Noncommutative Geometry and Noncommutative Invariant Theory: 22w5084

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Sept. 25, 2022 – Sept. 30, 2022

We are grateful to BIRS for supporting this five-day workshop, which brought together 22 in-person researchers (with an additional 22 virtual participants). The participants included a mix of PhD students, postdocs, and senior researchers from North America, Europe, and Asia. As a group, the participants had diverse research interests, which include noncommutative geometry, representation theory, the study of Hopf algebra actions, quantum groups, and other areas.

Overview of the field

Noncommutative algebra is a rich and diverse field that has influences rooted in algebraic and differential geometry, representation theory, algebraic combinatorics, mathematical physics, and other areas. Rapid developments in noncommutative algebra, especially in noncommutative algebraic geometry, influence many other mathematical disciplines. We briefly give an overview of some of the main trends that shape our field at this time, with an emphasis on the areas represented during the workshop.

In terms of the scope of the meeting, the main focus involved the following four connected sub-areas, for which we now give a quick overview.

(a) **Noncommutative Invariant Theory**

Classical Invariant Theory enjoys a long history, beginning with seminal work of Cayley, Hilbert, E. Noether, and others. Here, one examines subrings of polynomials that remain invariant under group actions. Much of this theory can be extended to the quantum setting where the polynomial ring is replaced by a suitable noncommutative analogue (typically an Artin-Schelter regular algebra or the quantized coordinate ring of a variety). One can also work more generally with actions of Hopf algebras instead of restricting to group actions. The key motivating question in this field is then: *how much of the classical theory can be lifted to this general setting?* This field has been a hugely active area of study and has undergone rapid advancement over the past decade, with many important results being proved in this time (see, for example, [27, 28, 29]). Key recent results in this area of research include work of Ferraro, Kirkman, Moore, and Peng [31], which looks at obtaining Noether's bound for noncommutative rings, and a noncommutative version of Knorrer's periodicity theorem by Conner, Kirkman, Moore, and Walton [25] (see also the work of Mori and Ueyama [46]). Moreover, noncommutative Kleinian singularities were defined recently in work of Chan, Kirkman, Walton, and Zhang [19, 20] via their development of a noncommutative McKay correspondence. Also, a version of Auslander's theorem that arises in the McKay correspondence has been studied in the noncommutative setting by several authors, including the work of Bao, He, and Zhang [6]; the work of Gaddis, Kirkman, Moore, and Won [32]; the work of Crawford [26]; and the work of Buchweitz, Faber, and Ingalls [12].

(b) **Artin-Schelter Regular Algebras: classification and applications**

Artin-Schelter regular (often abbreviated as AS regular) algebras are in some natural sense noncommutative analogues of polynomial algebras, and for this reason they play an integral role in both noncommutative invariant theory (see, for example, [38]) and in noncommutative geometry. In the three-dimensional case, these algebras were classified by Artin and Schelter [2] and Artin, Tate, Van den Bergh [5], and the resulting algebras have since appeared within many different contexts, including questions in noncommutative invariant theory and questions about the classification problem for Artin-Schelter regular algebras of dimension. In this case, some partial progress has been made by Lu, Palmieri, Wu, and Zhang [45], using techniques involving the A_∞-Koszul Dual.

New examples of Artin-Schelter regular algebras have also been given by Chirvasitu, Kanda, and Smith [21] and by Cassidy and Vancliff [14], in which the latter authors use skew Clifford algebras, with the aim of creating new AS-regular algebras. Further new interesting directions have also recently emerged. In particular, the noncommutative Zariski cancellation problem is now an active area of research with work of Bell and Zhang [8]; Lezama, Wang, and Zhang [42]; Bell, Hamidizadeh, Huang, and Venegas [7]; and Tang, Venegas, and Zhang [54], who have considered the classical Zariski cancellation problem in the more general context of Artin-Schelter regular algebras. Recent work of Walton and Zhang [56] looks at Artin-Schelter regularity of the quadratic dual of the Fomin-Kirillov algebras, which has important connections to algebraic combinatorics and provides a new approach to resolving a twenty-year-old conjecture of Fomin and Kirillov.

