The interaction of finite type and Gromov-Witten invariants

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This is the final report on the workshop titled "The interaction of finite type and Gromov-Witten invariants." This was a five-day workshop held at the Banff International Research station from November 15 - November 20, 2003.

Before five years ago, there was no interaction between these two distinct fields. In 1998 Gopakumar and Vafa suggested a relation between the GW invariants and certain (integer) counts of BPS states in M-theory [14]. Shortly thereafter work appeared showing that open string theory on $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$ is equivalent to Chern-Simons theory [1, 29, 24, 23]. This is a duality between Chern-Simons theory and open string theory built on the conjecture of t'Hooft relating large N gauge theories and string theories. These ideas developed very quickly in the theoretical physics literature. The resulting papers suggest strong links between these distinct mathematical areas. The mathematical communities working on finite-type invariants and Gromov-Witten invariants were and still are largely disjoint. The goal of the workshop was to bring together people working on finite-type invariants, people working on Gromov-Witten invariants and physicists who could explain the recent results in physics linking these two areas.

The workshop was an overwhelming success, and is sure to influence the development of mathematics in these two areas. We did succeed in bringing together mathematicians from these two areas and a physicist to tell us why we should interact. We predict that in another five years, the two mathematical disciplines will no longer be separate, but will have many close mathematical connections. Many of the participants at this workshop will contribute to this area of mathematical development, and the mathematical interaction started at this workshop will not just be limited to the participants. We are in the process of preparing a workshop proceedings that will communicate some of the exciting developments in this area to the world-wide mathematical community.

We will now give a brief technical overview of the subject matter covered at the workshop. The first area represented at the workshop was finite-type invariants and Chern-Simons theory. The theory of finite type invariants has roots in mathematical physics, statistical mechanics, operator algebras, topology and singularity theory.

The physical intuition for these invariants was first explained by E. Witten as a Topological quantum field theory based on the Chern-Simons invariant [39]. The invariant of a link in a 3-manifold is described as the path integral,

$$Z_k(M, L) = \int_{\mathcal{A}/\mathcal{G}} e^{\frac{i}{2g_s}\mathrm{CS}(A)} \prod \mathrm{Tr}_{\rho_j}(\mathrm{hol}_A(L_j))DA. \quad (1)$$

Here $g_s = \frac{2\pi}{k+N}$ is the string coupling constant. Motivated by properties that formally follow from the path integral, N. Reshetikhin and V. Turaev gave a mathematical definition of a 3-manifold invariant satisfying the same formal axioms [33]. A slightly weaker invariant was given by V. Turaev and O. Viro [37]. These invariants have been extended and studied see [20].

At the same time that the Witten-Reshetikhin-Turaev invariants were being developed, V. Vassiliev began the theory of finite type invariants by applying techniques from singularity theory to the space of knots [38]. The fundamental concept is the order of an invariant. An invariant, I, is said to have order less than n if

$$\sum_{\sigma}(-1)^{\sum_{k=1}^{n}\sigma_k} I(X_{\sigma_1,\sigma_2\ldots,\sigma_n}) = 0.$$

Here each σ_k is either 0 or 1, and $X_{\sigma_1\ldots\sigma_n}$ is an object obtained from X by making modifications to X at each of the locations labeled with a non-trivial σ. For example, if K is a knot projection with two labeled crossings K_{01} represents the knot obtained by changing the second crossing. The study of finite type invariants is tractable because the space of all order n invariants forms a finite dimensional space. The theory of finite type knot and link invariants was the further developed by J.Birman , X. Lin, and D. Bar-Natan. [5], [3].

Since the path integral of (1) is reminiscent of a Fourier integral operator, it is not surprising that the stationary phase approximation led to important new discoveries. This approximation was studied in [7], [34] and [21]. Higher order terms in the stationary phase approximation were similar to finite type invariants.

