Variance of Quasi-Coherent Torsion Cousin Complexes

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Grothendieck Duality is a subject having numerous applications in Algebraic Geometry, as well as its own intrinsic attractiveness. The basic ideas are well known, but because of the underlying complexity in the details, the situation with respect to full expositions is not yet entirely satisfactory.

The participants in this program have been working on some foundational matters in the area, with the intention of publishing a small book, now nearing completion, containing three separate papers. This volume constitutes a reworking of the main parts of Chapters VI and VII in Hartshorne's "Residues and Duality" [7], in greater generality, and by a local, rather than global, approach.

"Greater generality" signifies that we work throughout with arbitrary (quasi-coherent, torsion) Cousin complexes on (noetherian) formal schemes, not just with residual complexes on ordinary schemes. And what emerges at the end is a duality pseudofunctor on the category of composites of compactifiable maps between those formal schemes which admit dualizing complexes.[1]

"Local approach" signifies that the compatibilities between certain pseudofunctors associated to smooth maps on the one hand and to closed immersions on the other (base-change and residue isomorphisms. . .), compatibilities which underly the basic process of pasting together these two pseudofunctors, are treated by means of explicitly-defined—through formulas involving generalized fractions—maps between local cohomology modules over commutative rings. This way of dealing with compatibilities seems to us to have advantages over the classical one, not least of which is that the connection between local and global behaviors is made transparent, the latter being defined entirely in terms of the former. In regard to relative complexity, one might for instance compare Chapter 6 of [8], where the compatibilities we need are taken care of, with [2, Chap. 2, §7], where the compatibilities needed in the global approach of [7, Chap. VI, §2] are discussed. (To follow the global approach, one would have to redo everything for formal schemes, with the added complication introduced by the necessary presence of the derived torsion functor.)

The papers in this volume continue efforts, begun in [1], to generalize all of Grothendieck duality theory to noetherian formal schemes. Why formal schemes (aside from their just being there)? For one thing, the category F of formal schemes contains the category of ordinary schemes, that is, formal schemes whose structure sheaf has the discrete topology. Also, F contains the opposite category of the category of local homomorphisms of complete noetherian local rings. Thus the category of formal schemes offers, potentially, a framework for treating local and global duality results as aspects of a single theory.

In [1], the fundamental duality and flat base change theorems are proved for *pseudo-proper* formal-scheme maps. The most notable obstruction to dealing with more general separated pseudo-finite-type maps is that we know of no theorem to the effect that such a map is *compactifiable,* that

[1]Nagata showed that every separated finite-type map of (noetherian) schemes is compactifiable; this is not known to be so for formal schemes, and seems likely to be false.

is, factors as an open immersion followed by a pseudo-proper map. Nevertheless, we can still work with those pseudo-finite separated formal-scheme maps which can be built up from pseudo-proper maps and open immersions, i.e., consider the subcategory F^0 of F having the same objects, but only those maps which are compositions of compactifiable ones. The category F^0 includes all separated finite-type maps of ordinary noetherian schemes, since, by the above-mentioned theorem of Nagata, they are compactifiable. And indeed, we are able to extend the main theorem in [7] to F^0, as follows.

A basic problem is to paste together, in a natural way, the above pseudofunctor, denoted $(-)^!$, for pseudo-proper maps and the inverse image pseudofunctor $(-)^*$ on the category of open immersions into a pseudofunctor, still denoted $(-)^!$, on all of F^0. One would like to have a natural abstract pasting procedure in the spirit of Prop. 3.3.4 in [4, p. 318], a Proposition which, as indicated before, applies to ordinary schemes, but which cannot be applied to formal schemes because we don't know that the composition of two compactifiable maps is still compactifiable.

Nayak's paper "Pasting pseudofunctors and Grothendieck duality," provides *an applicable such procedure.*

Sastry's paper "Duality for Cousin complexes" provides, in many situations (see below), a *concrete, canonical realization of the pseudofunctor* $(-)^!$.

The approach taken overlaps—and was inspired by—that in [7, Chap. 7], but it is both more concrete and more general. It begins with the canonical pseudofunctor $(-)^\sharp$ to whose construction the joint paper "Pseudofunctorial behavior of Cousin complexes on formal schemes" of Lipman, Nayak and Sastry is devoted. Roughly speaking, $(-)^\sharp$ is defined over a suitable category $\mathbb{F}_c$ of formal schemes X with codimension functions Δ, assigning to each object (X, Δ) the category $\mathrm{Coz}_\Delta(X)$ of quasi-coherent torsion Δ-Cousin $\mathcal{O}_X$-complexes.

Briefly, having in mind that $(-)^\sharp$ is meant to be a concrete approximation to $(-)^!$, one first describes the functor $f^\sharp$ for f a closed immersion or a smooth map, by "Cousinifying" the usual concrete realizations (extended to formal schemes) in these cases. Then, noting that every $\mathbb{F}_c$-map factors locally as (smooth) $\circ$ (closed immersion), one defines $(-)^\sharp$ for such factorizable maps by pasting. All this is done canonically, so finally it is possible to define $(-)^\sharp$ globally by gluing the local definitions. Carrying this all out involves careful attention to a great many details, a good portion of which have already been dealt with by Huang in [8], where he constructed, in essence, the restriction of $(-)^\sharp$ to Cousin complexes with vanishing differentials.