(c) **Universal Quantum Groups**

Given an algebra, it is natural to ask is whether there is a universal object that controls the symmetries of the algebra. In the context of Hopf algebra coactions, this entity is a universal quantum group and these have proved to be profoundly useful in both the purely algebraic and C^*-algebraic settings. Many of these quantum groups that arise can in fact be realized as deformations of coordinate rings of affine algebraic groups, and can thus be shown to possess good algebraic properties. Recent work in this direction includes that of Chirvasitu, Walton, Wang [24]; that of Huang, Walton, Wicks, and Won [37]; and of Chakraborty and Saurabh [15], which studies the Gelfand-Kirillov dimension of cosemisimple universal quantum groups.

(d) **Noncommutative projective geometry and application of geometric methods in noncommutative algebra**

Noncommutative projective geometry first emerged as a distinct discipline during the late '80s, as part of the work of Artin, Schelter, Stafford, Tate, Van den Bergh, Zhang and others [4, 5, 2, 3, 57], who applied techniques from algebraic geometry to understand the Sklyanin algebras and related rings. The field has grown significantly since these beginnings and there have been many striking applications of geometric methods in noncommutative algebra over the past ten years, including work of Sierra and Walton [53] proving that the enveloping algebra of the Witt Lie algebra is non-noetherian; work of Chan and Ingalls [17, 18] on the "noncommutative minimal model program," work of Chirvasitu, Kanda, and Smith [21, 22, 23], and work of Rogalski, Sierra, and Stafford [48, 49, 50, 51] on the theory of birationally commutative surfaces. The chief open problem in the area is Artin's Conjecture [1], which gives a proposed birational classification of all noncommutative surfaces, and important recent progress towards this conjecture has been made by Faber, Ingalls, Okawa, and Satriano [30].

Presentation Highlights

The lectures delivered over the week of the workshop were uniformly of high quality, and many of the participants shared their experiences with us. We now include some of the highlights from these presentations. In particular, we are indebted to Lucas Buzaglo, Fabio Calderón, Jason Gaddis, Hongdi Huang, Frank Moore, Van Nguyen, Manny Reyes, Kent Vashaw, Padmini Veerapen, and Xingting Wang for sharing their thoughts with us about several presentations from the workshop.

- The opening talk of the workshop was delivered by James Zhang, who gave a particularly inspiring survey of open problems. The stated criteria for problems to be included in the list (aside from being of personal interest to the speaker) were: that the problems have significant consequences if they are answered, that they be motivated by fields of mathematics outside of noncommutative algebra, and that they have a strong influence on the development of the subject. It is noteworthy that most of the problems listed were proposed after 2000.

  After an opening discussion of important sources of noncommutative algebra, the first groups of problems centred on topics related to Hopf algebras. First came several problems about the classification of Hopf algebras of small GK-dimension. The next group of questions were clustered around homologically-flavoured questions related to Hopf algebras. This included the Brown-Goodearl conjecture of whether an affine noetherian Hopf algebra has finite injective dimension, along with several variants thereof. It also included the Etingof-Ostrik conjecture about finite generation of Ext-algebras of finite tensor categories, which includes as a special case the finite generation of the Ext-algebra of a Hopf algebra.

  At this point the talk shifted to questions related to Artin-Schelter (AS) regular algebras. The first set of questions related to actions of Hopf algebras on AS regular algebras. This included questions about when the associated Auslander map is an isomorphism, about classification of Hopf algebras acting on a given AS regular algebra, and classification of AS regular algebras acted upon by a given Hopf algebra. Then questions related to the Ozone group (recently defined by Chan, Gaddis, Won, and Zhang) of a AS regular algebra satisfying a polynomial identity were discussed.

  The survey concluded by recounting Artin's Conjecture [1] about the classification of finitely generated division algebras of transcendence degree 2 and providing a long list of questions that were not able to be discussed in the allotted time.

  This excellent talk will serve as a rich source of kindling to fuel a wide range of future research within the noncommutative algebra community for many years to come.

