rational function f on C for which $\mathrm{div}_C(f) = 0$. Then $\xi := \{(f, C)\} \in \mathrm{CH}^2(X, 1)$ defines a higher Chow cycle. If $\omega \in H^{2,0}(X)$, then the value $\Phi(\xi)(\omega)$ is described as follows. Consider $f : C \to \mathbf{P}^1$. Then one can argue that $f^{-1}[0, \infty]$ is a closed 1-cycle that bounds a real 2-dimensional membrane Γ. Then

$$\Phi(\xi)(\omega) = \int_\Gamma \omega \quad (\textit{modulo periods } H_2(X, \mathbf{Z})).$$

What can one say about the image $\Phi(V)$?

Problem 5. (Rolf Schimmrigk) Determine the relation between the conductor of modular Calabi-Yau variety and the level of the corresponding modular form.

Background: Weil's important contribution to the Shimura–Taniyama conjecture was his observation of the relation between the conductor of the modular elliptic curve and the level of the corresponding modular form. Recently it has been shown that many Calabi–Yau varieties are modular in the sense that the Mellin transform of the Hasse–Weil L-functions of the variety are modular forms of weight four at some level N. In order to understand this modularity of Calabi–Yau varieties more systematically, it would be useful to have an generalization of Weil's conductor observation to higher dimensional varieties, even if only conjecturally.

Problem 6. (Rolf Schimmrigk) Establish the relation between Kuga–Sato varieties and modular Calabi–Yau manifolds.

Background: Many Calabi–Yau threefolds have recently been shown to be modular. It has also been known for some time that cusp Hecke eigenforms forms of weight k admit a motivic interpretation in terms of the Kuga–Sato variety (Deligne, Scholl). We therefore encounter a situation which is rather different from the one encountered in the context of elliptic curves and abelian varieties, where it is known from Faltings' proof of the Tate conjecture that varieties over $\mathbf{Q}$ with the same L-function are isogeneous. In the context of Calabi–Yau varieties we have two different varieties with the same modular form. This raises the question what precisely the relation is between such modular Calabi-Yau varieties and the corresponding Kuga–Sato variety, associated to its modular form.

Problem 7. (Klaus Hulek) In [HV] we discuss two rigid (birational) Calabi-Yau varieties called X_1 and X_9 and prove that they are both modular with the same associated modular form f_6 which is the unique normalized newform of weight 4 and level 6. By a conjecture of Tate there should be a correspondence between X_1 and X_9 which explains this fact. We can actually prove that X_1 and X_9 are not birational. It would be interesting to exhibit a correspondence as predicted by Tate's conjecture.

Comment: The Calabi-Yau variety X_1 is also birational to the Barth-Nieto quintic and to Verrill's variety associated to the root lattice A_3 (cf [HSvGvS] and [SY]). The Barth-Nieto quintic has (birationally) a double cover $\tilde{X}_1$ which is again a Calabi-Yau variety with the same modular form. Here too X_1 and $\tilde{X}_1$ are not birational (they have different Euler numbers). But in this case the double cover gives the correspondence required by Tate's conjecture.

The following problem seems to have been posed by several people, and is posted here by Klaus Hulek.

Problem 8. (Hulek, et al.) For which Hecke eigenforms f of weight 4 does there exist a rigid Calabi-Yau variety X such that the L-series of X equals the Mellin transform of f (up to finitely many primes)?

Comment: Hulek became aware of this problem when B. Mazur asked him this question in Oslo in June 2003. He was meanwhile informed by D. van Straten that Straten himself had asked this question to B. van Geemen at an earlier occasion.

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Problems of Andrey Todorov

Introduction. Noriko Yui suggested to pose some problems about CY manifolds. The problems that I suggested are mainly connected to the construction of CY manifolds whose moduli space is a locally symmetric space. Up to now only one example is known, namely Borcea-Voisin manifold. The problems suggested in this note are related to the conjecture of Oort–Andre–Mazur which states that the set of manifolds of CM types are dense in the moduli space if and only if the moduli space is a locally symmetric manifold. This conjecture is related to some questions concerning rational conformal field theories.