In [7, Chap. 6, §3], Hartshorne describes the construction of a pseudofunctor $(-)^\Delta$ on residual complexes over noetherian schemes (i.e., those Cousin complexes which are "pointwise dualizing"). See also [2, §3.2]. Our pseudofunctor $(-)^\sharp$ is more general, because it operates on a larger class of Cousin complexes, and over formal schemes, but each $f^\sharp$ does take residual complexes to residual complexes. It should be said, however, that the basic elements of the strategy for constructing $(-)^\sharp$, as outlined in the preceding paragraph, can all be found in [7].

Let us return to Sastry's paper. The proof of the Duality Theorem in [7, Chapter 7] begins with a trace map $f_* f^\Delta K \to K$, of graded modules, defined when $f\colon X \to Y$ is a finite-type map of noetherian schemes and K is a residual $\mathcal{O}_Y$-complex. What is called there the Residue Theorem states that when the map f is proper, trace is a map of complexes. Using local residues, Sastry defines, for every $\mathbb{F}_c$-map $f\colon (X, \Delta_1) \to (Y, \Delta)$ and every Δ-Cousin $\mathcal{O}_Y$-complex F, a functorial trace

$$\mathrm{Tr}_f(F)\colon f_* f^\sharp F \to F\,;$$

and proves: *for pseudo-proper* f, $\mathrm{Tr}_f(F)$ *is a map of complexes* (Trace Theorem).

Via the basic properties of the functor $f^!$ constructed by Nayak (see above) for any composition $f\colon X \to Y$ of compactifiable maps, the Trace Theorem enables the construction of a canonical pseudo-functorial derived-category map

$$\gamma^!_f(F)\colon f^\sharp F \to f^! F \qquad \big(F \in \mathrm{Coz}_\Delta(Y)\big).$$

Applying the usual Cousin functor E makes this an *isomorphism* $f^\sharp F \cong E(f^! F)$. Moreover, $\gamma^!_f$ itself is an isomorphism whenever f is flat or F is an injective complex. One finds then, with Q the canonical functor from the category of complexes to the derived category, that if one restricts to flat maps and Cohen-Macaulay complexes (the derived-category complexes isomorphic to $Q(C)$ for some Cousin complex C), or to Gorenstein complexes (the derived-category complexes isomorphic to $Q(C)$ for some *injective* Cousin complex C), then, $Qf^\sharp E$ *is a pseudofunctor satisfying the expected conditions for a duality pseudofunctor.* Using $\gamma^!_f$, Sastry also proves a *canonical Duality Theorem* for pseudo-proper maps $f\colon (X,\Delta')\to(Y,\Delta)$ and Δ-Cousin $\mathcal{O}_Y$-complexes F: *the pair* $\big(f^\sharp F, \mathrm{Tr}_f(F\,)\big)$ *represents the functor* $\mathrm{Hom}_Y(f_*C,\ F\,)$ *of* Δ'*-Cousin* $\mathcal{O}_X$*-complexes* C.

In summary, $f^\sharp$ *is a canonical concrete approximation to the duality functor* $f^!$.

Finally, the canonicity of $\gamma^!_f$ and uniqueness properties of residual complexes enable one to draw closer to the holy grail of defining canonically a duality pseudofunctor $(-)^!$ for all pseudo-finite-type maps $f\colon X\to Y$, at least in the presence of bounded residual complexes (or equivalently, dualizing complexes), and under suitable coherence hypotheses. The idea, taken from [7], is to define $f^!$ as being dualization on Y with respect to a fixed residual complex $\mathcal{R}_Y$ (i.e., application of the functor $\mathcal{H}om^\bullet_Y(-,\mathcal{R})$), followed by $\mathbf{L}f^*$, followed by dualization on X with respect to the residual complex $f^\sharp\mathcal{R}$.

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References

[1] L. Alonso Tarrío, A. Jeremías López, and J. Lipman, Duality and flat base change on formal schemes. In *Studies in Duality on Noetherian formal schemes and non-Noetherian ordinary schemes,* Contemporary Mathematics, **244**, Amer. Math. Soc., Providence, RI, 1999, 1–87. See also Correction. . . , Proc. Amer. Math. Soc. **131** (2003), no. 2, 351–357.

[2] B. Conrad, *Grothendieck duality and base change,* Lecture Notes in Mathematics, **1750**, Springer-Verlag, New York, 2000.

[3] B. Conrad, *Deligne' notes on Nagata compactifications,* item **5** at < `http://www-math.mit.edu/~dejong/#brian` >

[4] P. Deligne, Cohomologie à supports propres. In *SGA 4 Tome 3,* Lecture Notes in Mathematics bf 305, 250–461, Springer-Verlag, New York, 1973.

[5] R. Hübl and P. Sastry, Regular differential forms and relative duality, *Amer. J. Math.* **115** (1993), no. 4, 749–787.

[6] A. Neeman, The Grothendieck duality theorem via Bousfield's techniques and Brown representability, *J. Amer. Math. Soc.* **9** (1996), 205–236.

[7] R. Hartshorne, *Residues and duality,* Lecture Notes in Math., **20**, Springer-Verlag, New York, 1966.

[8] I. C. Huang, *Pseudofunctors on modules with zero-dimensional support,* Memoirs Amer. Math. Soc. **114** (1995), no. **548**.

[9] W. Lütkebohmert, On compactification of schemes, *Manuscr. Math.* **80** (1993), no. 1, 95–111.

[10] J. L. Verdier, Base change for twisted inverse image of coherent sheaves. In *Algebraic Geometry,* 393–408, Oxford Univ. Press, London, 1969.

[11] A. Yekutieli, Smooth formal embeddings and the residue complex, *Canadian J. Math.* **50** (1998), no. 4, 863–896.