- Ellen Kirkman also gave an excellent talk, titled "Homological Regularities," based on her recent joint work with Robert Won and James Zhang [40]. Her talk gave an overview of homological measures for noncommutative graded algebras, especially those pertaining to regularity. The various measures highlighted were "Tor-regularity," "Castelnuovo-Mumford (CM) regularity," "AS regularity," "concavity," and weighted versions, with the three latter invariants being introduced by Kirkman, Won, and Zhang [40].

  In particular, Tor-regularity, first studied by Jørgensen, is used to study how far an algebra is being Koszul. One can also consider local cohomology modules and Castelnuovo-Mumford (CM) regularity to study Koszulity. These measures have a powerful application in finding bounds on degrees of generators of invariant rings in noncommutative invariant theory, particularly for actions of semisimple Hopf algebras acting on Artin-Schelter regular algebras.

  A new measure, called "AS regularity", can also be defined in terms of Tor-regularity and CM-regularity, equal to zero precisely when the algebra is AS-regular. In fact, there is an example of a 3-Koszul Artin-Schelter regular algebra that has Tor-regularity 1 and has CM-regularity -1, motivating the definition of AS regularity as being the sum of the two measures. The next measure introduced in the talk, "concavity," is given in terms of CM-regularity, and is zero for an noetherian Artin-Schelter regular algebra precisely when the algebra is Koszul. The name pertains to how far the corresponding noncommutative space is from being flat. Concavity can be used to show when certain noetherian Artin-Schelter regular algebras can be an invariant ring from an action of a semisimple Hopf algebra.

Lastly, weighted versions of both Tor-regularity and CM-regularity were used to refine the measures above. One application is that it provides a measure of how far a Veronese subring is from being Koszul. We expect that there will be *numerous* other applications of the measures introduced in Kirkman-Won-Zhang's work to understand noncommutative graded algebras in the near future.

- Xingting Wang gave a talk, "Twists of graded Poisson algebras and related properties," which described his joint work with Xin Tang and James Zhang [55]. Wang began by defining when a set of graded derivations of a graded Poisson algebra forms what he calls a "Poisson Twisting System." The talk presented the implications of the existence of such systems and gave methods for their construction for multivariate Poisson polynomial algebras.

  The first result presented made use of the notion of divergence of a smooth Poisson algebra, with an explicit formula given for it in the multivariate Poisson polynomial case; this led to the introduction of "semi-Poisson derivations," which are in turn a generalization of the classical notion of Poisson derivations. Importantly, the authors show that these new objects, much like their classical counterparts, form a Lie algebra. From here, Wang presented necessary and sufficient conditions on a set of graded derivations of a multivariate Poisson polynomial algebra to form a Poisson twisting system, which yield new Poisson algebras and which the authors call the "twisted Poisson algebras" associated to multivariate Poisson polynomial algebras.

  Using the theory of semi-Poisson derivations, Wang then defined the notion of "rigidity" for graded Poisson algebras, which in some sense measures how far an algebra is from being unimodular (that is, having trivial modular derivation) and showed implications of rigidity taking certain values. Wang ended the talk with a specific example: the Poisson polynomial algebra on three variables (together with a unimodular Poisson structure induced by a cubic polynomial) and illustrated how the concepts introduced in the talk relate to the Poisson centre, rigidity, and the Ozone group.

- Michael Wemyss gave an enlightening talk, "Local forms of noncommutative functions," which discussed his recent joint work with Gavin Brown [9]. Wemyss' talk focused on how one can adapt singularity theory to the noncommutative setting. In the commutative case, when studying isolated singularities, one often works with the completed local ring, invokes the Cohen Structure theorem, and then works with a multivariate power series algebra. A primary goal in this context is to study the classification problem among elements in this ring, or to determine when such a classification is impossible. Simple singularities are classified according to the classical ADE classification.

  Working in the more general noncommutative setting, one can now consider the formal noncommutative power series ring; i.e., the completion of the free associative algebra, On first glance, one might naively assume that this algebra is highly pathological. Surprisingly, Weymss makes the case that the process of completion introduces a huge group of units, and since these units are typically non-central, one can use inner automorphism to make changes of variables, which enable Brown and Weymss to obtain a striking analogue of the commutative theory. The classification problem in this setting is to determine when two elements of the completed free algebra are isomorphic; that is, when their corresponding Jacobi algebras are isomorphic. The Jacobi algebra is a local ring and by studying the growth rate of the filtration corresponding to its Jacobson radical, one obtains a new invariant, which is better behaved than Gelfand-Kirillov dimension in this setting.