A. Locally Symmetric Spaces as Moduli of Polarized CY Manifolds

Problem 9. Characterize all CY threefolds whose moduli spaces are locally symmetric spaces.

B. Gross in [gr] classified all the symmetric domains that are also tube domains and over them one can construct a variations of Hodge structure of weight three with $\dim H^{3,0} = 1$.

Problem 10. This is an open problem posed by B. Gross. Can one find a geometric realization of the variations of the Hodge structure described in [gr]?

Problem 11. Show that the moduli space of CY threefolds that are double cover of $\mathbf{CP}^3$ ramified over eight planes in a general position is a locally symmetric space associated with $\mathbf{SU}(3,3)/\mathbf{S}(\mathbf{U}(3)\times \mathbf{U}(3))$. If the moduli space of CY manifolds that are double covers of $\mathbf{CP}^n$ ramified over $2n+2$ planes is a locally symmetric space then it should be $\mathbf{SU}(n,n)/\mathbf{S}(\mathbf{U}(n)\times\mathbf{U}(n))$.

One can show that the moduli space of a CY threefold is a locally symmetric space of rank greater than or equal to two if and only if the Yukawa coupling has no quantum corrections. It is a well known fact that any moduli space of CY manifolds that is one dimensional is a locally symmetric space and the famous example of Candelas and coauthors shows that there exists a CY manifolds whose moduli space is one dimensional locally symmetric space and there are quantum corrections. In the Candelas example the action of the mapping class group on the upper half plane is not arithmetic. It will be interesting to construct CY manifolds whose moduli space is one dimensional and action of the mapping class group is arithmetic on the upper half plane. I do not know how arithmetically it is related to the existence of quantum corrections to Yukawa coupling.

Problem 12. This problem was also discussed in [Bo]. It was stated that I. Dolgachev conjectured that the moduli space of CY manifolds that are double covers of $\mathbf{CP}^n$ ramified over $2n+2$ planes is the tube domain $S_n(\mathbf{C})+\sqrt{-1}S_n^+(\mathbf{C})$, where $S_n(\mathbf{C})$ is the space of $n\times n$ Hermitian matrices and $S_n^+(\mathbf{C})$ is the space of positive Hermitian matrices.

The basis of proposing Problem 12 is the following lemma:

Lemma: Let $\mathbf{C}^{2n}$ be equipped with a Hermitian metric $\langle u,u\rangle$ with signature (n,n). Let $\mathbf{C}^{2n} = V\oplus\overline{V}$, where $\langle u,u\rangle$ when restricted to V is positive and on $\overline{V}$ is negative. Then $\wedge^n(V\oplus\overline{V})$ is a variation of Hodge Structures of weight n with $\dim_{\mathbf{C}} H^{n,0} = 1$. This variation of Hodge structures is parametrized by $\mathbf{SU}(n,n)/\mathbf{S}(\mathbf{U}(n)\times\mathbf{U}(n))$.

B. CY Manifolds, Conic Singularities and M. Reid Conjecture

It is natural to ask if the generic point of the discriminant locus of an odd dimensional CY manifold corresponds to a manifold with conic singularity. This is not true. Let us take double covers of $\mathbf{CP}^3$ ramified over eight planes in a general position. After the resolution of the singularities we will get a CY threefold. The discriminant locus corresponds to a double covering which is ramified over eight planes three of them meeting in one point. So the generic point of the discriminant locus does not correspond to a threefolds with conic singularities since the monodromy group around these points is finite. The monodromy group of a conic singularity is infinite.

Remark. The homological mirror conjecture predicts that CY threefolds should have always conic singularities. The example of double cover of $\mathbf{CP}^3$ ramified over eight planes in a general position shows that one needs to modify this part of the homological mirror conjecture.

Problem 13. Suppose that M is a CY manifold whose moduli space is not a locally symmetric space. Is it true in this case that the generic point of the discriminant locus corresponds to a CY manifold with a conic singularity?