The second area represented at this workshop is Gromov-Witten theory. Gromov-Witten (GW) invariants are invariants of a complex projective manifold X (more generally a symplectic manifold) that are invariant under deformations of the Kahler (more generally, almost-Kahler) structure of X. GW invariants are certain integrals over a moduli space of holomorphic maps of Riemann surfaces into X and can consequently often be related to the enumerative geometry of X ("counting curves" on X). They arose in physics as correlators in a certain topological string theory.

The mathematically rigorous foundations of Gromov-Witten theory have been developed over the last 15 years, drawing on techniques from algebraic geometry, geometric analysis, algebra, and topology (see for example [8] and the references therein). However, the ideas from physics have continued to be extremely influential, notably the idea of mirror symmetry. In physics, this is the statement that type IIA string theory compactified on a Calabi-Yau 3-fold M is equivalent to type IIB string theory compactified on a different Calabi-Yau 3-fold W (the "mirror manifold"). This leads to very surprising predictions for the number of rational curves on M in terms of the variation of Hodge structure on W. This aspect of the "mirror conjecture" has been recently proven for a wide class of Calabi-Yau 3-folds using mathematical localization techniques (consequently giving an understanding of the mirror symmetry phenomenon that is very different from the physics "proofs") [13, 28, 4].

The physically conjectured relationship between Chern-Simons theory and Gromov-Witten theory may be expressed using the free energy. For simplicity we restrict to the case of S^3. The Chern-Simons free energy is by definition,

$$F^{cs}(N, g_s) = \ln Z = \sum F_\Gamma W_{U(N)}(\Gamma) g_s^n(\Gamma).$$

Here Z is the partition function defined by the path integral (1), and the sum on the right arises from the stationary phase expansion of Z. The sum is taken over a collection of trivalent graphs, f_Γ are rational numbers that depend on the graph, $W_{U(N)}(\Gamma)$ is the $U(N)$ weight system applied to the graph and $n(\Gamma)$ is half the number of vertices of the graph. The weight system may be expressed as a sum of terms depending on the number of boundary components of surfaces obtained by thickening the graphs. These surfaces with boundary may be interpreted as open strings, however there is a way to combine the open strings into closed strings. Introduce a formal paramater, $t = g_s N$. Using this parameter the free energy may be expressed as $F^{cs}(t) = \sum_{g=0}^{\infty} F_g(t) g_s^{2g-2}$. This has the form of the free energy of a string theory. The relevant string theory is constructed via the "Conifold transition." The complex structure on the cotangent bundle of S^3 has a certain deformation whose limit is a complex 3-fold having an isolated singularity; this is the so-called "conifold". This singularity has a crepant resolution which is isomorphic to the total space of the bundle $X = \mathcal{O}(-1) \oplus \mathcal{O}(-1)$ on $\mathbb{P}^1$. Chern-Simons theory on S^3 is equivalent to Gromov-Witten theory on X. The moduli space of holomorphic maps of genus g in homology class β has a zero dimensional $\mathbb{Q}$-virtual class. The degree of this class a rational number, $N_{g,\beta}$. The Gromov-Witten free energy is then given by, $F^{GW}(t) = \sum_{g,\beta} N_{g,\beta} e^{t\beta} g_s^{2g-2}$. The most basic prediction of Chern-Simons String duality is that $F^{cs}(t) = F^{GW}(t)$.

The core of our workshop consisted of 23 invited 45-minute scientific talks and 16 15-minute question answer sessions.

Marcos Marino is a theoretical physicist in the high-energy theory department at CERN. He gave a series of four lectures on the physical arguments for Chern Simons string duality. His lectures were titled: CS invariants in the $1/N$ expansion, CS invarinats on S^3 knot theory and enumerative geometry, Extension to other 3-manifolds, and Gromov-Witten theory of CY manifolds and CS theory.

Hans Wenzl is a mathematician at UC San Diego. He gave a pair of lectures titled: Knot invariants from Quantum groups. In his talks, he reviewed the construction of topological invariants from modular tensor categories, and the construction of such categories from quantum groups.