Wemyss proposes that a classification is both possible and is ADE, and he uses the theory of preprojective algebras to directly relate the general problem to the classical ADE classification. Wemyss also argued that these "noncommutative function algebras" arise naturally through a theory of contractions and he defines a contraction algebra. Weymss finished by stating conjectures regarding how the classification of contraction algebras should correspond to that of Jacobi algebras. These problems will undoubtedly lead to further investigations and should reveal additional surprising connections between noncommutative algebra, algebraic geometry, deformation theory, and representation theory.

- Wendy Lowen gave a talk entitled "Enriching the nerve construction," based on her joint with Arne Mertens [44], which links noncommutative geometry and higher category theory, via the nerve functor. Her talk looked at quasi-categories over a monoidal category (e.g., the category of modules over a commutative ring), which are in some sense relaxations of dg-categories. The chief underlying question investigated during this talk was: *for linear categories can we define a nerve taking values in modules rather than sets?*

  The solution, starting with a monoidal category, is to construct a certain quiver on that category, which leads to the theory of templicial objects. From here, one can define a nerve functor from a category over the monoidal category into the category of templicial objects. The main theorem presented showed that this construction is compatible with the classical setting, and several concrete examples with diagrams were given during this talk, which provided additional motivation and provided greater understanding of the ideas involved in this work.

- Daniel Chan gave an inspiring talk on the noncommutative minimal model program for orders on surfaces. In the setting of classical algebraic geometry, the minimal model program (MMP), introduced by Mori in the late '80s, is a significant organizing paradigm, which has since had a revolutionary impact on the study of threefolds and higher-dimensional algebraic geometry. Mori's original motivation was to construct a birational model of a complex projective variety which is as simple as possible (or, in some natural sense, minimal), and has its origins in the work of Enriques and Castelnuovo giving a birational classification of surfaces.

  A Noncommutative version of this program was initiated by Chan and Ingalls [17] for orders on surfaces over a field (geometric surfaces) and by Chan et al. [16] in higher dimensions. Chan discussed joint work with Ingalls [18] on the MMP for orders on arithmetic surfaces X. This noncommutative theory can also be viewed purely algebro-geometrically, as algebraic geometry enriched by a Brauer class β, and Chan adopted this point of view during his talk. In this abstract setting, an arithmetic surface is taken to be a normal, separated, integral two-dimensional excellent scheme X which is quasi-projective over a noetherian affine scheme and has finite residue fields. Chan and Ingalls show in the case of prime index Brauer class that terminal resolutions exist. In addition, Chan and Ingalls give a classification of terminal singularities and Castelnuovo contractions where surprising new phenomena show up. Whereas for both the commutative theory of arithmetic surfaces and for orders over geometric surfaces, the theory closely mimics the classical situation arising in the work of Castelnuovo and Enriques, in this broader setting we find that if $\beta \neq 0$, regularity of X is neither a sufficient nor necessary condition for being terminal, and Castelnuovo contractions may or may not correspond to blowing up closed points even when X is regular.

- Charlotte Ure gave the talk "Twisting Comodules and Preregular Forms," which gave an overview of her interesting joint work with Hongdi Huang, Van Nguyen, Kent Vashaw, Padmini Veerapen, and Xingting Wang [34]. Given a graded algebra A, there are numerous ways to change (or "twist") the multiplicative structure to produce a new algebra defined on the same underlying vector space. The most common such technique is via a graded automorphism and is commonly called the Zhang twist [57]. In the case when H is a Hopf algebra, one may also twist the multiplication structures of H using a 2-cocycle defined on H. Ure's talk analyzed the effect performing such twists has on several important constructions from (co)representation theory and classification problems. More precisely, Manin introduced the universal quantum group of A, which is a Hopf algebra that universally (right) coacts on an algebra A. The authors' first result is that the twist of the universal quantum group of A by a 2-cocycle is the universal quantum group of a cocycle twist of A by the *inverse* of the original 2-cocyle.