This problem is closely related to Miles Ried's conjecture that the moduli spaces of all CY threefolds are connected. So one can ask the following question:

Problem 14. Is it true that a CY threefold such that its moduli space is a locally symmetric space then the moduli space is contained in the discriminant locus of the moduli space a CY manifold and the generic point of the discriminant locus corresponds to a manifold with a conic singularity?

C. The Analogue of the Dedekind Eta Function

In [To03] we proved the following Theorem:

Theorem. Let $\mathfrak{M}_L(\mathrm{M})$ be the moduli space of the polarized CY manifold. Let $\omega_{\mathcal{X}/\mathfrak{M}_L(\mathrm{M})}$ be the relative dualizing sheaf on $\mathfrak{M}_L(\mathrm{M})$. Let $\det(\Delta_{\tau,1})$ be the regularized determinant of the Laplacian of the CY metric acting on $(0,1)$ forms. Then locally we have: $\det(\Delta_{\tau,1}) = \langle\omega_\tau, \omega_\tau\rangle\,|\eta|^2$, where ω_τ is a family of holomorphic n forms and η is a holomorphic function.

As a Corollary we get:

Corollary. Let Γ_L is the subgroup in the mapping class group which preserve the polarization class. According to Sullivan and Kazdhan for CY manifolds the group $\Gamma_L/[\Gamma_L,\Gamma_L]$ is finite. (See [Bour].) Let $N = \#\Gamma_L/[\Gamma_L,\Gamma_L]$. Then there exist exists a section η^N of the line bundle $\left(\omega^*_{\mathcal{X}/\mathfrak{M}_L(\mathrm{M})}\right)^{\otimes N}$ such that $\det(\Delta_{\tau,1}) = \langle\omega_\tau, \omega_\tau\rangle\,|\eta|^2$.

Let $\tau \in \mathfrak{M}_L(\mathrm{M})$. Then we know that τ corresponds to a CY threefold M_τ. Let us denote by ω_τ a non-zero holomorphic threeform on M_τ. Let $\beta \in H_3(\mathrm{M},\mathbf{Z})$, then we will denote by

$$\langle\tau,\beta\rangle := \int_\beta \omega_\tau.$$

Problem 15. Can one find a product formula for the analogue of the Dedekind eta function of CY threefolds

$$\eta^N = \exp\left(2\pi\sqrt{-1}\,\langle\gamma,\tau\rangle\right) \times \prod_i \left(1 - \exp 2\pi\sqrt{-1}\,\langle\tau,\beta_i\rangle\right),$$

around points of maximal degenerations, which would mean that around such points the monodromy operator has an index of unipotency $n+1$, β_i are the vanishing invariant cycles of the monodromy operators of infinite order and $\tau = (\tau^1, ..., \tau^N)$ are the flat local coordinates? For a discussion of the product formulas for automorphic forms see [Bo1].

Problem 15 is closely related to paper [BCOV] and more precisely to the counting problem of elliptic curves on the CY threefold.

Problem 16. Prove that $\det\Delta_{\tau,1}$ is bounded on the moduli space $\mathfrak{M}_L(\mathrm{M})$ of any CY manifold M.

This problem will follow directly if one can prove that the coefficients a_k for $k = -n, ..., 1$ of the short term asymptotic expansion

$$Tr\left(\exp\left(-t\Delta_{\tau,1}\right)\right) = \sum_{k=-n}^{1} \frac{a_k}{t^k} + a_0 + ...$$

are constants. We prove that a_0 is a constant if M is a CY. The solution of Problem 16 will show that the analogue of the Dedekind eta function η^N vanishes on the discriminant locus. This will imply that the section $\overline{\eta}^N$ constructed in [To03] will be related to the algebraic discriminant as defined by Gelfand, Kapranov and Zelevinsky. The analogue of the Dedekind eta function for algebraic polarized K3 and Enriques surfaces was discussed in [Bo], [JT95], [JT96], [jt], [JT99] and [Y].

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