Chiu-Chu (Melissa) Liu is a mathematician at Harvard. She gave three lectures. Her lectures were titled Open Gromov-Witten theory, Virtual localization, and Formulas of one-partition and two partition Hodge integrals. Her talks described how to define open GW invariants in the S^1-equivariant case, how to apply localization to compute GW invariants, and provided a proof of conjectured formula of Marino-Vafa and Zhou that provide further evidence for CS string duality.

Justin Sawon is a mathematician at SUNY at Stony Brook. He gave a talk titled: Perturbative expansion of Chern-Simons theory. His talk covered the step from path integrals to Feynman diagrams and trivalent graphs.

Jun Li is a mathematician at Stanford. He gave a pair of lectures. His lectures described relative Gromov-Witten invariants.

Dror Bar-Natan is a mathematician at the University of Toronto. He gave a pair of lectures titled: Introduction to Perturbative Chern-Simons theory, and Indroduction to Khovanov Homology. His first talk covered the role of trivalent graphs in three related areas - Lie theory, perturbative CS theory, and Vassiliev theory. His second talk introduced Khovanov homology for tangles. The q-Euler characteristic of Khovanov homology is the Jones polynomial. One surprise of this workshop was the physical conjecture of a new interpretation of Khovanov homology that Marino gave in his lectures.

Conan Leung is a mathematician at the University of Minnesota. He gave a lecture on Branes and Instantons for vector cross products.

Stavros Garoufalidis is a mathematician at the Georgia Institute of Technology. He gave a lecture titled BPS invariants of links and a conjecture of Labastida-Marino-Ooguri-Vafa. Chern-Simons string duality predicts a relation between Link invariants and Gromov-Witten invariants. The resulting Gromov-Witten invariants should be related to BPS invariants (integer counts of embedded holomorphic curves) according to the Gopakumar-Vafa conjecture. The resulting integrality predictions for link invariants were expressed as the LMOV conjecture. Garoufalidis gave a proof of this conjecture.

Michael Hutchings is a mathematician at UC Berkeley. He gave a talk titled: The embedded contact homology of T^3.

Jozef Przytycki is a mathematician at George Washington University. He gave a talk titled Khovanov homology of tangles and I-bundles over surfaces.

Justin Roberts is a mathematician at UC San Diego. He gave a talk on Rozanski-Witten invariants.

Takashi Kimura is a mathematician at Boston University. He gave a talk titled: Admissible covers, equivariant topological field theories, and orbifolding.

Jacob Shapiro is a mathematician at the University of British Columbia. He gave a talk titled: On the Gopakumar-Vafa conjecture for a local elliptic K3 surface.

Lenny Ng is a mathematician at AIM. He gave a talk on contact homology of Ooguri-Vafa Lagrangians.

The participants at the workshop had many favorable things to say about the Banff International Research Station and the workshop.

Dagan Karp learned exciting new mathematics that is not available elsewhere, and began a new project/collaboration investigating one object from the point of view of Gromov-Witten theory, open Gromov-Witten theory, and Chern-Simons theory.

David Gay came away with two new research projects. He intends to study the Lagrangians in $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$ corresponding to various knots. He also intends on studying the relationship between G2 geometry and the degeneration of closed 2-forms along circles.

Hans Boden and Chris Herald proved a new theorem related to their ongoing project on Casson invariants.

Justin Roberts thought the timing of the meeting was perfect. He also thought having just one physicist was the right thing to do because it forced the physicist to interact with the mathematicians.

Kai Behrend said "[The workshop] is excellent. I'm getting lots of new ideas.

Melissa Liu commented that the subject of the workshop was directly related to her research and that she got a lot out of it.

Everyone commented about how spectacular the center was. Setting up a mathematics institute at an art center is a great way to encourage creativity in the mathematical sciences.

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