Additionally, Dubois-Violette showed that every N-Koszul Artin-Schelter regular algebra can be expressed as a derivation-quotient algebra. Motivated by this result, the authors show that the Zhang twist of such a derivation-quotient algebra can again be expressed as a derivation-quotient algebra with respect to a twisted version of the subspace used to define the original algebra. There were several other similar results discussed in the talk, which illuminated the behaviour of fundamental constructions under twisting operations. These new results will certainly be useful to researchers in the area.

- Robert Won, gave a presentation, titled "PI skew polynomial rings and their centers," which discussed currently ongoing work with Kenneth Chan, Jason Gaddis, and James Zhang. This talk gave interesting new results in noncommutative invariant ring theory, with a view towards understanding the centre of a skew polynomial ring S that is an Artin-Schelter regular algebra of finite global dimension and finite Gelfand-Kirillov dimension. This is approached via the study of the *Ozone group* of S, a subgroup of the full graded automorphism group consisting of those that are inner. The authors show that the Ozone group has a nice description involving only diagonal automorphisms. By its definition, the Ozone group fixes the centre of S and so properties possessed by the centre can be reinterpreted in the setting of noncommutative invariant theory.

  Kirkman, Kuzmanovich and Zhang [39] earlier had proved several invariant properties under group actions. Won's talk developed a parallel theory for the invariant theory of skew polynomial rings in relation to the Ozone group. In particular, the centre of such an algebra is regular if and only if the Ozone group is generated by reflections; and the centre is Gorenstein if and only if the Ozone group, modulo the normal subgroup generated by reflections, acts with trivial homological determinant.

  To give sufficient conditions for these desirable properties to hold, the authors introduce new algebraic invariants, including "Ozone Jacobians," "Ozone arrangements," and "Ozone discriminants." These notions coincide precisely with the corresponding notions of the Jacobian, arrangement, and the discriminant when the center is regular, introduced by Kirkman and Zhang [41]. These new invariants are then used to characterize explicitly when the Auslander map is an isomorphism, when the centre is Gorenstein, and when the centre is regular. These are among the most desirable algebraic properties in the context of noncommutative invariant theory.

  These new Ozone invariants will inevitably be extremely useful in both theoretical and computational settings when understanding Artin-Schelter regular algebras satisfying a polynomial identity.

- Lucas Buzaglo, a PhD student at the University of Edinburgh, gave a fascinating talk on the universal enveloping algebras of Krichever-Novikov algebras [13]. The theory of enveloping algebras of finite-dimensional Lie algebras is today well-understood, due to classical work of Dixmier and others. For infinite-dimensional Lie algebras, however, the theory of their enveloping algebras is murkier and little is known outside of some important families. A fundamental question in this vein is whether the enveloping algebra of an infinite-dimensional Lie algebra is necessarily non-noetherian. For years, the best hope for a potential counterexample was the enveloping algebra of the positive part of the Witt Lie algebra; this was shown to be non-noetherian, however, by Sierra and Walton [53] via methods from noncommutative projective geometry.

  An important class of Lie algebras comes from taking an irreducible complex affine variety V of positive dimension and looking at its vector fields. Bugzalo shows that enveloping algebras for all

Lie algebras arising in this manner are necessarily non-noetherian. Remarkably, the question reduces completely to the case when V is a curve, which can in turn be handled by reducing to the work of Sierra and Walton via a sequence of reductions making use of clever faithful flatness arguments.

- Van Nguyen gave the talk "Tensor representations of finite-dimensional Hopf algebras," which gave an overview of her joint work with Benkart, Biswal, Kirkman, and Zhu [10, 11]. Her talk focused on two distinct ways of studying tensor representations of finite-dimensional Hopf algebras. The first method involved studying McKay matrices associated to finite-dimensional Hopf algebras, which encode the relations for tensoring simple modules with a particular representation of the Hopf algebra. The second part of the talk focused on the study of the centralizer algebra of tensor representations of quasi-triangular Hopf algebras.

  The main motivation for studying McKay matrices is to find a relationship between them and an appropriate analogue of the character table for finite-dimensional Hopf algebras. Such a relationship was studied by Witherspoon in the late '90s for finite-dimensional almost cocommutative Hopf algebras, where it was shown that the characters provide eigenvectors for the McKay matrix. Several interesting results concerning McKay matrices were presented during the lecture. Of particular interest are certain identities involving the trace of the action of the Hopf algebra on the simple modules. This gives rise to the notion of left and right eigenvectors for the McKay matrix and allows the authors to recover a result of Grinberg, Huang, and Reiner as a special case.

  The study of centralizer algebras mainly focused on the case where the Hopf algebra is the Drinfeld double of a Taft algebra, which is an example of a non-semisimple quasitriangular Hopf algebra. The main result in this direction is that there is an injective algebra homomorphism from some Temperley-Lieb algebra to the centralizer algebra of a tensor power of a 2-dimensional module. This is an isomorphism for small values of the tensor power, giving a full description of the centralizer algebra. There will undoubtedly be more interesting future work done in further understanding how eigenvectors of the McKay matrix relate to the character theory of Hopf algebras.

- Dan Rogalski gave the talk "Results on infinite-dimensional weak Hopf algebras," which discusses his joint work with Rob Won and James Zhang [52]. This was a very clear presentation, which discussed the advancement of a program to prove analogues of the properties of finite-dimensional Hopf algebras in the setting of infinite-dimensional weak Hopf algebras. In particular, to prove: (1) the existence of (unique) integrals, which are realized as hom spaces in the category of modules for the Hopf algebra; (2) the Gorenstein property; and (3) the existence of a Nakayama automorphism and associated winding automorphism.

  Weak Hopf algebras can be motivated from the direction of monoidal categories. Hopf algebras occur precisely, via Tannakian reconstruction, from tensor categories with fibre functors to the category of vector spaces. On the other hand, when the category of vector spaces is replaced by bimodules for a finite-dimensional semisimple algebra, one reconstructs a weak Hopf algebra whose representation theory gives the original tensor category. Motivation for studying weak Hopf algebras also arises from recent work of Huang, Walton, Wicks, and Won [37], in which they arise from considering natural algebraic structures co-acting on a path algebra modulo relations. Yet another motivation for the study of weak Hopf algebras comes from their flexibility: while Hopf algebras are not closed under direct sum, weak Hopf algebras are, and so these algebras have become increasingly important objects of study within noncommutative algebra.

  On the other hand, the algebraic definitions of weak Hopf algebras are sometimes difficult to work with. Therefore, generalizations of (1), (2), and (3) above are desirable. For Noetherian infinite-dimensional Hopf algebras, a driving research question was the Brown-Goodearl conjecture, which states that an infinite-dimensional Noetherian Hopf algebra has finite injective dimension, and is Artin-Schelter Gorenstein. A partial analogue of this conjecture was recently proven in the weak Hopf setup by Rogalski, Won, and Zhang [52]. In particular, if a Noetherian weak Hopf algebra is finite over an affine centre, then it is proved to have finite injective dimension and to decompose as a direct sum of Artin-Schelter Gorenstein algebras.

Work in progress on the existence of integrals for infinite-dimensional weak Hopf algebras was also discussed. This approach builds on work of Lu, Wu, and Zhang [42], where integrals for infinite-dimensional Gorenstein Hopf algebras were given via Ext groups (as opposed to usual Hom groups in the finite-dimensional setting).

The novel results presented in this talk will serve to significantly advance the theory in an interesting direction. Since Hopf algebras have had major applications in representation theory, algebraic geometry, and topology, I expect that these results will motivate similar applications for weak Hopf algebras.

- Evelyn Lira-Torres, a PhD student, gave a talk called "Quantum Riemannian Geometry on the Fuzzy Sphere," which was based on her joint work with Majid [43]. This talk was unique in that it was at the interface of quantum physics and noncommutative algebra. The quantum spacetime hypothesis states that spacetime is not a continuum, and if one accepts this hypothesis, spacetime can then be effectively described via the use of a noncommutative coordinate algebra. In this talk, the speaker followed this line of thought and looked at the "fuzzy sphere," which is defined to be an algebra A which is the enveloping algebra of the angular momentum Lie algebra factored out by an ideal generated by a single relation. The goal of this talk was to explore the quantum Riemannian geometry of the fuzzy sphere, and this talk gave an interesting glimpse into connections between noncommutative algebra and mathematical physics.

- Jason Gaddis spoke on "Pointed Hopf actions on quantum generalized Weyl algebra," in joint work with Robert Won [33]. Gaddis' talk dealt with Hopf actions of pointed Hopf algebras on $\mathbb{Z}$-graded algebras. A special case of this had been looked at earlier by Cuadra, Etingof, and Walton [28], who showed that actions of semisimple Hopf algebras on Weyl algebras (which are $\mathbb{Z}$-graded) necessarily factor through group algebras. In light of the work of [28], it is natural to ask what happens for Hopf algebra actions on generalized Weyl algebras.

  The main result presented in this talk was the classification of inner-faithful actions of generalized Taft algebras on quantum generalized Weyl algebras which respect the $\mathbb{Z}$-grading. Towards the main result, other interesting questions were studied, notably *which cyclic subgroups of automorphism groups of generalized Weyl algebras are in fact realizable as restrictions of an inner-faithful action to the group of group-like elements?*

  Gaddis closed by computing the invariants for inner-faithful actions of Taft algebras on generalized Weyl algebras under Taft algebra actions and showed generically that these invariant rings are commutative rings whose associated graded rings are Kleinian singularities.

A complete list of talks can be found at the BIRS website.

Scientific Progress Made and Outcome of the Meeting

As mentioned at the beginning of this report, the workshop was a hybrid meeting, which had 22 in-person participants with various research interests connected to noncommutative algebra. As a result of this meeting, researchers working in different areas of noncommutative algebra were able to engage in often-fruitful discussions and many new projects were started during this workshop. This meeting had several younger researchers, postdocs, and graduate students and this meeting was particularly useful in allowing them to network and create new joint projects.

Additionally, several participants noted that this was a great workshop with many inspiring talks and new opportunities for collaboration, and many were appreciative of the return for face-to-face discussions. The program had several lectures highlighting recent progress on difficult problems and we include some testimonials from participants about research programs that they started at Banff and thoughts on results disseminated during the lectures.

First, several key advances in the field were disseminated during the workshop. Notably, Dan Rogalski discussed his recent proof of the Brown-Goodearl conjecture for module-finite weak Hopf algebras, with Robert Won and James Zhang [52]. In particular, they show for an affine weak Hopf algebra that is a finite module over its centre, one obtains several desirable features: finite injective dimension and Artin-Schelter regularity.

Hongdi Huang, Van C. Nguyen, Charlotte Ure, Kent B. Vashaw, Padmini Veerapen, and Xingting Wang recently began a sequence of collaborations [34, 35, 36] that began through an AIM-square. The authors all gave talks during the conference, with several of them discussing key parts of their recent accomplishments. Padmini Veerapen noted that the fact all of her collaborators were present allowed them to meet "on some nights and made good progress."

Padmini Veerapen added further remarks about how invaluable the BIRS environment and opportunity to meet with others was to her, saying, "I'm currently working on three projects and I was able to make good progress on all three while being at BIRS. With my teaching load, it can be hard to get research work done during the fall semester." In addition to the above-mentioned work with Huang and others, which is her first major project, she noted about the second project, "my collaborator and I spent a lot of time discussing background material with Dan Rogalski—I know that [for] some of the stuff Dan told us, it would have taken 2–3 months to figure this out on my own from books." While about her third project she said, "I was excited by the work of Kenta Ueyama and asked him if he would be willing to add something to a survey of twists paper that I'm writing and he happily accepted! I've only heard of Kenta and never met him but have always been super impressed by this work."

Charlotte Ure noted that she had a great and productive time at the workshop, saying, "I had many useful conversations with the other participants and got some new ideas for my current research projects. In particular, I've been working on a question related to the period-index problem on Jacobians that I was able to discuss. Additionally, as I'm usually collaborating over Zoom with Hongdi Huang, Van Nguyen, Padmini Veerapen, Kent Vashaw, and Xingting Wang, it was very productive to come together in person. We received some input and new questions after our talks that we've been discussing since."

Hongdi Huang noted that this workshop provider her with "a great chance for us to communicate mathematics in person, which is beneficial for our continuing collaboration and promoting our friendship as well." She also observed that there were many great and inspirational talks, and added, "during this conference, my collaborators ... obtained some new ideas on how to develop our future research direction. We also learned some new tools which are very useful to our current project."

James Zhang said, "I enjoyed the talks and the conversations with many participants. During the workshop Dan Rogalski and I were wondering if there are higher-Koszul AS algebras with finite Gelfand-Kirillov dimension. Now we might have some ideas to show that there is no N-Koszul AS regular algebra with finite (and large enough) Gelfand-Kirillov dimension for global dimension up to 8. I feel that in-person discussions are extremely helpful."

Ryan Kinser said that he "made several connections which contributed positively to my research program and career development during the meeting. One was the opportunity to speak with senior colleagues about their experiences writing and reviewing NSF research grant proposals. I received guidance which led to a higher quality proposal and a clearer idea of what is likely to be funded. Another was initial discussions with Matt Satriano about future collaboration (also with Jenna Rajchgot) to prove a 2009 conjecture of Anders Buch on equivariant K-classes of 'quiver cycles'. We plan to further collaborate on this, and explore the possibility of submitting a Research in Teams proposal to Banff in 2023."

Xin Tang wrote the following: "Though I have watched and been influenced by many recorded talks in BIRS, it is the first time that I am finally visiting BIRS. I feel lucky to be one of the participants, and I really enjoyed the 'view'. There are many great talks during the workshop, especially several of which are coming from an impressive collaborative group. I liked the talk 'Some Open Questions in Noncommutative Algebra' by James Zhang, which gives a wonderful survey of the current status. Milen Yakimov's talk 'Azumaya Loci of Root of Unity Quantum Cluster Algebras' is fascinating as his many other talks. These are just some examples, and there is much more to be said about many other illuminating talks. I have also had the chance to have extensive discussions with my collaborators about our projects concerning Poisson algebras. Motivated by the energetic talk 'Local Forms of Noncommutative Functions' by Michael Wemyss, we are led to consider a small addition in our current ongoing project. Everything is so convenient and well-organized. It had been a terrific week for me, and I hope that I will be able to visit BIRS again."

Jason Gaddis said, "I want to begin by thanking the organizers for the opportunity to attend, and the staff at BIRS for facilitating a very successful meeting. It was refreshing to be back in person. I very much appreciate the number of interesting talks and the ability to converse and share ideas with colleagues. One recurrent theme that stuck out to me was the idea of twisting in its various forms. The talks by Padmini Veerapen and Charlotte Ure both touched on the idea of graded ('Zhang') twists versus 2-cocycle twists in the setting of coalgebras. Xingting Wang discussed graded twists of Poisson algebras, while Kenta Ueyama talked on twisted Segre products. This is related to other material in the literature on graded twists and twisted tensor products, for example the work of Conner and Goetz."

He added, "In considering these topics, one question that came up in conversation was how to properly define a twisted tensor product of Poisson algebras. An Ore extension of automorphism type can be defined as a graded twist of a polynomial ring, but can also be defined as a twisted tensor product. The construction discussed by Wang leads to the first realization of a Poisson Ore extension as a graded Poisson twist. The question is whether there is an appropriate realization of the second type for Poisson Ore extensions. I believe that further discussions will lead to interesting collaborations on the subject."

Finally, a collaboration between Matt Satriano and Colin Ingalls also started, motivated by Satriano's question about an analogue of the wild automorphism question of Rogalski, Reichstein, and Zhang [47] in the setting of vector fields on a projective variety. Here one asks if X is a projective variety with a section s of the tangent bundle of X with the property that there are no proper subvarieties Y of X with $s|_Y$ restricting to a map from Y to its tangent bundle, then X must be an abelian variety. Ingalls and Satriano made significant progress towards a resolution of this problem during the week and they feel this will ultimately result in a joint publication along with Rahim Moosa.

We look forward to seeing the future progress made from the collaborations begun at this BIRS workshop and future research directions shaped by the presentations given.

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